Thresholds of Large N Factorization in CFT$_4$: Exploring bulk spacetime in AdS$_5$

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ABSTRACT

Large N factorization ensures that, for low-dimension gauge-invariant operators in the half-BPS sector of $\mathcal{N} = 4$ SYM, products of holomorphic traces have vanishing correlators with single anti-holomorphic traces. This vanishing is necessary to consistently map trace operators in the CFT$_4$ to a Fock space of graviton oscillations in the dual AdS$_5$. We investigate the regimes at which the CFT correlators do not vanish but become of order one in the large N limit, which we call a *factorization threshold*. Quite generally, we find the threshold to be when the product of the two holomorphic operator dimensions is of order $N \log N$. Our analysis considers extremal and non-extremal correlators and correlators in states dual to LLM backgrounds, and we observe intriguing similarities between the the energy-dependent running coupling of non-abelian gauge theories and our threshold equations. Finally, we discuss some interpretations of the threshold within the bulk AdS spacetime.

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Contents

1 Introduction

In the AdS/CFT correspondence [1, 2, 3], chiral primary operators of small dimension in $\mathcal{N} = 4$ super Yang-Mills theory are dual to Kaluza-Klein graviton excitations in Type IIB supergravity

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on $AdS_5 \times S^5$ [3]. A remarkable early success of AdS/CFT was the explicit large N calculation and matching of the three-point correlators of gauge theory operators with the associated graviton correlators in supergravity [4]. On the gauge theory side, the operators are symmetric traceless combinations of the six adjoint scalar fields, and the correlator can be calculated at zero gauge coupling g_{YM}^2. On the supergravity side, the corresponding fields arise from the Kaluza-Klein reduction along the 5-sphere of excitations of the metric and the self-dual 5-form field strength. The agreement between these correlators on both sides of the correspondence is possible because three-point functions of chiral primary operators are not renormalized [5, 6, 7, 8, 9, 10].

The half-BPS sector of chiral primary operators is described by a single holomorphic matrix $Z = \Phi_1 + i\Phi_2$, formed from the complex combination of two adjoint hermitian scalars [12, 13, 14]. Single trace operators consisting of a small number of Z matrices can be matched to single particle bulk graviton states, and multi-trace operators can be matched to multi-graviton states. The number of matrices J in a single trace operator corresponds to the angular momentum of the Kaluza-Klein graviton in the S^5 directions. For a three-point extremal correlator of the form

$$\langle \mathrm{tr} Z^{J_1}(x_1) \mathrm{tr} Z^{J_2}(x_2) \mathrm{tr} Z^{\dagger J_1 + J_2}(y) \rangle, \tag{1.1}$$

the conformal symmetry allows the spacetime dependence of the correlator to be factored out completely. The remaining factor is purely combinatoric, and gives the CFT inner product between the 2-graviton state and the 1-graviton state. This combinatoric factor is known exactly for finite N, arising directly from Wick contraction combinatorics of matrices [15, 16]. With an appropriate normalization, this free-field correlator goes to zero in the limit of large N when the operator dimensions J_i ($i = 1, 2$) are kept fixed. This is an example of a general property of large N physics called *large N factorization*. It is necessary for a weakly-coupled Fock space description of the bulk theory to be valid. The single trace operators can be matched with a set of graviton oscillators

$$tr Z^J \leftrightarrow \alpha_J, \tag{1.2}$$

with the commutation relations $[\alpha_{J_1}, \alpha_{J_2}^\dagger] = \delta_{J_1, J_2}$, which annihilate the AdS vacuum state $\alpha_J |0\rangle = 0$. The excitations of the vacuum state form a Fock space, and correlators of states with different numbers of excitations are orthogonal:

$$\langle 0 | \alpha_{J_1} \alpha_{J_2} \alpha_{J_1 + J_2}^\dagger | 0 \rangle = 0, \tag{1.3}$$

which is in agreement with the CFT correlator at large N.

In this paper we will be interested in the growth of the operator dimensions J_i which leads to the failure of factorization. We find that if the J_i grow sufficiently rapidly with N, the normalized correlator diverges as $N \to \infty$. We undertake a detailed study of the *factorization threshold*, defined to be the submanifold of the space of parameters (dimensions and global symmetry charges of the operators, and N) on which the normalized correlator is equal to a

constant c, chosen for convenience to be 1 in most formulae. At the threshold, c can be as small as we like but independent of N. It therefore makes sense in this regime to associate single traces to single objects and multi-traces to multiple objects, just as it does below the threshold. However, a standard Fock space structure as the starting point for a $1/N$ expansion is not appropriate at the threshold. Here, composite states made of a pair of gravitons have non-vanishing quantum correlations with states consisting of a single graviton, even as N is taken to infinity. This motivates the detailed characterization and interpretation of the threshold, which we undertake in this paper. Above the threshold, associating single traces to single objects of any sort probably does not make sense. Certainly, for J_i of order N, it is known that the gravitons are represented semiclassically by D3-branes wrapping a sphere [17], and cannot be represented as single traces [12]. The correct basis for single and giant gravitons is given by Schur polynomials, indexed by Young diagrams [13].

The aim of this paper is to introduce and investigate the threshold of factorization for several cases of correlators in the half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills, and to explore the implications of these in the dual $AdS_5 \times S^5$ spacetime. We focus on three types of correlator in particular: an extremal three-point correlator with one independent angular momentum J, an extremal three-point correlator with two independent angular momenta J_1 and J_2, and a non-extremal three-point correlator. We also consider briefly some extensions concerning extremal correlators on non-trivial backgrounds and extremal correlators with a large number of operators.

In Section 2 we give an overview of our results, introducing the definition of the factorization threshold and stating without detailed calculation the form of the threshold in the simplest case. The local gauge invariant operators are functions of a four-dimensional spacetime position and an energy J, which is equal to angular momentum because of the BPS condition. We explain an interesting aspect of our results, namely the similarity of the dependence of the threshold on separations in spacetime and on differences in energy. We elaborate on the departure from the usual Fock space structure associated with traces at large N and raise the question of a spacetime effective field theory derivation of the properties of the threshold. This is one of our motivations for performing detailed studies of the threshold.

In the subsequent sections, we present the details of the calculations of the thresholds. In Section 3 we review and introduce some notation on large N asymptotics for describing the thresholds precisely, and give a complete calculation of the extremal three-point correlator with one independent angular momentum J. We also discuss in this section some links between the form of the threshold equations with running gauge coupling equations and instanton expansions. In Section 4, we present a calculation of the three-point extremal correlator when the operator dimensions are not equal. In Section 5, we calculate a non-extremal three-point correlator, and discuss how it differs from the extremal cases.

We discuss in Section 6 some other tractable examples of extremal correlators that could shed more light on the general nature of factorization thresholds. We consider the case of a correlator with k holomorphic insertions, and also the case of a three-point correlator on a non-trivial background dual to an LLM geometry [18]. We conclude by summarizing what has been

shown about factorization within this paper, and discussing some other examples of correlators that could tell us more about the general nature of factorization thresholds in the future.

2 Factorization thresholds and bulk interpretations

In this section we describe the factorization threshold for the simplest case: the transition of two gravitons with the same angular momentum J going to a single graviton of angular momentum $2J$. This is followed by a discussion of the physics at the threshold in the bulk AdS space. This motivates further investigations of thresholds, which we outline, along with the qualitative results. The details of these investigations are presented in subsequent sections.

2.1 Thresholds of factorization in the gauge theory

Our starting point is the three-point correlator of two holomorphic single trace operators and an antiholomorphic single trace operator,

$$\langle \mathrm{tr} Z^{J_1}(x_1) \mathrm{tr} Z^{J_2}(x_2) \mathrm{tr} Z^{\dagger J_1 + J_2}(y) \rangle. \tag{2.1}$$

This correlator is not renormalized [4], and so a calculation in the free field limit will hold for all values of the coupling g_{YM}^2. The position-dependence of the correlator can be factored out by conformal symmetry:

$$\langle \mathrm{tr} Z^{J_1}(x_1) \mathrm{tr} Z^{J_2}(x_2) \mathrm{tr} Z^{\dagger J_1 + J_2}(y) \rangle = \frac{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle}{|x_1 - y|^{2J_2} |x_2 - y|^{2J_2}}. \tag{2.2}$$

The factor in the numerator of this expression is position-independent and can be calculated using character expansions [15]. If we apply an inversion $y' = \frac{y}{|y|^2}$, and transform the anti-holomorphic operator to the primed frame, while taking $x \to 0, y' \to 0$, then the position dependence disappears, and we are left with the purely combinatoric factor which can be interpreted as an inner product of the double trace state and the single trace state. This correlator is *extremal* as the sum of the holomorphic operator dimensions $J_1 + J_2$ is equal to the antiholomorphic operator dimension. In the following sections, we focus on the inner product

$$\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle. \tag{2.3}$$

A natural normalization for these correlators is the *multiparticle normalization*, in which each operator is divided by the square root of its two-point function,

$$\langle\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle\rangle = \frac{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle}{\sqrt{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{\dagger J_1} \rangle \langle \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_2} \rangle \langle \mathrm{tr} Z^{J_1 + J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle}}. \tag{2.4}$$

This normalization is used in comparing AdS and CFT calculations of the 3-point functions [4]. We have introduced the double-bracket notation $\langle\langle \cdot \rangle\rangle$ to refer to a multiparticle-normalized

correlator. It is known [4, 19] that when the operator dimensions J_i are sufficiently small, then

$$\langle\langle \operatorname{tr} Z^{J_1} \operatorname{tr} Z^{J_2} \operatorname{tr} Z^{\dagger J_1 + J_2} \rangle\rangle \sim \frac{\sqrt{J_1 J_2 (J_1 + J_2)}}{N} \tag{2.5}$$

in the large N limit. This clearly tends to zero at large N, and so the single trace and double trace operators are orthogonal at large N.

Large N orthogonality of the operators can still hold when J_1 and J_2 increase with N. By calculating the correlator explicitly at finite N, it can be shown that (2.5) is still valid when J_1 and J_2 are functions of N, provided that $J_1, J_2 \leq \sqrt{N}$ at large N. However, this formula is not valid when J_1 and J_2 grow large enough with N. For large enough J_i, the normalized correlator grows exponentially with N, and factorization of the operators no longer holds. The aim of this paper is to investigate and interpret the threshold partitioning these two distinct large N limits of the normalized correlator.

For simplicitly, we initially consider in Section 3 a correlator in which the holomorphic operator dimensions are equal. Setting $J_1 = J_2 = J$, we define

$$G_3(J, N) = \langle\langle \operatorname{tr} Z^J \operatorname{tr} Z^J \operatorname{tr} Z^{\dagger 2J} \rangle\rangle. \tag{2.6}$$

To gain some insight into the large N behaviour of this correlator when J depends on N, we can plug in a simple trial function $J(N)$ and find the asymptotic behaviour of the correlator when N is large. If we set $J = N^\alpha$, where α is a constant, then a finite N calculation [19] shows that

$$\boxed{\begin{aligned} G_3(N^\alpha, N) &\to 0, & 0 < \alpha \leq \frac{1}{2}, \\ G_3(N^\alpha, N) &\to \infty, & \frac{1}{2} < \alpha < 1. \end{aligned}} \tag{2.7}$$

If J grows as a power of N larger than $\frac{1}{2}$, then the correlator will diverge and factorization breaks down. However, a simple power-law scaling is not sufficient to deduce the exact growth of J that is required for the correlator to diverge. A more general N-dependence can be found, intermediate between the cases $\alpha = \frac{1}{2}$ and $\alpha > \frac{1}{2}$, for which the correlator tends to a constant value.

Our main approach to considering the threshold between factorization and breakdown is to look for a solution to the equation

$$G_3(J, N) = 1. \tag{2.8}$$

We call this the *factorization threshold equation*. It defines a curve $J(N)$ in the parameter space with axes labelled (J, N). For large enough N, this curve divides the parameter space into two regions: the factorization region, where the correlator is less than one, and the breakdown region, where the correlator is greater than one. The *threshold* $J_t(N)$ is the exact solution of
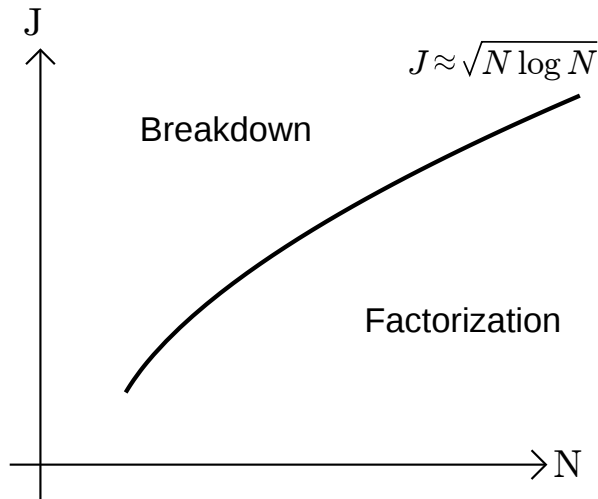
Figure 1: A sketch of the threshold curve $J_t(N)$ in (J, N) parameter space for large N. Away from the origin, the curve partitions the parameter space into the factorization region $G_3(J_t(N), N) < 1$, and the breakdown region $G_3(J_t(N), N) > 1$.

the equation $G_3(J_t(N), N) = 1$. A sketch of this threshold curve in (J, N) parameter space is shown in Figure 1.

The trial function approach shows that the threshold must scale with N at a faster rate than $\sqrt{N}$, but at a slower rate than $N^{\frac{1}{2}+\delta}$ for any constant δ. Provided that J lies in the range $N^{\frac{1}{2}} < J < N^{\frac{1}{2}+\delta}$, we show in Section 3.2 that the correlator G_3 has the asymptotic behaviour

$$G_3(J, N) \sim \sqrt{\frac{J}{2N}} \exp\left(\frac{J^2}{2N}\right). \tag{2.9}$$

Using this asymptotic form of the correlator, we can invert the equation $G_3(J_t(N), N) = 1$ to derive an asymptotic solution of $J_t(N)$, the threshold of factorization. In Section 3.3 we show that the large N solution is

$$J_t(N) = \sqrt{\frac{1}{2}N \log N} \left[1 - \frac{\log \log N}{2 \log N} + \frac{\log 8}{2 \log N} + \mathcal{O}\left(\frac{\log \log N}{\log N}\right)^2\right]. \tag{2.10}$$

Neglecting the constant term, the leading-order behaviour is simply

$$J_t^2 \approx N \log N. \tag{2.11}$$

This is the solution that divides (J, N)-space into the regions where factorization holds and breaks down.

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0   J $2J$ Graviton energy

2-graviton state $\begin{cases} \mathrm{tr}Z^J \\ \circ \\ \circ \\ \mathrm{tr}Z^J \end{cases}$ $\mathrm{tr}Z^{2J}$ $\circ$ 1-graviton state
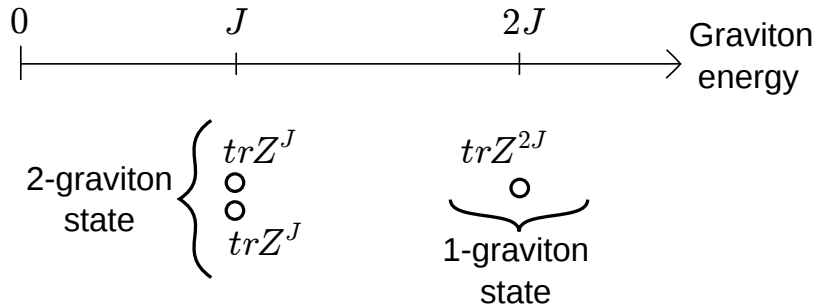
Figure 2: The single and multi-graviton states within energy space.

2.2 The breakdown of bulk effective field theory at the threshold

The correlator $\langle \mathrm{tr}Z^J \mathrm{tr}Z^J (\mathrm{tr}Z^\dagger)^{2J} \rangle$ is not renormalized [4]. It is an inner product of the double trace state with the single trace state, normalized by the appropriate factors given above. A sketch of these two states in energy space is given in Figure 2. In the CFT computation, this is a non-trivial inner product which mixes trace structures according to a non-trivial function of J and N. This inner product can equally be computed for J of order one in the large N limit in the dual supergravity. The supergravity computation can be understood as relying on a Fock space structure for gravitons, where at leading large N single gravitons are orthogonal to multi-gravitons, hence single traces are orthogonal to multi-traces. This Fock space structure is used to set up perturbation theory where there are $\frac{1}{N}$ interactions. The N-corrected inner product coming from CFT is then recovered with the help of the supergravity interactions. At the factorization threshold, the leading large N overlap is not vanishing; it is order one. So a Fock space structure with single gravitons corresponding to single traces, being orthogonal to multi-gravitons corresponding to multi-traces, cannot be the right spacetime structure for computing the leading large N behaviour of the correlator. There should be a modification of the spacetime effective field theory which reproduces the correlators at threshold. This modification is unknown, but hints about its nature can be obtained by studying the detailed properties of the threshold.

Once the angular momenta J are sufficiently large that we are well past the threshold and into the region of broken factorization, we eventually reach the region of $J \approx N$, where the best way to think about the physics is in terms of giant gravitons [12]. The basis of Schur polynomial operators, which are non-trivial linear combinations of multi-traces, becomes the best way to match bulk states and CFT states [13]. The region where $J \approx N$ was indeed earlier identified as an interesting region in connection with the fact that finite N relations allow single traces to be expressed in terms of multi-traces via Cayley-Hamilton relations [20]. This lead to a stringy exclusion principle, suggestive of some form of algebraic deformation of the spacetime algebra of functions [21].

Here we focus instead on the threshold near $J \approx \sqrt{N \log N}$, where the large N correlator is

not infinite, but fixed at $G_3 = 1$. We could even take $G_3 = c$ for a small c, say 10^{-5}, but not going to zero as N approaches infinity. So it is very plausible that a spacetime picture in terms of elementary objects whose number matches the number of traces, e.g. gravitons or gravitons stretched into BMN strings, is the right framework for understanding the precise nature of the threshold and the form of the interactions in this threshold region.

With these motivations spelt out, we turn to some qualitative outcomes of our detailed studies of how the thresholds are approached when various parameters in the graviton system are tuned. An intriguing result we find is that, as we explain further in the next subsection, in some of its effects on the factorization threshold, separation in J-space is similar to separation in coordinate space. It is tempting to interpret this by associating the J-quantum number of a graviton to the radial AdS dimension, in the spirit of the UV-IR relation [22, 23]. This line of argument was adopted in the first version of the paper. This turns out to be rather subtle.[4] It is true that we can make an argument relating spatial extents to graviton energies by considering the LLM picture [18]. The trace is a superposition of Schur polynomials corresponding to hook representations, interpolating between a single row and a single column Young diagram. This is a superposition of states in the free fermion picture involving excitation of a fermion from some depth k below the top of the Fermi sea to a level $(J - k)$ above the Fermi sea, with k varying from 0 to $(J-1)$. Since the fermion energy levels translate to radial positions in the LLM plane, with large radial positions of the excited fermion being closer to the boundary, this is in line with the naive UV-IR argument. However, consideration of normalizable modes in the global coordinates shows that gravitons at higher energy J become more localized near the centre [24]. This suggests that the interpretation of half-BPS correlators in terms of gravitons requires care regarding the distinction between normalizable and non-normalizable modes of the same field, and between the Lorentzian versus Euclidean picture of AdS. It is therefore prudent to postpone a detailed spacetime interpretation of the thresholds at this stage. It is nevertheless clear that this breakdown of the standard Fock space structure of effective spacetime field theory is an important new window where the gauge theory can provide valuable information to guide the spacetime understanding.

2.3 Refined investigations of the factorization thresholds

In Section 4 we investigate the more general extremal normalized three-point correlator

$$G_3(J_1, J_2, N) = \langle\langle \text{tr} Z^{J_1} \text{tr} Z^{J_2} \text{tr} Z^{\dagger J_1 + J_2} \rangle\rangle \tag{2.12}$$

where $J_1 \neq J_2$. We define the threshold to be the surface in the three-dimensional parameter space (J_1, J_2, N) that satisfies

$$G_3(J_1, J_2, N) = 1. \tag{2.13}$$

Making the assumption that both J_1 and J_2 grow at least as large as a positive power of N, then we find in Section 4 that the correlator decays to zero if the product of the angular momenta

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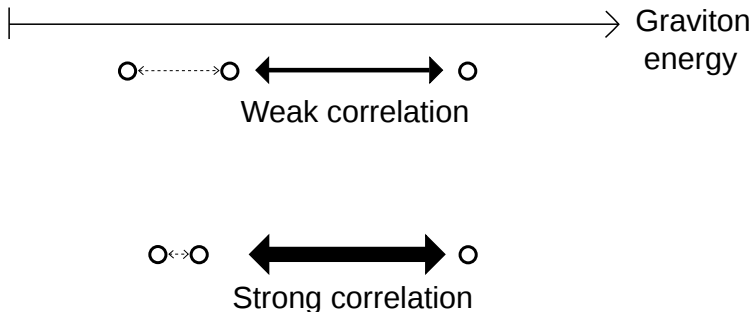[4]We thank the JHEP referees for comments on this point.

Figure 3: Two systems of gravitons with different energy differences but the same total energy. Graviton states become strongly correlated when the separation of the graviton energies decreases.

$J_1 J_2$ is less than N at large N, and grows exponentially if $J_1 J_2$ grows faster than $N^{1+\delta}$ with N, where δ is any positive constant. If the angular momenta are constrained to lie in the range $N < J_1 J_2 < N^{1+\delta}$, then an asymptotic form of the correlator can be found. We find that at large N in this regime, the threshold lies at

$$J_1 J_2 \approx N \log N, \tag{2.14}$$

where we have dropped a constant multiplicative factor.

In the bulk picture, single trace operators with different dimensions correspond to gravitons at different energies. The combined energy of the two gravitons with energies J_1 and J_2 is equal to the energy of the other graviton $(J_1 + J_2)$. If we fix N and the energy $(J_1 + J_2)$ of the more energetic graviton, but vary the *difference* in the energies of the less energetic gravitons $\Delta J = |J_1 - J_2|$, then we find that we can move within parameter space *from the factorization region to the threshold by decreasing the difference in energies of the two gravitons*. This is illustrated in Figure 3.

Another related set-up is a strongly-correlated system of gravitons at the threshold in which N and the value of the correlator $G_3(J_1, J_2, N) = 1$ are fixed but the separation of the graviton energies is varied. Once N is fixed and we are constrained to the threshold surface, there is only one available free parameter in the system, which we take to be the separation of the graviton energies $|J_1 - J_2|$. It can be shown that *increasing the separation in energies $|J_1 - J_2|$ of the two gravitons at the threshold corresponds to an increase in the energy $(J_1 + J_2)$ of the single graviton state*. This system is shown in Figure 4.

We extend the investigation of factorization thresholds to the case of non-extremal correlators. In particular we study in detail the multiparticle-normalized correlator

$$\langle\langle \mathrm{Str}(Z^{J_1} Y) \mathrm{Str}(Z^{J_2} Y^\dagger) \mathrm{tr}(Z^{\dagger J_1 + J_2}) \rangle\rangle \tag{2.15}$$

and find a sensible extension of the discussion of factorization thresholds from the extremal case. In the discussion of extremal correlators above, we did not pay much attention to the
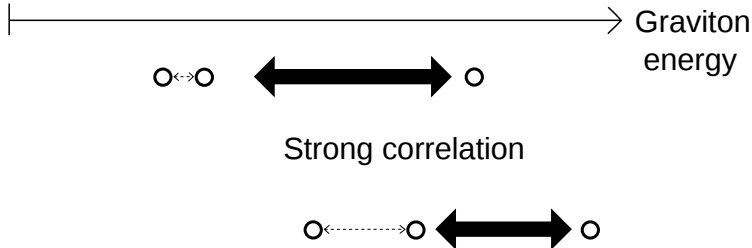
Figure 4: Two systems of gravitons at the threshold with different energy separations. Graviton states become strongly correlated at lower energies (further from the boundary) when the separation of the graviton energies is smaller.

spatial dependences of correlators. There is a simple reason for this. In the extremal case, we can set the two holomorphic operators at one point x_1 and the anti-holomorphic operator at another point x_2. This has the standard dependence $|x_1 - x_2|^{-2(J_1+J_2)}$. The spatial dependence can be removed by taking the anti-holomorphic operator to infinity, changing frame by the inversion $y = \frac{x_2}{|x_2|^2}$. In this limit the correlator is computing an inner product of states and all position dependences disappear after we take into account the conformal transformation of the anti-holomorphic operator. In the above non-extremal case we can set the first operator at x_1, the second at $x_1 + \epsilon$ and take the third operator to infinity by applying an inversion. The only position dependence left is ϵ^{-2}. So the above correlator is a dimensionful quantity and it does not make sense to ask when it is equal to one in the large N limit.

We can introduce a dimensionful energy cutoff Λ in the CFT. This dimensional cutoff will not change the CFT calculation if we take $\epsilon\Lambda \gg 1$. The correct quantity to use to define the threshold is then Λ^{-2} times the non-extremal correlator above. This will be dimensionless, will contain the dimensionless parameter $\epsilon\Lambda \equiv \mathcal{R}$ and can be compared to one to define a factorization threshold. In the region of J_i of order one and $\mathcal{R} \approx 1$ there is factorization, but appropriate growth of J_i with N can cause breakdown of factorization, the details of the threshold depending on the dimensionless $\mathcal{R}$. We find that *decreasing $\epsilon\Lambda$, within the regimes where the correlator calcuations are valid, can cause the transition from factorization to breakdown.* This is in line with the discussion in [25], where short distances were argued to explore large energies which have to be low enough in relation to N for factorization to hold. Another interesting aspect of this nearly-extremal correlator is that when $\mathcal{R} = \epsilon\Lambda$ is large and fixed, or only varies with N as a power or less, then the threshold is of the same form as the extremal correlator; we find the threshold lies at $J_1 J_2 \approx N \log N$.

Later in the paper, we consider the transition from multiple holomorphic traces to a single anti-holomorphic trace, or equivalently multiple gravitons going to a single graviton. If we have k starting gravitons, with k order 1, we find that the threshold depends on the *largest pairwise product $J_i J_j$*, and occurs at $J_i J_j \sim k^{-1} N \log N$. The threshold of factorization *decreases* as the number of gravitons in the multi-graviton state *increases*.

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Another generalization of the threshold investigation involves considering three-point extremal correlators corresponding to graviton scatterings on an LLM background given by M maximal giant gravitons, as in [26]. When M is of the same order as N, then the factorization threshold is $J_1 J_2 \approx (M + N) \log(M + N)$. If M is chosen to have a fixed linear dependence on N, then the leading order behaviour of the threshold is again $J_1 J_2 \approx N \log N$, up to a constant factor.

We conclude that another striking property of the thresholds is the universality of the leading large N behaviour of the form $J_i J_j \sim N \log N$.

3 The extremal three-point correlator with $J_1 = J_2$

In this section we present a detailed calculation of the asymptotic form of the three-point correlator

$$G_3(J, N) = \langle\langle \mathrm{tr} Z^J \mathrm{tr} Z^J \mathrm{tr} Z^{\dagger 2J} \rangle\rangle \tag{3.1}$$

in the relevant region $N^{\frac{1}{2}} < J < N^{\frac{1}{2}+\delta}$, where δ is any small positive constant. We then asymptotically solve the threshold equation

$$G_3(J_t(N), N) = 1 \tag{3.2}$$

in the large N limit, deducing that at leading order the threshold behaves as

$$J_t \sim \sqrt{\frac{N}{2} \log N}. \tag{3.3}$$

Further, we find explicitly the all-orders asymptotic expansion of the threshold, and attempt to extend this result past perturbation theory by deriving a transseries expansion. Finally, we discuss some links between the form of the threshold solution and running couplings in QCD.

3.1 Review of asymptotics and series

We start by briefly reviewing and clarifying some definitions, and introducing some new notation. Throughout this paper, we will be using the precise mathematical definition of the asymptotic symbol '$\sim$', the 'little o' order symbol o, and asymptotic series. We will also be using a precise definition of the 'big O' order symbol $\mathcal{O}$ that differs slightly from that used in the literature, but which is stronger than the commonly-used definition.

For two N-dependent functions $f(N)$ and $g(N)$, then we say that $f \sim g$ at large N if

$$\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1. \tag{3.4}$$

Note that with this definition the ratio of these two functions must tend to one, and not to any other constant. We use the notation $f = o(g)$ if f is a function that satisfies

$$\lim_{N \to \infty} \frac{f(N)}{g(N)} = 0, \tag{3.5}$$

i.e. if f is much smaller than g at large N. From these definitions, the following two statements are equivalent:

$$\begin{aligned} f(N) &\sim g(N) \\ f(N) &= g(N)(1 + o(1)). \end{aligned} \tag{3.6}$$

We shall also use the notation $f \ll g$ if $f = o(g)$, and conversely $f \gg g$ if $g = o(f)$. An *asymptotic series* at large N is formally defined by a set of functions $\{\phi_k(N)\}$ and constant coefficients $\{a_k\}$ with the property that

$$\phi_{k+1} = o(\phi_k) \tag{3.7}$$

for any $k \geq 0$. We say that

$$f(N) \sim \sum_{k=0}^{\infty} a_k \phi_k \tag{3.8}$$

if, for any $n \geq 0$, we have

$$f - \sum_{k=0}^{n} a_k \phi_k = o(\phi_n). \tag{3.9}$$

This definition of an asymptotic series does not allow for terms which are subleading to all the ϕ_k. Later, we shall also employ an extended version of an asymptotic series called a *transseries*. This type of series contains extra terms that tend to zero faster than all terms in a classical asymptotic series, but can still be assigned meaning when considered as a formal sum. Transseries are commonly used in describing instanton corrections to series expansions generated in QFTs, in which the instanton-dependent terms are exponentially suppressed in the coupling constant. We discuss this more in Section 3.5.

In this paper, we write $f = \mathcal{O}(g)$ (or occasionally $f \approx g$) if there exists some positive constant C such that

$$\lim_{N \to \infty} \frac{|f(N)|}{|g(N)|} = C. \tag{3.10}$$

This is a departure from the $\mathcal{O}$ (big O) notation in common use which only requires the ratio f/g to be bounded from above at large N. This modified definition is a stronger condition as it not only implies that f/g is bounded from above, but is also bounded from below too.

This is useful for keeping track of the errors and assumptions made at each step within our calculations.

The $\mathcal{O}$ notation is used for expressing the errors of an N-dependent function, or corrections to an asymptotic series, or for giving a coarse expression of the leading-order behaviour of a function. It is used in the following for representing functions whose explicit forms are unknown or irrelevant, but whose leading-order behaviours at large N are important. Generally, when an upper bound on the leading-order behaviour of a correction is known but a lower bound is not, then we will use the o (little o) symbol. In general, we shall write equations as equalities when the corrections or errors are present, and use '$\sim$' for equations when the error terms have been dropped.

3.2 Asymptotics of the three-point correlator

To solve the threshold equation (3.2), we need to find an asymptotic form of the normalized correlator (3.1) at large N and large J, with small J/N. The form of this expression will change depending on how quickly J grows with N, so it is necessary to carefully specify at each stage what possible behaviour J can take. We will find that the breakdown threshold is located at J just larger than $\mathcal{O}(\sqrt{N})$, and so we will look for a large N asymptotic form of the correlator $G_3(J, N)$ that is valid in this region. It suffices to impose $J \ll N^{2/3}$ to describe the asymptotic form of the correlator around the threshold.

The position-independent two-particle and three-particle correlators are known precisely for finite N [15]. We recall that the two-point function at zero coupling is

$$\langle \text{tr}Z^J \text{tr}Z^{\dagger J} \rangle = J! \left[\binom{N+J}{J+1} - \binom{N}{J+1} \right],$$
(3.11)

and the three-point function (for general operator dimensions J_1 and J_2) is

$$\langle \text{tr}Z^{J_1} \text{tr}Z^{J_2} \text{tr}Z^{\dagger J_1 + J_2} \rangle = (J_1 + J_2)! \left[\binom{N+J_1+J_2}{J_1+J_2+1} - \binom{N+J_1}{J_1+J_2+1} \right.$$
$$\left. - \binom{N+J_2}{J_1+J_2+1} + \binom{N}{J_1+J_2+1} \right], \quad (3.12)$$

All the terms in the finite N correlator expressions are of the form

$$J! \binom{N+\Lambda}{J+1} = \frac{(N+\Lambda-J)}{(J+1)} \frac{(N+\Lambda)!}{(N+\Lambda-J)!},$$
(3.13)

where Λ is either 0 or J for the terms in the two-point function. Taking N and J to be large, but keeping J/N small, we apply Stirling's approximation

$$n! = e^{-n} n^{n+\frac{1}{2}} \sqrt{2\pi} \left(1 + \mathcal{O}\left(\frac{1}{n}\right) \right)$$
(3.14)

to find that

$$J!\binom{N+\Lambda}{J+1} \sim \frac{(N+\Lambda-J)}{J+1}N^J e^{-J}\sqrt{1+\frac{\Lambda}{N}}\sqrt{1+\frac{\Lambda-J}{N}}\left(1+\frac{\Lambda}{N}\right)^{N+\Lambda}\left(1+\frac{\Lambda-J}{N}\right)^{-N-\Lambda+J} \quad (3.15)$$

$$\sim \frac{N^{J+1}e^{-J}}{J}\left(1+\frac{\Lambda}{N}\right)^{N+\Lambda}\left(1+\frac{\Lambda-J}{N}\right)^{-N-\Lambda+J}. \quad (3.16)$$

Here, we have dropped some error terms of order $\mathcal{O}\left(\frac{1}{J}\right)$ and $\mathcal{O}\left(\frac{J}{N}\right)$. We expand the terms in the brackets by taking logs, and using the fact that $\Lambda < N$ to perform a series expansion. We find that

$$\log\left(1+\frac{\Lambda}{N}\right)^{N+\Lambda} = -N\left(1+\frac{\Lambda}{N}\right)\sum_{k=1}^{\infty}\frac{(-\Lambda)^k}{kN^k} \quad (3.17)$$

$$= \Lambda + \sum_{k=1}^{\infty}\frac{(-\Lambda)^{k+1}}{k(k+1)N^k}. \quad (3.18)$$

Hence, replacing Λ with $\Lambda - J$ in the second bracketed factor of (3.16), we find

$$J!\binom{N+\Lambda}{J+1} \sim \frac{N^{J+1}}{J}\exp\left(\sum_{k=1}^{\infty}\frac{(-\Lambda)^{k+1}-(-\Lambda+J)^{k+1}}{k(k+1)N^k}\right). \quad (3.19)$$

We can simplify this expression by dropping the terms in the infinite sum that tend to zero with large N. The kth term in the sum scales like J^{k+1}/N^k for some integer k, so if we impose that $J \ll N^{2/3}$, then all terms with $k \geq 2$ are small. With this condition, we can drop the subleading terms of order $\mathcal{O}(J^3/N^2)$ and write

$$\boxed{J!\binom{N+\Lambda}{J+1} \sim \frac{N^{J+1}}{J}\exp\left(\frac{-J^2}{2N}+\frac{J\Lambda}{N}\right).} \quad (3.20)$$

This expression, which is valid for any $\Lambda \leq J \ll N^{2/3}$, is used repeatedly in the following sections to derive the asymptotics of finite N correlators. Including both terms in (3.11) with $\Lambda = J$ and $\Lambda = 0$ respectively, we can now state that two-point function has the asymptotic form

$$\langle \text{tr}Z^J \text{tr}Z^{\dagger J}\rangle \sim \frac{N^{J+1}}{J}e^{\frac{J^2}{2N}}\left(1-e^{-\frac{J^2}{N}}\right). \quad (3.21)$$

This approach generalizes in a straightforward manner to the three-point function. Replacing J with $2J$ and allowing Λ to take the values 0, J, and $2J$, we find that (3.12) becomes

$$\langle \text{tr}Z^J \text{tr}Z^J \text{tr}Z^{\dagger 2J}\rangle \sim \frac{N^{2J+1}}{2J}\left(e^{\frac{2J^2}{N}}-2+e^{\frac{-2J^2}{N}}\right) \quad (3.22)$$

$$\sim \frac{N^{2J+1}}{2J}e^{\frac{2J^2}{N}}\left(1-e^{-\frac{2J^2}{N}}\right)^2. \quad (3.23)$$

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These expressions allow us to read off the asymptotic form of the normalized three-point function (3.1). We find that

$$G_3 \sim \sqrt{\frac{J}{2N}} \exp\left(\frac{J^2}{2N}\right) \frac{\left(1 - e^{-\frac{2J^2}{N}}\right)^2}{(1 - e^{-\frac{J^2}{N}})\sqrt{\left(1 - e^{-\frac{4J^2}{N}}\right)}}. \tag{3.24}$$

This expression is valid for any behaviour of J provided that $J \ll N^{2/3}$.

To find a more tractable version of this formula at large N, we need to state how J^2/N grows with N. There are three cases to consider: J^2/N going to zero with large N, J^2/N going to a constant, and J^2/N going to infinity. In the first case where J^2/N is small, we can use

$$(1 - e^{-\frac{kJ^2}{N}}) \sim \frac{kJ^2}{N}, \qquad \exp\left(\frac{J^2}{2N}\right) \sim 1 \tag{3.25}$$

where $k \in \{1, 2, 4\}$, to see that

$$G_3 \sim \frac{\sqrt{JJ(2J)}}{N}, \tag{3.26}$$

which is the known behaviour of the normalized three-point correlator for $J \ll \sqrt{N}$. The assumption $J^2/N \to 0$ means that the correlator will tend to zero in this limit, and so factorization holds in this case. Alternatively, in the case that J^2/N tends to a constant value, i.e. $J = \mathcal{O}(\sqrt{N})$, then (3.24) will scale as $\mathcal{O}(N^{-\frac{1}{4}})$ with large N. This means that factorization will still hold in this case. However, in the case that J^2/N grows large with N, then we have

$$(1 - e^{-\frac{kJ^2}{N}}) \sim 1, \qquad \exp\left(\frac{J^2}{2N}\right) \to \infty, \tag{3.27}$$

and thus

$$\boxed{G_3 = \langle\langle \mathrm{tr}Z^J \mathrm{tr}Z^J \mathrm{tr}Z^{\dagger 2J} \rangle\rangle \sim \sqrt{\frac{J}{2N}} \exp\left(\frac{J^2}{2N}\right).} \tag{3.28}$$

This correlator will grow to infinity if J grows quickly enough with N. In particular, if $J \geq N^{\frac{1}{2}+\delta}$ for some small constant $\delta > 0$ at large enough N i.e. if J grows faster than $\sqrt{N}$ by a positive power, then the exponential term dominates and the correlator will tend to infinity. We deduce that the threshold - that is, the growth of J with N which keeps the correlator finite and non-zero at large N - lies in the range

$$N^{\frac{1}{2}} < J < N^{\frac{1}{2}+\delta}, \tag{3.29}$$

where δ is any small positive number. This is the relevant region for solving asymptotically the factorization threshold equation

$$G_3(J_t(N), N) = 1. \tag{3.30}$$

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3.3   Solving the factorization threshold equation

We can use (3.28) in the region (3.29) to find a function $J(N)$ that solves the threshold equation (3.30) at large N. To do this, we write down the *exact* equation

$$G_3 = \sqrt{\frac{J}{2N}} \exp\left(\frac{J^2}{2N}\right) e^{-\frac{1}{4}r}, \tag{3.31}$$

where the error function $r(J, N)$ is implicitly defined by this equation (the factor of $\frac{1}{4}$ here is chosen for later convenience). All the large N approximations that were taken in generating the asymptotic expression (3.28) are encoded in this error function, so it must tend to zero with N (provided that we remain in the range (3.29)). To find the leading-order behaviour of r, we collate the terms dropped at various stages in the previous section. In (3.16) and (3.20), we have dropped terms of order $\mathcal{O}\left(\frac{1}{J}\right)$, $\mathcal{O}\left(\frac{J}{N}\right)$, and $\mathcal{O}\left(\frac{J^3}{N^2}\right)$. As J^2/N is large, all these errors are $\mathcal{O}\left(\frac{J^3}{N^2}\right)$. Also, in performing the approximation

$$\left(1 - e^{-\frac{kJ^2}{N}}\right) \sim 1 \tag{3.32}$$

for various values of k, we have dropped terms of order $\mathcal{O}(e^{-\frac{J^2}{N}})$. At present, we have not specified tight enough constraints on J to determine which is the larger, so we keep both remainders. We write

$$G_3 = \sqrt{\frac{J}{2N}} \exp\left(\frac{J^2}{2N}\right) \left(1 + \mathcal{O}\left(\frac{J^3}{N^2}\right) + \mathcal{O}\left(e^{-\frac{J^2}{N}}\right)\right) \tag{3.33}$$

and so we have

$$e^{-\frac{1}{4}r} = 1 + \mathcal{O}\left(\frac{J^3}{N^2}\right) + \mathcal{O}\left(e^{-\frac{J^2}{N}}\right) \tag{3.34}$$

This means that the error function r is bounded by

$$r = \mathcal{O}\left(\frac{J^3}{N^2}\right) + \mathcal{O}\left(e^{-\frac{J^2}{N}}\right). \tag{3.35}$$

Again, we know that this function tends to zero, but can't yet deduce its leading-order behaviour before solving the threshold equation. Rearranging (3.31), we can write the threshold equation $G_3(J_t(N), N) = 1$ as

$$\left(\frac{2J_t^2}{N} \exp\left(\frac{2J_t^2}{N}\right) \frac{1}{8Ne^r}\right)^{\frac{1}{4}} = 1. \tag{3.36}$$

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This equation cannot be solved exactly in terms of elementary functions (e.g. exponentials, logarithms and powers of z), but it can be rewritten and approximated by using the Lambert W-function. The Lambert W-function is defined by the equation

$$W(z)e^{W(z)} = z. \tag{3.37}$$

It is a multivalued function, but here we just consider the principle branch of the function, where $W(z)$ is positive and real for positive real z. In this regime, a large z asymptotic expansion of the function is known to all orders [27, 28]. Equation (3.36) is solved in terms of the W-function by

$$\frac{2J_t^2}{N} = W(8Ne^r), \tag{3.38}$$

which can be written

$$J_t = \sqrt{\frac{N}{2}W(8Ne^r)}. \tag{3.39}$$

To find a more tractable version of the threshold expressed in terms of elementary functions, we can expand the W-function by using its asymptotic series. The large z expansion of the W-function is [28]

$$W(z) \sim \log z - \log\log z + \sum_{n=1}^{\infty} \left(\frac{-1}{\log z}\right)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ n-k+1 \end{bmatrix} \frac{(-\log\log z)^k}{k!}, \tag{3.40}$$

where the coefficients in the square brackets are the Stirling cycle numbers (of the first kind); the notation $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the number of permutations of n elements composed of k disjoint cycles. We can find the leading-order behaviour of the threshold by truncating this series. However, to guarantee that the truncated solution still satisfies $G_3(J_t(N), N) = 1$ in the large N limit, we need to keep all the terms in the series that do not tend to zero. The first two terms in the series are large as $z \to \infty$, and the remaining terms in the infinite series all go to zero, and so we keep the first two terms and find that the large N solution of (3.39) is

$$\frac{J_t^2}{N} = \frac{1}{2}\left[\log(8Ne^r) - \log\log(8Ne^r) + \mathcal{O}\left(\frac{\log\log N}{\log N}\right)\right]. \tag{3.41}$$

We can now extract out the N-dependence of the remainder function at the threshold, $r(J_t(N), N)$. Since

$$\frac{J_t^2}{N} = \frac{1}{2}\left(\log 8N - \log\log N + o(1)\right), \tag{3.42}$$

we find that

$$e^{-\frac{J_t^2}{N}} \sim \sqrt{\frac{\log N}{8N}}, \qquad \frac{J_t^3}{N^2} \sim \sqrt{\frac{(\log N)^3}{8N}}, \tag{3.43}$$

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and so to leading order in N,

$$r(J_t(N), N) = \mathcal{O}\left(\sqrt{\frac{(\log N)^3}{N}}\right). \tag{3.44}$$

This term is smaller than $N^{-\frac{1}{2}+\delta}$ for any constant $0 < \delta < \frac{1}{2}$, and so all powers of r are subleading to all logarithm-dependent terms in the expansion. We can therefore discard these r-dependent terms as they are 'exponentially suppressed' in terms of the parameter $\log N$. The full asymptotic series expansion of the threshold is

$$J_t^2 \sim \frac{1}{2} N \left[\log(8N) - \log\log(8N) + \sum_{n=1}^{\infty} \left(\frac{-1}{\log(8N)}\right)^n \sum_{k=0}^{n} \left[\begin{matrix} n \\ n-k+1 \end{matrix}\right] \frac{(-\log\log(8N))^k}{k!}\right]. \tag{3.45}$$

Taking square roots and moving out the constant factors in the logs, we deduce that the leading-order terms in the expansion of the threshold are

$$J_t = \sqrt{\frac{1}{2} N \log N}\left[1 - \frac{\log\log N}{2\log N} + \frac{\log 8}{2\log N} + \mathcal{O}\left(\frac{(\log\log N)^2}{(\log N)^2}\right)\right]. \tag{3.46}$$

This is the leading-order solution to

$$G_3(J_t(N), N) := \langle\langle \mathrm{tr}Z^{J_t} \mathrm{tr}Z^{J_t} \mathrm{tr}Z^{2J_t}\rangle\rangle = 1 \tag{3.47}$$

for large N and large J_t.

In (3.46), we have given the first three terms in the expansion of the threshold. This is the necessary degree of accuracy of the threshold $J_t(N)$ for which the truncated series still satisfies the threshold equation in the large N limit. That is, if we take the truncated threshold

$$\tilde{J}(N) = \sqrt{\frac{1}{2} N \log N}\left[1 - \frac{\log\log N}{2\log N} + \frac{\log 8}{2\log N}\right] \tag{3.48}$$

and plug this into the exact expression (3.31), we have

$$G_3(\tilde{J}(N), N) = \exp\left[\frac{1}{16\log N}\left(\log\left(\frac{8}{\log N}\right)\right)^2 - \frac{1}{4}r\right], \tag{3.49}$$

which tends to one in the large N limit. If we had only taken the first term in the threshold solution $\tilde{J} = \sqrt{\frac{1}{2} N \log N}$ and plugged this into (3.31), we would have found that $G_3(\tilde{J}(N), N)$ actually grows logarithmically with N, and so the threshold equation cannot hold for arbitrarily large N. Similarly, truncating the series at the second term causes the correlator $G_3(\tilde{J}(N), N)$ to converge to a different constant than 1 at large N.

We remark that the factors of 8 appearing in the logs have come from choosing the factorization threshold to be at $G_3 = 1$. If we had instead chosen $G_3(J, N) = c$ for some constant c, then the threshold solution would be

$$\frac{J_t^2}{N} = \frac{1}{2}W(8c^4 N e^r),\tag{3.50}$$

and the leading-order behaviour after expansion would be

$$J_t = \sqrt{\frac{1}{2}N \log N}\left[1 - \frac{\log \log N}{2 \log N} + \frac{\log 8 + 4 \log c}{2 \log N} + \mathcal{O}\left(\frac{\log \log N}{\log N}\right)^2\right].\tag{3.51}$$

3.4 Similarities to the running coupling of non-abelian gauge theories

We pause here to discuss some similarities between our threshold solution and the the running coupling of non-abelian gauge theories. The beta function of $\alpha_s(Q^2)$ from QCD gauge theory is

$$Q^2 \frac{d\alpha_s}{dQ^2} = \frac{d\alpha_s}{dL} = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \beta_2 \alpha_s^3 + \mathcal{O}(\alpha_s^4),\tag{3.52}$$

where Q^2 is the energy scale, β_i is the beta function at loop order $(i + 1)$, and $L = \log(\frac{Q^2}{\Lambda^2})$. This has been solved perturbatively [29, 30] for the running coupling $\alpha_s(Q^2)$,

$$\alpha_s(Q^2) + \frac{1}{\beta_0 \log L} = \frac{\beta_1}{\beta_0^2 L}(\log L) + \frac{\beta_1^2}{\beta_0^4 L^2}\left((\log L)^2 - \log L - 1 + \frac{\beta_0 \beta_2}{\beta_1^2}\right)$$
$$+ \frac{\beta_1^3}{\beta_0^6 L^3}\left((\log L)^3 - \frac{5}{2}(\log L)^2 - (2 - 3\frac{\beta_0 \beta_2}{\beta_1^2})\log L + \frac{1}{2} - \frac{\beta_0^2 \beta_3}{\beta_1^3}\right) + \mathcal{O}\left(\frac{(\log L)^4}{L^4}\right).\tag{3.53}$$

The threshold solution can be recast into a form which reveals a striking similarity with the expansion of $\alpha_s(Q^2)$. Starting from the definition of the W-function and its asymptotic series (3.40), we can write

$$\log W(z) = \log z - W(z)\tag{3.54}$$
$$\sim \log \log z - \sum_{n=1}^{\infty}\left(\frac{-1}{\log z}\right)^n \sum_{k=0}^{n}\begin{bmatrix}n\\n-k+1\end{bmatrix}\frac{(-\log \log z)^k}{k!},\tag{3.55}$$

where the factors $\begin{bmatrix}n\\k\end{bmatrix}$ are Stirling cycle numbers of the first kind. Introducing the new variables $y = \log J_t$ and $v = \log N$, we can take logs of the exact solution

$$J_t = \sqrt{\frac{N}{2}W(8N e^r)}\tag{3.56}$$

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and plug in the first few Stirling numbers to find

$$2y = v + \log v - \log 2 + \frac{1}{v}\left(-\log v + \log 8\right)$$

$$+ \frac{1}{v^2}\left[-\frac{1}{2}(\log v)^2 + (1 + \log 8)\log v - \frac{1}{2}(\log 8)(\log 8 + 2)\right] + \mathcal{O}\left(\frac{(\log v)^3}{v^3}\right) \quad (3.57)$$

$$\sim v + \log v - \log 2 + \sum_{l=1}^{\infty} \frac{P_0^l(\log v)}{v^l}, \quad (3.58)$$

where P_0^l are polynomials of order l, and we have dropped the subleading r-dependent terms. All but the first three terms in this sum tend to zero in the large v (i.e. large N) limit, so we can define the variable $Y = 2y - v - \log v + \log 2$, which has the perturbative expansion

$$Y = \frac{1}{v}\left(-\log v + \log 8\right) + \frac{1}{v^2}\left[-\frac{1}{2}(\log v)^2 + (1 + \log 8)\log v - \frac{1}{2}(\log 8)(\log 8 + 2)\right]$$

$$+ \mathcal{O}\left(\frac{(\log v)^3}{v^3}\right). \quad (3.59)$$

We can now see that both (3.53) and (3.59) are manifestly of the same form. Each bracketed term in the first series can be written $L^{-n}\tilde{P}^n(\log L)$, and each bracketed term in the second series can be written $v^{-n}P^n(\log v)$, where $\tilde{P}$ and P are polynomials of order n. The similarity between these series is intriguing, and it would be of interest to find out if there is a physical explanation.

3.5   Expansion of the threshold as a transseries

We have given in equation (3.45) an infinite asymptotic series expansion of the threshold $J_t(N)$ in terms of powers of $\log N$ and $\log \log N$. We can go beyond this classical asymptotic series approach to the threshold by considering the *non-perturbative corrections*, generated by the subleading terms in r that were previously neglected. This type of series is known as a *transseries*, and is perhaps most commonly seen in theoretical physics to describe instanton corrections in quantum field theory.

When considering asymptotic expansions from path integrals in quantum field theory, we are interested in not only the original perturbative series in the coupling constant, but also the exponentially-suppressed instanton correction terms. These typically come from saddle-points in the path integral. A typical asymptotic series in a quantum field theory with small coupling constant $g \to 0$ and instanton corrections has the form

$$\sum_n a_n g^n + e^{-A/g} \sum_n a_n^{(1)} g^n + \mathcal{O}\left(e^{-2A/g}\right). \quad (3.60)$$

The definition of an asymptotic series given in Section 3.1 cannot be used to describe the exponential contributions, as they are subleading to all powers of the coupling g. We make sense of a series with instanton corrections by thinking of it as a purely formal sum, in which g and $e^{-A/g}$ are treated as independent variables. Once the formal transseries is constructed, there are approaches that can recover the *exact* full form of the path integral from the series; this is called the theory of *resurgence*. The lecture notes [31] give a review of transseries and resurgence in QFT and string theory.

In our analysis of the threshold, the series we have found has not come from a path integral, but still has exponentially-suppressed corrections. Rather than corresponding to saddle-points, the exponential corrections arise from the corrections to the asymptotics of the finite N correlators. We can see the analogy between thresholds and instanton expansions by changing variables from N to $v = \log N$ in our threshold expressions; the remainder term r is then proportional to $e^{-v/2}$. We show in the following that the general form of a transseries of the threshold can be found, in terms of $e^{-v/2}$, v and $\log v$.

An interesting possible future research direction would be to use the transseries expansion to search for an effective field theory description of gravitons at the threshold. The threshold expansions with exponential corrections strongly resemble instanton expansions of field theoretic partition functions, and so they could well contain valuable hints about the nature of such an effective field theory.

We start by writing the threshold in terms of the variables $y = \log J_t$ and $v = \log N$ introduced in the previous section, but retain the r-dependent terms in the series expansion. With the r-corrections, the series (3.57) becomes

$$2y = v + \log v - \log 2 + \frac{1}{v}\left(-\log v + \log 8\right) + \frac{r}{v} + \mathcal{O}\left(\frac{r^2}{v^2}\right)$$
$$+ \frac{1}{v^2}\left[-\frac{1}{2}(\log v)^2 + (1 + \log 8)\log v - \frac{1}{2}(\log 8)(\log 8 + 2)\right] + \mathcal{O}\left(\frac{(\log v)^3}{v^3}\right) \quad (3.61)$$

$$\sim v + \log v - \log 2 + \sum_{l=1}^{\infty}\frac{P_0^l(\log v)}{v^l} + \mathcal{O}\left(\frac{r}{v}\right), \quad (3.62)$$

All the terms that depend on the error function r are subleading to any power of $\log v$ and v. To find the exponentially-supressed contributions to the threshold and extend the asymptotic series to a transseries, we need to find a more precise expression for r near the threshold. In the previous section, the function $r(J, N)$ was defined by the exact equation

$$G_3 = \sqrt{\frac{J}{2N}}\exp\left(\frac{J^2}{2N}\right)e^{-\frac{1}{4}r}. \quad (3.63)$$

The next-to-leading order corrections to the remainder function r were estimated in (3.35). A more careful calculation shows that the next-to-leading order behaviour of the correlator near

the threshold is

$$G_3 = \sqrt{\frac{J}{2N}} \exp\left(\frac{J^2}{2N}\right) \left[1 - \frac{J^3}{N^2} + e^{-\frac{J^2}{N}} + \mathcal{O}\left(\frac{J^6}{N^4}\right)\right], \tag{3.64}$$

and so

$$r(J, N) = 4\left(\frac{J^3}{N^2} - e^{-\frac{J^2}{N}} + \mathcal{O}\left(\frac{J^6}{N^4}\right)\right). \tag{3.65}$$

Plugging in the leading-order behaviour of the threshold $J \sim \sqrt{\frac{1}{2}N\log N}$ gives us the leading-order behaviour of r as a function purely of N, or as a function of v. We find

$$r(J_t(N), N) = \sqrt{\frac{2(\log N)^3}{N}} \left[1 - \frac{1}{\log N}\left(\log\log N - \frac{3}{2}\log 8 + 1\right) + \mathcal{O}\left(\frac{\log\log N}{\log N}\right)^2\right] \tag{3.66}$$

$$= \sqrt{2}v^{\frac{3}{2}}e^{-\frac{v}{2}}\left[1 - \frac{1}{v}\left(\log v - \frac{3}{2}\log 8 + 1\right) + \mathcal{O}\left(\frac{\log v}{v}\right)^2\right] \tag{3.67}$$

This correction can be reintroduced into (3.61) to give the first exponential correction of the threshold,

$$2y = v + \log v - \log 2 + \frac{1}{v}\left(-\log v + \log 8\right) + \mathcal{O}\left(\frac{\log v}{v}\right)^2$$
$$+ \sqrt{2ve^{-v}}\left[1 + \frac{1}{v}\left(\frac{3}{2}\log 8 - 2 - \log v\right) + \mathcal{O}\left(\frac{\log v}{v}\right)^2\right] + \mathcal{O}\left(ve^{-v}\right). \tag{3.68}$$

The remainder r has an asymptotic expansion at the threshold as a series of powers of $v^{\frac{3}{2}}e^{-\frac{v}{2}}$ multiplied by powers of $\log v$ and inverse powers of v. From considering the structure of the terms in (3.55), and writing $8Ne^r = e^{v+\log 8 + r}$, it can be seen that a kth power of r in the asymptotic expansion of $W(8Ne^r)$ is accompanied by a kth inverse power of v, followed by positive powers of $\log v$ and inverse powers of v. Noting that the subleading terms in the asymptotic expansion of r can also contribute, we can deduce the all-orders form of the asymptotic series with exponential corrections, although it is difficult to calculate coefficients explicitly beyond the first few terms. The general form of the transseries form of the threshold is

$$2y \sim v + \log v - \log 2 + \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(\sqrt{ve^{-v}})^k \frac{P_k^n(\log v)}{v^n}, \tag{3.69}$$

where the P_k^n are polynomials of order n, and $P_0^0(\log v) = 0$.

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This series gives an alternative expression for the threshold $J_t = e^y$ in terms of $v = \log N$. Only the first three terms do not go to zero in the large N limit, so we can exponentiate this expression to derive an infinite asymptotic series for the threshold. We find that

$$J_t \sim \sqrt{\frac{1}{2}N \log N} \left[1 + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left(\sqrt{\frac{\log N}{N}} \right)^k \frac{P_k'^n(\log \log N)}{(\log N)^n} \right] \tag{3.70}$$

where the polynomials have been modified, but the form of the series has not. As remarked at the end of subsection 3.3, for a truncated threshold series $\tilde{J}_t(N)$ to satisfy $G_3(\tilde{J}_t(N), N) \to 1$ at large N, we must include the next-to-leading order term,

$$\frac{P_0'^1(\log \log N)}{\log N} = \frac{-\log \log N}{2 \log N} + \frac{\log 8}{2 \log N}. \tag{3.71}$$

As a final remark, we note again that changing the threshold from $G_3 = 1$ to $G_3 = c$ will not alter the form of the series, but will modify the polynomials and constants. From (3.50), we see that shifting the threshold equation to $G_3 = c$ will transform the series as

$$2y \quad \sim v + \log v - \log 2 + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (\sqrt{v + c^4} e^{-\frac{v + c^4}{2}})^k \frac{P_k^n(\log(v + c^4))}{(v + c^4)^n} \tag{3.72}$$

$$\sim \quad v + \log v - \log 2 + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (\sqrt{v} e^{-\frac{v}{2}})^k \frac{\tilde{P}_k^n(\log v)}{v^n}. \tag{3.73}$$

The three leading-order terms and the highest-order terms in the polynomials are unaffected by the shift.

4 The extremal three-point correlator with $J_1 \neq J_2$

In the previous section we solved the equation $G_3(J(N), N) = 1$ at large N by finding the asymptotic form of the three-point function G_3 and solving for $J(N)$. In this section we consider the threshold of factorization for the more general three-point function,

$$\begin{aligned} G_3(J_1, J_2, N) \quad &:= \quad \langle\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle\rangle \\ &= \quad \frac{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle}{\sqrt{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{\dagger J_1} \rangle \langle \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_2} \rangle \langle \mathrm{tr} Z^{J_1 + J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle}}, \end{aligned} \tag{4.1}$$

and examine the behaviour of $J_1(N)$, $J_2(N)$ with N for which the threshold equation

$$G_3(J_1(N), J_2(N), N) = 1 \tag{4.2}$$

is satisfied at large N. Using similar methods as in the previous section, the asymptotic form of $G_3(J_1(N), J_2(N), N)$ can be found at large N, and can be used to invert the threshold

equation (4.2) to retrieve a simple leading-order constraint on the functions $J_1(N), J_2(N)$ at the threshold. We find quite generally that (4.2) is solved in the large N limit by solutions $J_1(N), J_2(N)$ that have the leading-order behaviour

$$J_1 J_2 \approx N \log N, \tag{4.3}$$

where we have omitted a constant of proportionality. In fact, this constant of proportionality depends on the N-dependent behaviour of the *smaller* of the two angular momenta J_1 and J_2.

In the following subsection, we present the calculation of the large N behaviour of the correlator $G_3(J_1, J_2, N)$, and invert the threshold equation $G_3(J_1(N), J_2(N), N) = 1$ to find the result $J_1 J_2 = \mathcal{O}(N \log N)$. Following that, we discuss how the threshold from the bulk perspective relates the separation of the graviton energies $\Delta J = |J_1 - J_2|$ to the energy of the single graviton $E = (J_1 + J_2)$.

4.1 Scaling limits and the threshold equation

We start from the expressions for the two and three-point correlators in Section 3.2. These generalize in a straightforward manner to give the expression, valid for large N and $1 \ll J_1, J_2 \ll N^{\frac{2}{3}}$:

$$G_3(J_1, J_2, N) \sim \sqrt{\frac{J_1 J_2}{(J_1 + J_2)N}} \exp\left(\frac{J_1 J_2}{2N}\right) \frac{(1 - e^{-\frac{J_1(J_1+J_2)}{N}})(1 - e^{-\frac{J_2(J_1+J_2)}{N}})}{\sqrt{(1 - e^{-\frac{J_1^2}{N}})(1 - e^{-\frac{J_2^2}{N}})(1 - e^{-\frac{(J_1+J_2)^2}{N}})}}. \tag{4.4}$$

Without loss of generality, we assume throughout that $J_1 \le J_2$.

We can find bounds on the threshold region by considering the large N behaviour of the product of the angular momenta $J_1 J_2$. If $J_1 J_2/N$ goes to zero with N, then the assumption $J_1 \le J_2$ means that J_1^2/N must also go to zero with N. We note that

$$
\begin{aligned}
1 - e^{-\frac{J_1^2}{N}} &\sim \frac{J_1^2}{N}, \\
1 - e^{-\frac{J_1(J_1+J_2)}{N}} &\sim \frac{J_1(J_1 + J_2)}{N}, \\
1 - e^{-\frac{J_2(J_1+J_2)}{N}} &\sim 1 - e^{-\frac{J_2^2}{N}}, \\
1 - e^{-\frac{(J_1+J_2)^2}{N}} &\sim 1 - e^{-\frac{J_2^2}{N}}
\end{aligned}
\tag{4.5}
$$

to deduce that the the correlator behaves as

$$G_3 \sim \sqrt{\frac{J_1 J_2}{N}} \sqrt{\frac{J_1 + J_2}{N}} \ll 1. \tag{4.6}$$

The correlator thus decays to zero at large N. On the other hand, if $J_1 J_2/N$ grows with N to infinity at a faster rate than some small positive power of N, i.e. $J_1 J_2 \ge N^{1+\delta}$ for some

small positive constant δ, then the $\exp(J_1 J_2/2N)$ factor scales at least as quickly as $\exp(N^\delta)$, an exponential of a positive power of N. All other factors in the expression are bounded by powers of N, and so the exponential term dominates and G_3 must tend to infinity. Summarizing the above, we have

$$
\boxed{
\begin{aligned}
G_3(J_1, J_2, N) &\to 0, & \frac{J_1 J_2}{N} &\to 0, \\
G_3(J_1, J_2, N) &\to \infty, & \frac{J_1 J_2}{N^{1+\delta}} &\to \infty \quad \text{for some } \delta > 0.
\end{aligned}
}
\tag{4.7}
$$

These limits extend the relations given in (2.7) to the more general case. We deduce that a large N solution to the equation $G_3 = 1$ could only exist when the product $J_1 J_2$ lies somewhere in the range

$$
N < J_1 J_2 < N^{1+\delta},
\tag{4.8}
$$

for any small positive constant δ.

By constraining $J_1 J_2$ to lie within this range, the expression for the three-point correlator (4.4) can be simplified. Since we require $J_1 J_2$ to be grow larger than N, and have constrained both J_1 and J_2 to be less than $N^{\frac{2}{3}}$, we must have that $J_1 \gg N^{\frac{1}{3}}$, i.e. J_1 grows at least as quickly as a positive power of N. Also, the factors of the form $(1 - e^{-x})$ in (4.4) tends to 1 if x tends to ∞, so we can use the facts that $J_1 J_2/N \to \infty$ near the threshold and $J_1 \le J_2$ to neglect several factors and write

$$
G_3 \sim \sqrt{\frac{J_1 J_2}{(J_1 + J_2)N}} \exp\left(\frac{J_1 J_2}{2N}\right) \left(1 - e^{-\frac{J_1^2}{N}}\right)^{-\frac{1}{2}}.
\tag{4.9}
$$

We can keep track of the errors generated in approximating the asymptotic form of the correlator by writing the *exact* expression,

$$
G_3 = \sqrt{\frac{J_1 J_2}{(J_1 + J_2)N}} \exp\left(\frac{J_1 J_2}{2N}\right) \left(1 - e^{-\frac{J_1^2}{N}}\right)^{-\frac{1}{2}} e^{-\frac{r}{2}},
\tag{4.10}
$$

where again the remainder function $r(J_1, J_2, N)$ is defined implicitly by this equation, and the J_i scale with N in the range $N^{\frac{1}{3}} \ll J_1 \le J_2 \ll N^{\frac{2}{3}}$. This remainder function tends to zero with N, but its leading-order behaviour will in general change depending on the scaling behaviour of J_1 and J_2. We will later show that, near the threshold, the remainder function is of the order

$$
r = \mathcal{O}\left(\frac{(\log N)^2}{J_1}\right),
\tag{4.11}
$$

and so decays to zero at a faster rate than some inverse power of N.

We wish to simplify the equation

$$G_3 = \sqrt{\frac{J_1 J_2}{(J_1 + J_2)N}} \exp\left(\frac{J_1 J_2}{2N}\right) \left(1 - e^{-\frac{J_1^2}{N}}\right)^{-\frac{1}{2}} e^{-\frac{r}{2}} = 1 \tag{4.12}$$

in the large N limit. A convenient way to do this is by using the Lambert W-function, and its large argument expansion. Equation (4.12) is solved exactly (with the implicit remainder term r) by

$$\frac{J_1 J_2}{N} = W\left((J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})e^r\right). \tag{4.13}$$

The argument of the W-function changes depending on the behaviour of J_1^2/N with increasing N, but will grow to infinity in all relevant cases, allowing us to use the large argument asymptotic expansion of the W-function,

$$W(z) = \log z - \log \log z + \mathcal{O}\left(\frac{\log \log z}{\log z}\right). \tag{4.14}$$

To proceed, we must consider three possible scaling behaviours of J_1^2/N in turn: the case when J_1^2/N tends to zero, the case when J_1^2/N tends to a constant, and the case when J_1^2 tends to infinity.

First, consider the case where $J_1^2/N \to 0$. We have

$$(1 - e^{-\frac{J_1^2}{N}}) = \frac{J_1^2}{N} + \mathcal{O}\left(\frac{J_1^2}{N}\right)^2 \tag{4.15}$$

so

$$(J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}}) = \frac{J_1 J_2}{N} J_1 \left(1 + \frac{J_1}{J_2}\right)\left(1 + \mathcal{O}\left(\frac{J_1^2}{N}\right)\right) \tag{4.16}$$

which must tend to infinity since $J_1 J_2/N$ and J_1 are large. Neglecting the remainder term r for the moment, we expand out the W-function to find the threshold equation

$$\frac{J_1 J_2}{N} = \log J_1 - \log \log J_1 + \log\left(\frac{J_1 J_2}{N}\right)$$
$$+ \log\left(1 + \frac{J_1}{J_2}\right) - \log\left[1 + \frac{1}{\log J_1}\left(\log\left(\frac{J_1 J_2}{N}\right) + \log\left(1 + \frac{J_1}{J_2}\right)\right)\right]$$
$$+ \mathcal{O}\left(\frac{J_1^2}{N}\right) + \mathcal{O}\left(\frac{\log \log(J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})}{\log(J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})}\right) \tag{4.17}$$

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This fairly involved expression can be substantially simplified as follows: first, we simplify the final error term by giving its leading behaviour in terms of N. Next, we show that all terms on the second line are small at large N, which allows us to deduce that the leading-order behaviour of the expression is $\log J_1$. Finally, by plugging in $\log J_1(1+o(1))$ into the expressions for $J_1 J_2/N$ on the RHS of (4.17), we will find that the $\log \log J_1$ term cancels, and that only one large term remains in its asymptotic series expansion.

First, we consider the latter remainder term. We know that J_1 and J_2 scale with N at a larger rate than some positive power of N, so $\log J_1$ is $\mathcal{O}\left(\log N\right)$ to leading order. We've also required $J_1 J_2/N$ to scale to infinity at a slower rate than any positive power of N, as this is required for the threshold solution to $G_3 = 1$ to be valid at large N. This means that $\log(J_1 J_2/N)$ must be $o(\log N)$. We deduce that

$$\log \left[(J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}}) \right] = \log J_1 + \log \left(\frac{J_1 J_2}{N} \right) + \log \left(1 + \frac{J_1}{J_2} \right) + \mathcal{O}\left(\frac{J_1^2}{N} \right) \quad (4.18)$$

$$= \mathcal{O}(\log N), \quad (4.19)$$

and hence

$$\mathcal{O}\left(\frac{\log \log (J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})}{\log (J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})} \right) = \mathcal{O}\left(\frac{\log \log N}{\log N} \right). \quad (4.20)$$

Both this term and the $\mathcal{O}\left(\frac{J_1^2}{N} \right)$ term are small in the large N limit. Next, we can see that all terms on the second line of (4.17) must be small. Noting that

$$\frac{J_1}{J_2} = \frac{J_1^2}{N} \frac{N}{J_1 J_2} \to 0 \quad (4.21)$$

since $J_1^2/N \to 0$ and $J_1 J_2/N \to \infty$, we have that

$$\log \left(1 + \frac{J_1}{J_2} \right) \to 0. \quad (4.22)$$

Also, it was required that $J_1 J_2/N$ grows to infinity with N, but not as a positive power of N or greater, so $\log(J_1 J_2/N) = o(\log N)$. Since $\log J_1 = \mathcal{O}(\log N)$, this means that

$$\frac{1}{\log J_1} \log \left(\frac{J_1 J_2}{N} \right) \to 0, \quad (4.23)$$

and so the second term in the second line of (4.17) is also small. The largest term in (4.17) must therefore be $\log J_1$, which is of order $\mathcal{O}(\log N)$. Using this and (4.21), we see that J_1/J_2 must be smaller than $\mathcal{O}(1/\log N)$, and so we can collate all the remainders in the threshold expression into two terms; we find

$$\frac{J_1 J_2}{N} = \log J_1 - \log \log J_1 + \log \left(\frac{J_1 J_2}{N} \right) + \mathcal{O}\left(\frac{\log \log N}{\log N} \right) + \mathcal{O}\left(\frac{J_1^2}{N} \right). \quad (4.24)$$

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By plugging in this expression for $J_1 J_2/N$ into the third term, we can cancel the $\log \log J_1$ and obtain the leading-order threshold equation

$$\frac{J_1 J_2}{N} = \log J_1 + \mathcal{O}\left(\frac{\log \log N}{\log N}\right) + \mathcal{O}\left(\frac{J_1^2}{N}\right). \tag{4.25}$$

This formula is valid at the threshold, provided that $J_1^2/N \to 0$ with large N. There are two different remainder terms in this expression as we have not imposed enough conditions on J_1 to state which term is larger. Constraining the scaling behaviour of J_1 with N would allow us to deduce which term is subleading. For example, if we set $J_1 \sim \sqrt{N}(\log \log N)/\sqrt{\log N}$, then the $\mathcal{O}(J_1^2/N)$ term is the leading error, but if $J_1 \sim N^{5/12}$ then the $\mathcal{O}\left(\frac{\log \log N}{\log N}\right)$ term is the largest error.

Next, we consider the case where J_1^2/N tends to a constant. Starting from threshold equation

$$\frac{J_1 J_2}{N} = W((J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})e^r), \tag{4.26}$$

the argument of the W-function is clearly large since $J_1 + J_2$ grows with N. Again neglecting the remainder term, we can use the large argument expansion of the W-function and write

$$\frac{J_1 J_2}{N} \sim \log\left((J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})\right) - \log \log\left((J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})\right)$$
$$+ \mathcal{O}\left(\frac{\log \log\left((J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})\right)}{\log\left((J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})\right)}\right) \tag{4.27}$$

Since $\log((J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})) = \mathcal{O}(\log N)$, we can simplify this remainder term and expand out the second term to write

$$\frac{J_1 J_2}{N} \sim \log(J_1 + J_2) - \log \log(J_1 + J_2) + \log\left(1 - e^{-\frac{J_1^2}{N}}\right) + \mathcal{O}\left(\frac{\log \log N}{\log N}\right). \tag{4.28}$$

In writing this expression, we have dropped a term of $\mathcal{O}\left(\frac{1}{\log N}\right)$ as it is subleading to the $\mathcal{O}\left(\frac{\log \log N}{\log N}\right)$ remainder term. The first two terms in this expression grow large with increasing N, and the third term tends to a constant.

Finally, we consider the case where J_1^2/N tends to infinity with N. Again we find that (4.28) still holds, but that the third term now tends to zero. From the series expansion of the logarithm, we have

$$\log\left(1 - e^{-\frac{J_1^2}{N}}\right) = \mathcal{O}\left(e^{-\frac{J_1^2}{N}}\right), \tag{4.29}$$

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so we write the final expression

$$\frac{J_1 J_2}{N} \sim \log(J_1 + J_2) - \log\log(J_1 + J_2) + \mathcal{O}\left(e^{-\frac{J_1^2}{N}}\right) + \mathcal{O}\left(\frac{\log\log N}{\log N}\right). \tag{4.30}$$

Again, we have two remainder terms, as we have not specified how quickly J_1^2/N scales to infinity with N and so cannot state which is the larger.

Summarizing the above, we have three different threshold equations for the different regimes of J_1^2/N. Listed in order of increasing J_1^2/N, we have:

$$\frac{J_1 J_2}{N} = \begin{cases} \log J_1 + \mathcal{O}\left(\frac{\log\log N}{\log N}\right) + \mathcal{O}\left(\frac{J_1^2}{N}\right) & \frac{J_1^2}{N} \to 0 \\ \log(J_1 + J_2) - \log\log(J_1 + J_2) + \log\left(1 - e^{-\frac{J_1^2}{N}}\right) + \mathcal{O}\left(\frac{\log\log N}{\log N}\right) & \frac{J_1^2}{N} \to \text{const.} \\ \log(J_1 + J_2) - \log\log(J_1 + J_2) + \mathcal{O}\left(\frac{\log\log N}{\log N}\right) + \mathcal{O}\left(e^{-\frac{J_1^2}{N}}\right) & \frac{J_1^2}{N} \to \infty. \end{cases} \tag{4.31}$$

In all cases, the explicitly-given terms are non-zero in the large N limit, and the higher-order terms are small. All these large terms are necessary to describe the threshold accurately at large N; if we plug (4.31) into (4.12) with the remainder terms and r discarded, then the correlator tends to one at large N in each case.

The angular momenta J_1 and J_2 grow at least as quickly as a positive power of N, so the leading-order term in the threshold is always proportional to $\log N$. If we assume that the power-dependence of J_1 on N is simple enough that it can be separated out into the form $J_1 = N^{\alpha_1} e^{\delta_1}$, where α_1 is a constant and $|\delta_1(N)| \ll \log N$, then the leading-order term of the threshold solution is

$$J_1 J_2 \sim \alpha_1 N \log N. \tag{4.32}$$

We have so far neglected the error parameter r without discussion, but we can now justify this. To derive the equation

$$G_3 \sim \sqrt{\frac{J_1 J_2}{(J_1 + J_2)N}} \exp\left(\frac{J_1 J_2}{2N}\right)\left(1 - e^{-\frac{J_1^2}{N}}\right)^{-\frac{1}{2}} \tag{4.33}$$

near the threshold, we have dropped corrections of at most order $\mathcal{O}\left(1/J_1\right)$ and $\mathcal{O}(J_1 J_2(J_1 + J_2)/N^2)$. Near the threshold, J_1 and J_2 satisfy

$$\frac{J_1 J_2(J_1 + J_2)}{N^2} \sim \frac{1}{J_1}\left(\frac{J_1 J_2}{N}\right)^2 = \mathcal{O}\left(\frac{(\log N)^2}{J_1}\right). \tag{4.34}$$

The remainder parameter r, defined in (4.12), must contain all the corrections to the correlator near the threshold. We can therefore state that, near the threshold, the largest corrections to r must be

$$r = \mathcal{O}\left(\frac{(\log N)^2}{J_1}\right), \tag{4.35}$$

which decays to zero with N at a faster rate than some inverse power of N. If we reintroduce this remainder when expanding out the W-function in (4.13), we will modify each equation in (4.31) by the addition of an r term, plus $\mathcal{O}(r^2)$ corrections. However, this term must be smaller than $\mathcal{O}\left(\frac{\log \log N}{\log N}\right)$, and in fact is smaller than any power of $(\log \log N / \log N)$: in terms of the parameter $v = \log N$, the contributions from r are exponentially suppressed in v. As a consequence, we can always drop these terms from the solution.

4.2 A change of variables

The threshold equation $G_3(J_1, J_2, N) = 1$ defines a two-dimensional threshold surface in three-dimensional (J_1, J_2, N)-space. We can develop some insight into the relation between this surface and the physical properties of the correlator by changing the parameter space variables. If we take N to be fixed but large enough that the remainder $\mathcal{O}\left(\frac{\log \log N}{\log N}\right)$ is small, then we can use (4.31) to rewrite the threshold as a curve in $E = J_1 + J_2$ and $\Delta J = |J_2 - J_1|$. For the region where $J_1^2/N \to 0$, i.e. $(E^2 - \Delta J^2)/N \to 0$, then the threshold of factorization is

$$\frac{E^2 - \Delta J^2}{4N \log (E - \Delta J)} \sim 1 + \mathcal{O}\left(\frac{\log \log N}{\log N}\right), \tag{4.36}$$

and for the region where $J_1^2/N = (E^2 - \Delta J^2)/N$ does not tend to zero, then the threshold is

$$\frac{E^2 - \Delta J^2}{4N(\log E - \log \log E)} \sim 1 + \mathcal{O}\left(\frac{\log \log N}{\log N}\right), \tag{4.37}$$

where all the discarded terms are small.

We can say something about how perturbations away from the threshold in $(E, \Delta J, N)$ space affect the factorization of the correlator by rewriting the correlator in the form

$$G_3(E, \Delta J, N) = \left[\frac{E^2 - \Delta J^2}{4NE} \exp\left(\frac{E^2 - \Delta J^2}{4N}\right) \left(1 - e^{-\frac{1}{4N}(E - \Delta J)^2}\right)^{-1} e^{-r}\right]^{\frac{1}{2}}. \tag{4.38}$$

It is convenient to work with $\log(G_3)^2$, and allow E and ΔJ to be independent of N. Taking the differential of $\log(G_3)^2$, we have

$$d\log(G_3)^2 = \frac{2}{G_3}dG_3 = \frac{2}{G_3}\left[\frac{\partial G_3}{\partial E}dE + \frac{\partial G_3}{\partial (\Delta J)}d(\Delta J) + \frac{\partial G_3}{\partial N}dN\right], \tag{4.39}$$

Expressing the coefficients of the differentials in terms of J_1, J_2 and N for convenience, we have

$$d\log(G_3)^2 = \frac{1}{2}dE\left[\frac{J_2}{N} + \frac{1}{J_1} + \frac{J_1}{N} + \frac{1}{J_2} - \frac{2}{J_1+J_2} - \frac{2J_1}{N(e^{\frac{J_1^2}{N}}-1)}\right]$$

$$+\frac{1}{2}d(\Delta J)\left[-\frac{J_2}{N} - \frac{1}{J_1} + \frac{J_1}{N} + \frac{1}{J_2} + \frac{2J_1}{N(e^{\frac{J_1^2}{N}}-1)}\right]$$

$$-\frac{dN}{N}\left[\frac{J_1 J_2}{N} + 1 - \frac{J_1^2}{N(e^{\frac{J_1^2}{N}}-1)}\right] - dr. \quad (4.40)$$

At large N and near the threshold $J_1 J_2 = \mathcal{O}(N\log N)$, the largest term in the coefficient of dE is J_2/N, which is of order $\mathcal{O}\left(\frac{\log N}{J_1}\right)$. This means that $\frac{\partial G_3}{\partial E}$ is positive at large N. Similarly, the largest term in the coefficient of $d(\Delta J)$ is $-J_2/N$, which is order $\mathcal{O}\left(\frac{\log N}{J_1}\right)$, and so $\frac{\partial G_3}{\partial \Delta J}$ is negative at large N. The corrections to dE and $d(\Delta J)$ from the differential of the error function dr are order $\mathcal{O}\left(\frac{\log N}{J_1^2}\right)$ at the threshold, and so are subleading.

The signs of the partial derivates of G_3 with respect to E and ΔJ gives us some interesting insights into factorization near the threshold. If we consider N to be large and fixed, and take E and ΔJ near to the threshold, then a small increase in the energy of the single graviton E will increase the correlator G_3, and move the correlator into the breakdown region. On the other hand, if the separation between the gravitons ΔJ in the multi-graviton state is increased by a small amount, then G_3 will decrease, and the correlator will move into the factorization region.

5 Non-extremal correlators

We can consider the existence of a threshold of factorization for a non-extremal three-point function with operators formed from the complex scalar fields $Z = \phi_5 + i\phi_6$ accompanied by a small number of $Y = \phi_3 + i\phi_4$ insertions. Consider a correlator of symmetrized trace operators inserted at the points x_1, x_2, and y:

$$\langle \text{Str}(Z^{J_1}Y^{J_3})(x_1)\text{Str}(Z^{J_2}Y^{\dagger J_3})(x_2)\text{tr}(Z^{\dagger J_1+J_2})(y)\rangle. \quad (5.1)$$

In a similar manner to the extremal correlator consisting of only Z-fields, we can use the conformal symmetry to separate out a position-independent correlator by a particular choice of operator insertion locations. Under the inversion $y \to y' = y/|y|^2$, the antiholomorphic operator transforms as

$$\text{tr}(Z^{\dagger J_1+J_2})(y) \to \text{tr}'(Z^{\dagger J_1+J_2})(y')$$
$$= |y|^{J_1+J_2}\text{tr}(Z^{\dagger J_1+J_2})(y). \quad (5.2)$$

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By taking $x_1 \to 0$ and $y' \to 0$ i.e. $y \to \infty$, the correlator becomes

$$\langle \text{Str}(Z^{J_1}Y^{J_3})(0)\text{Str}(Z^{J_2}Y^{\dagger J_3})(x_2)\text{tr}'(Z^{\dagger J_1 + J_2})(0)\rangle$$
$$= \frac{\langle \text{Str}(Z^{J_1}Y^{J_3})\text{Str}(Z^{J_2}Y^{\dagger J_3})\text{tr}(Z^{\dagger J_1 + J_2})\rangle}{|x_2|^{2J_3}}. \qquad (5.3)$$

We have separated out a combinatoric factor which can be evaluated by a matrix model calculation. Unlike the extremal correlator, however, the separation $|x_2|$ between the operators inserted at $\text{Str}(Z^{J_1}Y^{J_3})$ and $\text{Str}(Z^{J_2}Y^{\dagger J_3})$ is still present in this correlator. Introducing the notation $\epsilon \equiv |x_2|$ for the magnitude of the separation between these two operators, and $\| \mathcal{O} \| = \sqrt{\langle \mathcal{O}\mathcal{O}^\dagger \rangle}$ for the norm of a matrix model operator $\mathcal{O}$, then the multiparticle-normalized correlator is

$$G_3(J_i, N; \epsilon) = \frac{\langle \text{Str}(Z^{J_1}Y^{J_3})\text{Str}(Z^{J_2}Y^{\dagger J_3})\text{tr}(Z^{\dagger J_1 + J_2})\rangle}{\epsilon^{2J_3} \| \text{Str}(Z^{J_1}Y^{J_3}) \| \| \text{Str}(Z^{J_2}Y^{J_3}) \| \| \text{tr}(Z^{J_1 + J_2}) \|} \qquad (5.4)$$

The appearance of this position-dependence means that the three-point correlator is dimensionful, and so it is not meaningful to define the threshold as being when the correlator approaches a fixed number at large N. However, if we introduce an arbitrary mass scale Λ, then we can instead consider the combination $\Lambda^{-2J_3}G_3(J_i, N; \epsilon)$, which is dimensionless. We define the *non-extremal threshold* as the solution to the equation

$$\Lambda^{-2J_3}G_3(J_i, N; \epsilon) = 1. \qquad (5.5)$$

A natural choice of Λ would be a UV cutoff of the CFT. This will modify correlators in general, and the ϵ^{-2J_3} factor will be modified to

$$\frac{1}{\epsilon^{2J_3}}\left(1 + o(\epsilon^{-1}\Lambda^{-1})\right). \qquad (5.6)$$

The higher-order corrections can be neglected if we require that the separation ϵ is much larger than the cutoff length Λ^{-1}. We can do this by setting $\epsilon\Lambda$ to be large and independent of N, or by allowing $\epsilon\Lambda$ to grow large with N. It is convenient in the following to define $\mathcal{R} := \epsilon\Lambda$ as the dimensionless ratio between the cutoff separation and the length scale. This is required to be large for the higher-order corrections to ϵ to be absent. The non-extremal threshold equation can then be written in the form

$$\Lambda^{-2J_3}G_3(J_i, N; \epsilon) = \mathcal{R}^{-2J_3}\frac{\langle \text{Str}(Z^{J_1}Y^{J_3})\text{Str}(Z^{J_2}Y^{\dagger J_3})\text{tr}(Z^{\dagger J_1 + J_2})\rangle}{\| \text{Str}(Z^{J_1}Y^{J_3}) \| \| \text{Str}(Z^{J_2}Y^{J_3}) \| \| \text{tr}(Z^{J_1 + J_2}) \|} = 1. \qquad (5.7)$$

To investigate the threshold of this non-extremal correlator, we look for an exact finite N expression of the correlator that is valid when some of the operator dimensions are large. There are three matrix model correlator expressions that we need in order to evaluate the correlator:

$$\| \text{Str}(Z^{J_1}Y^{J_3}) \|^2 = \langle \text{Str}(Z^{J_1}Y^{J_3})\text{Str}(Z^{\dagger J_1}Y^{\dagger J_3})\rangle,$$
$$\| \text{tr}(Z^{J_1 + J_2}) \|^2 = \langle \text{tr}(Z^{J_1 + J_2})\text{tr}(Z^{\dagger J_1 + J_2})\rangle,$$
$$\langle \text{Str}(Z^{J_1}Y^{J_3})\text{Str}(Z^{J_2}Y^{\dagger J_3})\text{tr}(Z^{\dagger J_1 + J_2})\rangle. \qquad (5.8)$$

The norm $\| \operatorname{tr}(Z^{J_1+J_2}) \|^2$ is known explicitly, but we have not found a closed form of the other correlators for general operator dimensions. However, exact evaluations of the correlator can be found for small values of J_3, where there is only a small number of Y-insertions; in the following we focus on the 'near-extremal' case when $J_3 = 1$.

5.1 The 'near-extremal' correlator

We set $J_3 = 1$ in (5.7) and consider the correlator

$$G_3(J_i, N; \epsilon) = \frac{\langle \operatorname{Str}(Z^{J_1}Y)\operatorname{Str}(Z^{J_2}Y^\dagger)\operatorname{tr}(Z^{\dagger J_1+J_2})\rangle}{\epsilon^2 \| \operatorname{Str}(Z^{J_1}Y) \|\| \operatorname{Str}(Z^{J_2}Y) \|\| \operatorname{tr}(Z^{J_1+J_2}) \|}. \tag{5.9}$$

The norm $\| \operatorname{tr}(Z^{J_1+J_2}) \|^2$ was known previously [15] and used in Sections 3 and 4:

$$\langle \operatorname{tr}(Z^{J_1+J_2})\operatorname{tr}(Z^{\dagger J_1+J_2})\rangle = (J_1 + J_2)! \left[\binom{N + J_1 + J_2}{J_1 + J_2 + 1} - \binom{N}{J_1 + J_2 + 1} \right]. \tag{5.10}$$

For $J_3 = 1$, there is only one pair of Y-matrices, so the contraction of the three-point function can be performed immediately. The unnormalized three-point correlator becomes

$$\langle \overbrace{\operatorname{Str}(Z^{J_1}Y)\operatorname{Str}(Z^{J_2}Y^\dagger)}\operatorname{tr}(Z^{\dagger J_1+J_2})\rangle = \langle \operatorname{tr}(Z^{J_1+J_2})\operatorname{tr}(Z^{\dagger J_1+J_2})\rangle, \tag{5.11}$$

where we have used the fact that $\operatorname{Str}(Z^{J_1+J_2}) = \operatorname{tr}(Z^{J_1+J_2})$. This means that (5.9) reduces to

$$G_3(J_i, N; \epsilon) = \frac{\| \operatorname{tr}(Z^{J_1+J_2}) \|}{\epsilon^2 \| \operatorname{Str}(Z^{J_1}Y) \|\| \operatorname{Str}(Z^{J_2}Y) \|}. \tag{5.12}$$

The other correlators can be determined by tensor space methods. In Appendix B, we have derived the equation

$$\langle \operatorname{Str}(Z^{J_1}Y^{J_2})\operatorname{Str}(Z^{\dagger J_1}Y^{\dagger J_2})\rangle = J_1!J_2! \left[\binom{N + J_1 + J_2}{J_1 + J_2 + 1} - \binom{N}{J_1 + J_2 + 1} \right]. \tag{5.13}$$

Substituting in the relevant values of J_1 and J_2 in to the correlators in the denominators of (5.9), we find that

$$\| \operatorname{Str}(Z^{J_1}Y) \| = \sqrt{J_1!} \left[\binom{N + J_1 + 1}{J_1 + 2} - \binom{N}{J_1 + 2} \right]^{\frac{1}{2}}, \tag{5.14}$$

$$\| \operatorname{Str}(Z^{J_2}Y) \| = \sqrt{J_2!} \left[\binom{N + J_2 + 1}{J_2 + 2} - \binom{N}{J_2 + 2} \right]^{\frac{1}{2}}, \tag{5.15}$$

and so

$$\Lambda^{-2}G_3(J_i, N; \epsilon) = \mathcal{R}^{-2} \left(\frac{(J_1 + J_2)!}{J_1!J_2!} \right)^{\frac{1}{2}} \left[\binom{N + J_1 + J_2}{J_1 + J_2 + 1} - \binom{N}{J_1 + J_2 + 1} \right]^{\frac{1}{2}} \times$$
$$\times \left[\binom{N + J_1 + 1}{J_1 + 2} - \binom{N}{J_1 + 2} \right]^{-\frac{1}{2}} \left[\binom{N + J_2 + 1}{J_2 + 2} - \binom{N}{J_2 + 2} \right]^{-\frac{1}{2}}. \tag{5.16}$$

This is the finite N expression of the non-extremal correlator when $J_3 = 1$. It is valid for small or large J_1 and J_2, provided that $J_1, J_2 \ll N$. As in the extremal case, we wish to find the asymptotic form of this expression when J_1, J_2, and N are large, but the ratios J_1/N and J_2/N are small. Making the assumptions that $J_1 \le J_2 \ll N^{\frac{2}{3}}$, then equation (3.20) still holds with J replaced by $J_1 + 1$. Dropping the subleading corrections, we find that

$$J_1! \left[\binom{N + J_1 + 1}{J_1 + 2} - \binom{N}{J_1 + 2} \right] \sim \frac{N^{J_1 + 2}}{J_1^2} \exp\left(\frac{J_1^2}{2N} \right) \left(1 - e^{-\frac{J_1^2}{N}} \right), \tag{5.17}$$

and similarly for J_2. The full large N expression for the correlator for $1 \ll J_1 \le J_2 \ll N$ is therefore

$$\Lambda^{-2} G_3 \sim \mathcal{R}^{-2} \sqrt{\frac{J_1^2 J_2^2}{(J_1 + J_2) N^3}} \exp\left(\frac{J_1 J_2}{2N} \right) \sqrt{\frac{\left(1 - e^{-\frac{(J_1 + J_2)^2}{N}} \right)}{\left(1 - e^{-\frac{J_1^2}{N}} \right) \left(1 - e^{-\frac{J_2^2}{N}} \right)}} \tag{5.18}$$

We can argue that the correlator must decay to zero if $J_1 J_2/N$ is small as follows: If $J_1 J_2/N$ tends to zero with N, then the exponential term tends to 1. The factor $\mathcal{R}^{-2}$ has already been taken to be small. Since $J_1 J_2/N$ is small and we have assumed that $J_1 \le J_2$, we know that J_1^2/N is also small and so

$$1 - e^{-\frac{(J_1 + J_2)^2}{N}} \quad \sim \quad 1 - e^{-\frac{J_2^2}{N}},$$
$$\frac{J_1^2/N}{1 - e^{-\frac{J_1^2}{N}}} \quad \sim \quad 1, \tag{5.19}$$

and thus we can deduce that

$$\Lambda^{-2} G_3 \sim \mathcal{R}^{-2} \frac{J_2}{N} \frac{1}{\sqrt{J_1 + J_2}}. \tag{5.20}$$

The correlator must therefore tend to zero when $J_1 J_2/N$ is small.

On the other hand, consider the case when $J_1 J_2/N$ grows larger than a positive power of N, i.e. $J_1 J_2 > N^{1+\delta}$ for some $\delta > 0$. The exponential term will dominate the expression, as it will grow to infinity exponentially quickly with N as compared to the other factors of J_1, J_2, and N outside of the exponential. In this case, the correlator must definitely grow to infinity (provided that $\mathcal{R}$ is does not grow with N at a faster than a power of N). Summarizing the above, we have

$$\boxed{\begin{aligned} G_3(J_1, J_2, N; \mathcal{R}) &\to 0, & \frac{J_1 J_2}{N} &\to 0, \\ G_3(J_1, J_2, N; \mathcal{R}) &\to \infty, & \frac{J_1 J_2}{N^{1+\delta}} &\to \infty \quad \text{for some } \delta > 0. \end{aligned}} \tag{5.21}$$

The threshold must therefore be constrained to lie in the region

$$N < J_1 J_2 < N^{1+\delta}. \tag{5.22}$$

In this range, the large N behaviour of the correlator is simply

$$\Lambda^{-2} G_3(J_1, J_2, N; \epsilon) \sim \mathcal{R}^{-2} \sqrt{\frac{J_1^2 J_2^2}{(J_1 + J_2) N^3}} \exp\left(\frac{J_1 J_2}{2N}\right) \left(1 - e^{-\frac{J_1^2}{N}}\right)^{-\frac{1}{2}}. \tag{5.23}$$

We can encompass all the errors present in approximating this expression by the function r, defined by the equation

$$\Lambda^{-2} G_3(J_1, J_2, N; \epsilon) = \mathcal{R}^{-2} \sqrt{\frac{J_1^2 J_2^2}{(J_1 + J_2) N^3}} \exp\left(\frac{J_1 J_2}{2N}\right) \left(1 - e^{-\frac{J_1^2}{N}}\right)^{-\frac{1}{2}} e^{-\frac{r}{2}}, \tag{5.24}$$

and attempt to solve asymptotically the threshold equation

$$\Lambda^{-2} G_3(J_1, J_2, N; \epsilon) = 1. \tag{5.25}$$

We consider the cases $J_1 = J_2$ and $J_1 \neq J_2$ separately.

5.2 $J_1 = J_2$

If we consider the non-extremal correlator when $J_1 = J_2 = J$, then the threshold equation with error function r becomes

$$\mathcal{R}^{-2} \sqrt{\frac{J^3}{2N^3}} \exp\left(\frac{J^2}{2N}\right) \left(1 - e^{-\frac{J^2}{N}}\right)^{-\frac{1}{2}} e^{-\frac{r}{2}} = 1. \tag{5.26}$$

This has an exact solution in term of the W-function,

$$\frac{J_t^2}{N} = \frac{3}{2} W\left[\frac{2^{5/3}}{3} N \mathcal{R}^{8/3} \left(1 - e^{-\frac{J_t^2}{N}}\right)^{\frac{2}{3}} e^{\frac{2r}{3}}\right]. \tag{5.27}$$

The argument of the W-function must be large, so we can again expand it in terms of logarithms. The factors of $(1 - e^{-\frac{J^2}{N}})$ and $e^{\frac{2r}{3}}$ must be subleading, and so a short calculation shows that the threshold expands out to

$$\frac{J_t^2}{N} = \frac{3}{2} \log N + 4 \log \mathcal{R} - \frac{3}{2} \log\left[\log N + \frac{8}{3} \log \mathcal{R}\right] + \frac{1}{2} \log\left(\frac{32}{27}\right) + o(1). \tag{5.28}$$

In this large N expansion of the threshold, we have the two parameters N and $\mathcal{R} \equiv \epsilon\Lambda$. If we take $\mathcal{R}$ to be large but independent of N, then it must become subleading in the large N limit, and the threshold becomes

$$
\frac{J_t^2}{N} = \frac{3}{2}\log N - \frac{3}{2}\log\log N + 4\log\mathcal{R} + \frac{1}{2}\log\left(\frac{32}{27}\right) + o(1) \tag{5.29}
$$

$$
\sim \frac{3}{2}\log N. \tag{5.30}
$$

Alternatively, we can allow the ratio $\mathcal{R}$ to grow large with N, by letting either the separation of the operators ϵ or the cutoff scale Λ grow with N. The $\log\mathcal{R}$ terms are subleading and the above expression still holds if $\mathcal{R}$ scales to infinity at a slower rate than a power of N. If $\mathcal{R}$ grows like a power of N, then it can influence the leading constant of the threshold, but it is still logarithmically dependent on N. In all these cases, the leading-order behaviour of the threshold is simply

$$
J^2 = \mathcal{O}(N\log N), \tag{5.31}
$$

as was the case for the extremal correlator.

The expansion of the threshold given in (5.29) tells us something new about the factorization thresholds for non-extremal correlators. The $(4\log\mathcal{R})$ term, which did not appear in the extremal threshold, means that the threshold in the non-extremal case depends on the separation of the correlators in the boundary directions. If we considered a system at the threshold at fixed large N and fixed large $\mathcal{R}$, then a decrease in $\mathcal{R}$ will lead to an increase in $\Lambda^{-2}G_3$, and an increase in $\mathcal{R}$ will lead to a decrease in $\Lambda^{-2}G_3$. From the bulk AdS perspective, this means that we move from factorization to breakdown when the gravitons are moved closer together in the boundary directions, perpendicular to the AdS radius.

5.3 $J_1 \neq J_2$

When J_1 and J_2 are not equal, but lie in the region $N^{\frac{1}{3}} \ll J_1 \leq J_2 \ll N^{\frac{2}{3}}$, then equation (5.24) has the solution

$$
\frac{J_1 J_2}{N} = 2W\left[\sqrt{\frac{1}{4}\mathcal{R}^4 N(J_1 + J_2)(1 - e^{-\frac{J_1^2}{N}})e^r}\right]. \tag{5.32}
$$

The form of the expansion of the W-function depends on the scaling behaviour of the smallest angular momentum with N, which we have chosen to be J_1. We consider separately three cases: J_1^2/N tends to zero, J_1^2/N tends to a constant, and J_1^2/N tends to infinity.

If $J_1^2/N \to 0$, then the leading terms in the expansion of the W-function are

$$
\frac{J_1 J_2}{2N} \sim \frac{1}{2}\log\left[\frac{1}{4}\mathcal{R}^4 N(J_1 + J_2)\frac{J_1^2}{N}\right] - \log\left[\frac{1}{2}\log\left[\frac{1}{4}\mathcal{R}^4 N(J_1 + J_2)\frac{J_1^2}{N}\right]\right] + o(1) \tag{5.33}
$$

$$= \frac{1}{2} \log \left(\frac{J_1 N \mathcal{R}^4}{4} \right) + \log \left(1 + \frac{J_1}{J_2} \right) + \log \left(\frac{J_1 J_2}{N} \right) - \log \left[\frac{1}{2} \log \left[\frac{1}{4} \mathcal{R}^4 N (J_1 + J_2) \frac{J_1^2}{N} \right] \right] \quad (5.34)$$

Plugging in $J_1 J_2 / N$ into the third term, the log-log cancels and we have

$$\frac{J_1 J_2}{2N} = \frac{1}{2} \log \left(\frac{J_1 N \mathcal{R}^4}{4} \right) + \log \left(1 + \frac{J_1}{J_2} \right) + \mathcal{O}(1). \quad (5.35)$$

Since $J_1 \leq J_2$, the second term is $\mathcal{O}(1)$, hence

$$\frac{J_1 J_2}{N} = \log(J_1 N) + 4 \log \mathcal{R} + \mathcal{O}(1). \quad (5.36)$$

If J_1^2 / N tends to a constant at large N, then the expansion becomes

$$\frac{J_1 J_2}{2N} = \frac{1}{2} \log \left[\frac{\mathcal{R}^4 N}{4} (J_1 + J_2) \right] + c - \log \left[\frac{1}{2} \log \left[\frac{\mathcal{R}^4 N}{4} (J_1 + J_2) \right] + c \right] + o(1), \quad (5.37)$$

where c is some constant (order 1 with respect to N). Hence

$$\frac{J_1 J_2}{N} = \log((J_1 + J_2)N) + 4 \log \mathcal{R} - \log \log((J_1 + J_2)N) + \mathcal{O}(1). \quad (5.38)$$

If J_1^2 / N tends to infinity with N, then the above equation also holds but with c replaced by zero.

We can collate these three cases into a single equation by taking the leading scaling-behaviour of J_1 to be fixed, i.e. assuming $J_1 = N^{\alpha_1} e^{\delta_1}$ for subleading δ_1 and constant α_1. The threshold can then be written in all cases as

$$\frac{J_1 J_2}{N} = (1 + \alpha_1) \log N + 4 \log \mathcal{R} + o(\log N). \quad (5.39)$$

As in the extremal case, decreasing the difference between the angular momenta $\Delta J = |J_2 - J_1|$ will move the correlator from the threshold to the breakdown region. In addition, from the structure of the correlator in (5.24), it is clear that decreasing $\mathcal{R}$ while fixing N, J_1, and J_2 will move the correlator from the threshold to the breakdown region. From the bulk AdS point of view, non-extremal correlators correspond to the interactions of Kaluza-Klein gravitons with angular momenta in perpendicular directions in the S^5. We can move from the threshold to the breakdown region by moving the gravitons closer together in the boundary directions, or by decreasing the separation in the graviton energies.

6 Multi-gravitons and non-trivial backgrounds

In the previous sections we have studied in detail the thresholds of some simple extremal and non-extremal three-point functions. In this section we briefly discuss two other examples of

extremal correlators for which we have found explicit expressions of the threshold: a correlator corresponding to a $k + 1$-graviton system, and a correlator corresponding to gravitons in an LLM background. We find a very similar form of the thresholds to the previous examples in both cases. In the future, developing the tools to calculate more general correlators in the half-BPS sector could give us more insight into general properties of thresholds, and thus also shed light on the behaviour of high-momentum graviton systems in supergravity.

6.1 The $k + 1$-graviton correlator

We can calculate the extremal correlator associated to k gravitons scattering into a single graviton,

$$\langle\langle \prod_{i=1}^{k} (\mathrm{tr}Z^{J_i}) \mathrm{tr}Z^{\dagger \sum J_i} \rangle\rangle \tag{6.1}$$

and take the large dimensions limit using similar techniques. An outline of the derivation of the $k \to 1$ correlation function and its large N limit is given in Appendix B. In the regime where all $J_i \ll N^{\frac{2}{3}}$ for all $i = 1, 2, \ldots k$, then the correlator is asymptotic to

$$\sqrt{\frac{J_1 \ldots J_k}{N^{k-1} \sum_i J_i}} \frac{(1 - e^{-\frac{J_1 \sum J_i}{N}}) \ldots (1 - e^{-\frac{J_k \sum J_i}{N}})}{\sqrt{(1 - e^{-\frac{J_1^2}{N}}) \ldots (1 - e^{-\frac{J_k^2}{N}})(1 - e^{-\frac{(\sum J_i)^2}{N}})}} \exp\left(\sum_{i<j} \frac{J_i J_j}{2N}\right). \tag{6.2}$$

The factors in front of the exponential tend to zero as a power of N when $1 \ll J_i \ll N^{\frac{2}{3}}$. If *all* pairs of dimensions satisfy $J_i J_j \lesssim N$, then the exponential term is small, and the correlator decays to zero. However, if *any* pair of distinct dimensions satisfy $J_i J_j \geq N^{1+\delta}$ for some $\delta > 0$, then the exponential term dominates any power of N, and so the correlator tends to infinity. We can deduce that the factorization threshold when $G_3 = 1$ should be located when the product of the largest two operators grows logarithmically larger than N:

$$J_i J_j = \mathcal{O}(N \log N). \tag{6.3}$$

In the case when all the J_i are taken to be equal to J, then we can solve the threshold explictly at leading order. The correlator for $N^{\frac{1}{2}} < J < N^{\frac{1}{2}+\delta}$ is asymptotically

$$G_{k+1} \sim \sqrt{\frac{J^{k-1}}{kN^{k-1}}} \exp\left(\frac{k(k-1)}{4} \frac{J^2}{N}\right), \tag{6.4}$$

and the leading-order terms in the expansion of the threshold satisfying $G_{k+1}(J_t(N), N) = 1$ are

$$J_t^2 = \frac{N}{k}\left[\log N - \log\log N + \left(\frac{k+1}{k-1}\right)\log k + o(1)\right]. \tag{6.5}$$

This can be interpreted as saying that as the number of gravitons *increases*, the region in which factorization holds *shrinks*. When more gravitons are added to a system, they will start behaving like a single particle located further away from the boundary.

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6.2 Factorization thresholds for large backgrounds

Thresholds of factorization can be considered in more general half-BPS bulk backgrounds, specified in the dual description by Schur Polynomials. For a background described by a Young tableau B with n boxes, the associated Schur polynomial χ_B is a $U(N)$ character [13],

$$\chi_B(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_B(\sigma) \mathrm{tr}(\sigma Z^{\otimes n}). \tag{6.6}$$

The CFT state corresponding to such a background is $|B\rangle = \chi_B(Z^\dagger)|0\rangle$, and the operator in this background are defined by [26]

$$\langle \mathcal{O} \ldots \mathcal{O} \rangle_B = \frac{\langle B | \mathcal{O} \ldots \mathcal{O} | B \rangle}{\langle B | B \rangle}. \tag{6.7}$$

This gives us the definition of a three-particle normalized correlator in the state,

$$
\begin{aligned}
G_3(J_1, J_2; N, M)_B &= \frac{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1+J_2} \rangle_B}{\sqrt{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{\dagger J_1} \rangle_B \langle \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_2} \rangle_B \langle \mathrm{tr} Z^{J_1+J_2} \mathrm{tr} Z^{\dagger J_1+J_2} \rangle_B}} \\
&= \sqrt{\langle B | B \rangle} \frac{\langle B | \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1+J_2} | B \rangle}{\sqrt{\langle B | \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{\dagger J_1} | B \rangle \langle B | \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_2} | B \rangle \langle B | \mathrm{tr} Z^{J_1+J_2} \mathrm{tr} Z^{\dagger J_1+J_2} | B \rangle}}. \tag{6.8}
\end{aligned}
$$

One of the easiest ones backgrounds in which to perform the threshold calculation is the background corresponding to a large rectangular Young diagram with N rows of length M, where M is of the same order as N. In [26], it was shown by performing manipulations of Schurs that the large rectangular background modifies the normalized correlator by shifting the matrix rank parameter from N to $M+N$. That is, we have

$$
\begin{aligned}
G_3(J_1, J_2; N, M)_B &= \left[\frac{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1+J_2} \rangle}{\sqrt{\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{\dagger J_1} \rangle \langle \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_2} \rangle \langle \mathrm{tr} Z^{J_1+J_2} \mathrm{tr} Z^{\dagger J_1+J_2} \rangle}} \right]_{N \to N+M} \tag{6.9} \\
&= G_3(J_1, J_2, N + M). \tag{6.10}
\end{aligned}
$$

Hence, the correlator in a large rectangular background only differs from the normalized correlator in that the argument N is replaced by $N+M$. This means that, in this background, the threshold of factorization is at

$$J_1 J_2 \approx (N + M) \log(N + M). \tag{6.11}$$

We interpret this as evidence that the presence of a background can *increase* the size of the region in which factorization is valid.

7 Conclusions and Outlook

We have undertaken a detailed study of the thresholds where multi-particle Kaluza-Klein gravitons have order one correlations at large N with single gravitons. The angular momenta of the gravitons in $AdS_5 \times S^5$ must grow large with N for the correlator to approach the threshold, and the precise form of this growth was worked out in several cases. The large N growth at the threshold region for the case of two gravitons of angular momentum J being correlated with a single graviton of angular momentum $2J$ is $J \approx \sqrt{N \log N}$. The breakdown of factorization is a breakdown of the usual perturbative scheme for computing graviton interactions in spacetime, which relies on a multi-graviton Fock space with states of different particle number being orthogonal. In this usual framework, the mixing between different particle numbers arises in $1/N$ corrections which are suppressed at large N for small enough J. We have found quantitative description of several factors which can move a correlator from the regime factorization to the threshold, such as:

- Increasing the total energy of the gravitons,

- Decreasing the separation in the energies of the two gravitons,

- Decreasing the separation of gravitons in the boundary directions,

- Increasing the number of gravitons.

Another qualitative outcome of interest is that for k gravitons being correlated with a single graviton, the threshold can be expressed in terms of the two largest momenta among the k gravitons, taking the form $J_i J_j \approx N \log N$. In these investigations, we have found a rich variety of applications of the Lambert W-function. We have seen intriguing similarities between asymptotic threshold equations and running gauge couplings in non-abelian gauge theories. The large N approximations have also involved transseries of the kind seen in instanton-corrected perturbation expansions of quantum field theory.

We also investigated the factorization thresholds in the presence of LLM backgrounds associated with rectangular Young diagram backgrounds. The presence of these backgrounds *increases* the region of graviton momenta that are consistent with factorization. There are indications that triangular Young diagrams can be used to model thermal black hole-like backgrounds [32]. We expect that, in the presence of black holes, the regime of validity of effective field theory should be smaller than in the absence of black holes. This would suggest that factorization in triangular Young diagram backgrounds should occur in a more limited regime of graviton angular momenta than factorization in the vacuum. This is a very concrete problem in the combinatorics of CFT correlators, and an interesting research direction for the future.

In our study of factorization thresholds, we have consistently found thresholds when the angular momenta are of the form $J_i J_j \approx N \log N$, which suggests that there is some form of *universality* of the threshold. An interesting future direction would be to consider the thresholds calculated in the 'overlap-of-states' norm from [12, 33], as opposed to the 'multiparticle' norm

used in this paper. In the overlap normalization, the correlators are bounded by one from above and cannot grow exponentially with N, but they may well tend to a finite non-zero constant at large N if their angular momenta grow quickly enough. We could define a threshold in the overlap normalization as the surface where a correlator is equal to some fixed constant between zero and one. Evidence from shifting the factorization threshold at the end of Section 3.3 suggest that the form of the threshold will not change when going to the overlap norm, and will remain $J^2 \approx N \log N$. This is another interesting problem for the future that involves non-trivial asymptotics of finite N CFT correlators, and could well provide further evidence for the universality of the $N \log N$ threshold.

In Section 5 we showed how the 'nearly-extremal' correlator has a threshold which depends on the separation of the CFT-insertions in the 4D spacetime directions, as well as exhibiting the dependences on total energy and energy differences of the corresponding gravitons. We considered two gravitons in AdS with angular momenta $(J_3, J_1), (-J_3, J_2)$ where the first entry refers to the Y-plane and the second to the Z-plane. We studied the correlation with a single graviton with angular momenta $(J_1 + J_2, 0)$. The explicit calculations were done for $J_3 = 1$, with J_1, J_2 growing with N. A generalization to the case of J_3 also growing with N would be very interesting, as it would show the effect on the quantum correlations at threshold between two gravitons and a single graviton, when the two gravitons annihilate a large amount of Y-momentum and the correlator is no longer near-extremal.

We hope to have convinced the reader that the theme of thresholds between different behaviours is a fruitful way to explore the bulk AdS physics encoded in the correlators of the CFT. Since

$$\frac{1}{N} = g_s \frac{l_s^4}{R^4} \,, \tag{7.1}$$

for fixed R/l_s, finite N is finite string coupling, which is non-perturbative from the point of view of strings in the bulk spacetime. Hence, finite N calculations in CFT contain valuable information about strongly quantum gravitational effects. The generic $J_i J_j \approx N \log N$ we found, which in spacetime variables is

$$J_i J_j \approx N \log N = \left(\frac{R^4}{g_s l_s^4} \right) \log \left(\frac{R^4}{g_s l_s^4} \right) ,$$

is an intriguing result that should be understood better from the bulk point of view, either from a first principles string calculation in $AdS_5 \times S^5$ or from a phenomenological model of quantum gravitational spacetime constructed to reproduce the CFT result. As we observed, the threshold corresponds to a region where the Fock space of spacetime field modes breaks down. The broader issue of the breakdown of perturbative effective field theory is central to questions in black hole physics [25, 34, 35]. In particular, black hole complementarity is related to the structure of Hilbert spaces needed to describe infalling observers and outgoing radiation. We propose that a convincing spacetime understanding of the thresholds derived here would be a highly instructive step in understanding the departures from effective field theory in quantum

gravity. Insights from earlier work on bulk spacetime in AdS in connection with gauge-string duality, such as in [36, 37], might be useful. Alternatively, the methods of collective field theory [38] could help with a derivation of the large N effective field theory. Another possible approach towards better understanding the thresholds from the spacetime point of view would be to make use of a combination of semi-classical tools, exploiting high energy eikonal approximations or physical effects such as the tidal stretching of high energy gravitons into strings, for example along the lines of [39, 40].

The study of Schur operators as the description of giant gravitons was motivated by the observed departure from orthogonality between multi-graviton and single graviton states at large J [12]. Schur operators give a weakly-coupled description of giant gravitons in the regime of $J \approx N$, but become strongly-interacting as J is decreased [19]. In this paper, we have focused on the approach to the threshold in the regime near $J \approx \sqrt{N}$ by studying single and multi-trace graviton operators. It would be very interesting to study thresholds between weak and strong interactions in giant graviton physics as the angular momenta are decreased from $J \approx N$. The detailed investigations of the one-loop and multi-loop dilatation operators around giant graviton backgrounds should provide useful data for this purpose [41, 42, 43].

The fact that the thresholds are near $\sqrt{N}$ rather than $N^{1/4}$ is rather intriguing. This has been discussed in [19]. Angular momenta of $J \approx N^{1/4}$ correspond to momenta comparable to the ten-dimensional Planck scale. This may be a sign that $AdS_5 \times S^5$ physics is just very different from expectations derived from effective field theory in flat space $\mathbb{R}^{9,1}$. On the other hand, it could be that a clever interpretation of the link between the extremal correlators and flat space scattering would account for the thresholds we see from the CFT. Potentially, the correct interpretation has to recognise that extremal correlators correspond to collinear graviton scatterings. We would need to consider the flat space expectations in the light of collinear effective theories of gravitons, along the lines developed in [44], to understand the difference between the threshold scale and the Planck scale. An early discussion of the subtleties of connecting bulk AdS spacetime physics to the flat space limit is given in [45].

There is a lot of fun to be had with factorization thresholds in AdS/CFT: there is a wealth of quantitative information about graviton correlations at threshold available via finite N CFT computations and their large N asymptotics. The lessons we draw from these are very likely to be important for questions we would like to answer in black hole physics and quantum gravity.
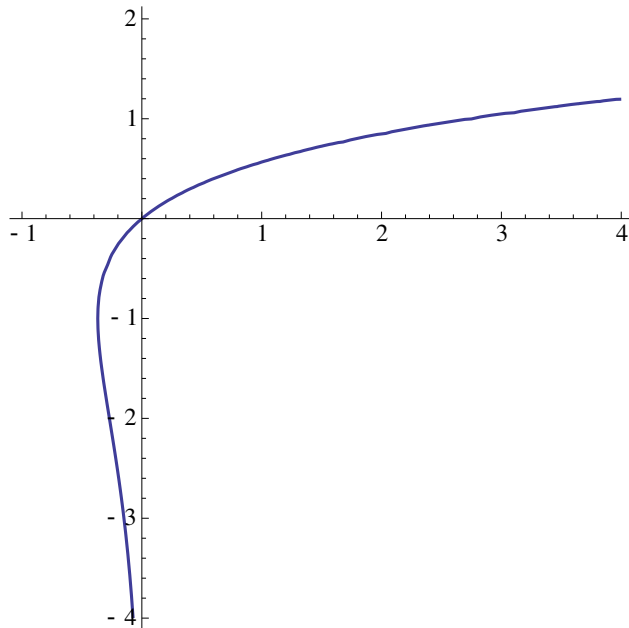
Figure 5: The Lambert W-function $W(x)$ for real x is multivalued: the principal branch W_0 takes values greater than -1, and the other branch W_{-1} is defined for $W < -1$.

Appendix

A   The Lambert W-function

The Lambert W-function is, by definition, the solution to the equation

$$W(z)e^{W(z)} = z. \tag{A.1}$$

This equation cannot be solved in a closed form in terms of elementary functions, but a Taylor series can be found near $z = 0$, and its asymptotic series can be derived for large positive z.

There are many solutions $W(z)$ to the equation (A.1), which means that the Lambert W-function is multivalued. However, only two solutions take real values when z is real, and these are the only relevant solutions in this paper. One of these solutions is the principal branch $W_0(z)$, which is real and satisfies $W_0(z) \geq -1$ on its domain $z \in [-e^{-1}, \infty)$. The other is the $W_{-1}(z)$ branch, which takes values in the range $W_{-1}(z) \leq -1$ and is defined on the domain $z \in [-e^{-1}, 0)$. The two real branches of the W-function are shown in figure 5.

The large z expansion of the principal branch of the W-function is

$$W(z) \sim \log z - \log\log z + \sum_{n=1}^{\infty} \left(\frac{-1}{\log z}\right)^n \sum_{k=1}^{n} \left[\begin{matrix} n \\ n-k+1 \end{matrix}\right] \frac{(-\log\log z)^k}{k!}, \tag{A.2}$$

where the coefficients in the square brackets are the (unsigned) Stirling cycle numbers of the first kind. The notation $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the number of permutations of n elements composed of k disjoint cycles. (For example, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ refers to the number of permutations in the symmetric group S_4 composed of two disjoint cycles. There are six permutations in S_4 composed of a 3-cycle and a 1-cycle, and three permutations composed of a pair of disjoint 2-cycles, and these are the only permutations composed of two disjoint cycles in S_4. Hence, $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 6 + 3 = 9$.)

B Combinatoric calculations using character sums

In this appendix we present some finite N calculations of correlators using matrix model techniques. The extremal correlator $\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \mathrm{tr} Z^{\dagger J_1 + J_2} \rangle$ was calculated in [15], and using character sums in [16]. We use the methods of [16] to calculate the norm of the operator $\mathrm{Str}(Z^{J_1} Y^{J_2})$, and to calculate the $k \to 1$ correlator $\langle \mathrm{tr} Z^{J_1} \mathrm{tr} Z^{J_2} \ldots \mathrm{tr} Z^{J_k} \mathrm{tr} Z^{\dagger \sum_i J_i} \rangle$. We then find an expression for the normalized $k + 1$-point correlator at large N.

B.1 The non-extremal operator norm

Consider the non-extremal two-point function which is the norm of a mixed operator consisting of two types of adjoint fields,

$$\| \mathrm{Str}(Z^{J_1} Y^{J_2}) \|^2 = \langle \mathrm{Str}(Z^{J_1} Y^{J_2}) \mathrm{Str}(Z^{\dagger J_1} Y^{\dagger J_2}) \rangle. \tag{B.1}$$

The symmetrized trace of a string of matrices in the adjoint representation of the gauge group $U(N)$ is

$$\mathrm{Str}(Z^{J_1} Y^{J_3}) = \frac{1}{(J_1 + J_3 - 1)!} \sum_{\sigma \in [J_1 + J_3]} X_{i_{\sigma(1)}}^{i_1} X_{i_{\sigma(2)}}^{i_2} \ldots X_{i_{\sigma(J_1)}}^{i_{J_1}} Y_{i_{\sigma(J_1+1)}}^{i_{J_1+1}} \ldots Y_{i_{\sigma(J_1+J_3)}}^{i_{J_1+J_3}}. \tag{B.2}$$

The sum is performed over all permutations in $[J_1 + J_3]$, the conjugacy class in $S_{J_1 + J_3}$ consisting of all the cyclic permutations with a single cycle of length $(J_1 + J_3)$. All matching pairs of adjoint matrix indices i_l are implicitly summed. This expression can be written more concisely in tensor space notation [16] as

$$\mathrm{Str}(Z^{J_1} Y^{J_3}) = \frac{1}{(J_1 + J_3 - 1)!} \sum_{\sigma \in [J_1 + J_3]} \mathrm{tr}(\sigma X^{\otimes J_1} \otimes Y^{\otimes J_3}). \tag{B.3}$$

This two-point function can be calculated by using diagrammatic tensor space techniques

[16]:

$$\| \operatorname{Str}(Z^{J_1}Y^{J_2}) \|^2 \;\; = \;\; \frac{1}{(J_1+J_2-1)!^2} \sum_{\sigma_1,\sigma_2\in[J_1+J_2]} \left\langle \begin{array}{c} Z^{J_1}\ Y^{J_2} \quad\quad Z^{\dagger J_1}\ Y^{\dagger J_2} \\ \boxed{\sigma_1} \qquad\qquad \boxed{\sigma_2} \end{array} \right\rangle \tag{B.4}$$

$$= \frac{1}{(J_1+J_2-1)!^2} \sum_{\substack{\sigma_1,\sigma_2\in[J_1+J_2]\\ \gamma_1\in S_{J_1}\\ \gamma_2\in S_{J_2}}} \begin{array}{c} \boxed{\gamma_1}\ \boxed{\gamma_2} \quad \boxed{\gamma_1^{-1}}\ \boxed{\gamma_2^{-1}} \\ \boxed{\sigma_1} \qquad \boxed{\sigma_2} \end{array} \tag{B.5}$$

$$\frac{J_1!J_2!}{(J_1+J_2-1)!^2} \sum_{\sigma_1,\sigma_2\in[J_1+J_2]} \begin{array}{c} \boxed{\sigma_1} \\ \boxed{\sigma_2} \end{array} \tag{B.6}$$

We can replace the permutation sums with sums over representations with projectors on the group algebra,

$$\| \operatorname{Str}(Z^{J_1}Y^{J_2}) \|^2 = J_1!J_2! \sum_{R_1,R_2\vdash(J_1+J_2)} \frac{\chi_{R_1}([J_1+J_2])\chi_{R_2}([J_1+J_2])}{d_{R_1}d_{R_2}} \begin{array}{c} \boxed{R_1} \\ \boxed{R_2} \end{array} , \tag{B.7}$$

where $\chi_{R_1}([J_1+J_2])$ is the character in R_1 of a permutation in the conjugacy class $[J_1+J_2]$. Representation projectors satisfy the identity $P_{R_1}P_{R_2} = \delta_{R_1R_2}P_{R_1}$, and $\operatorname{tr}P_R = dim_N(R)d_R$. From the Murnaghan-Nakayama lemma [46], the character of a (J_1+J_2)-cycle in $S_{J_1+J_2}$ is ± 1 if the diagram is a *hook*, and zero otherwise. A hook representation corresponds to a Young tableau where all the boxes are in the first row or the first column, as in Figure 6. We find
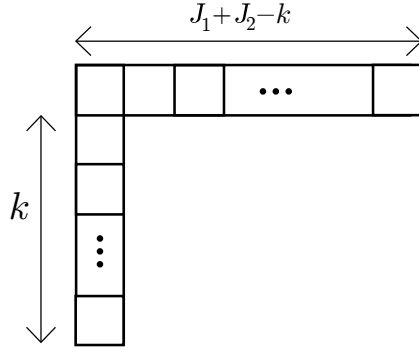
45

Figure 6: A Young diagram with $J_1 + J_2$ boxes corresponding to a hook rep with hook length k.

$$\| \operatorname{Str}(Z^{J_1} Y^{J_2}) \|^2 = J_1! J_2! \sum_{R \vdash (J_1 + J_2)} \frac{\chi_R([J_1 + J_2])^2}{d_R^2} \operatorname{tr}(P_R) \tag{B.8}$$

$$= J_1! J_2! \sum_{R \text{ a hook rep}} \frac{dim_N(R)}{d_R}. \tag{B.9}$$

This sum is weighted by the dimension of a hook rep of $U(N)$ divided by the dimension of the corresponding hook rep in $S_{J_1 + J_2}$. Parametrizing the hook lengths by the hook length k, where $k = 0, 1, \ldots (J_1 + J_2 - 1)$, we find that the ratio of the dimensions is

$$\frac{dim_N(R)}{d_R} = \binom{N + J_1 + J_2 - k - 1}{J_1 + J_2}, \tag{B.10}$$

and hence the correlator is

$$\| \operatorname{Str}(Z^{J_1} Y^{J_2}) \|^2 = J_1! J_2! \sum_{k=0}^{J_1 + J_2 - 1} \binom{N + J_1 + J_2 - k - 1}{J_1 + J_2} \tag{B.11}$$

$$= J_1! J_2! \sum_{k=0}^{J_1 + J_2 - 1} \binom{N + k}{J_1 + J_2}. \tag{B.12}$$

Finally, we employ the general identity

$$\sum_{k=0}^{n-1} \binom{N + k}{m} = \binom{N + n}{m + 1} - \binom{N}{m + 1} \tag{B.13}$$

to deduce the final exact answer,

$$\| \operatorname{Str}(Z^{J_1} Y^{J_2}) \|^2 = J_1! J_2! \left[\binom{N + J_1 + J_2}{J_1 + J_2 + 1} - \binom{N}{J_1 + J_2 + 1} \right]. \tag{B.14}$$

B.2 The $k + 1$-graviton correlator character sum

In this section we present a calculation of the $k + 1$-graviton correlator in the gauge theory. A similar calculation was done previously in [47]. The representation sum of the general extremal correlator was stated in [16] as being

$$\langle \prod_{i=1}^{k} (\mathrm{tr} Z^{J_i}) \mathrm{tr} Z^{\dagger J} \rangle = \sum_{R \vdash J} f_R \chi_R([J_1 \ldots J_k]) \chi_R([J]). \tag{B.15}$$

We adopt the notation $J = \sum_i J_i$ throughout this subsection. Using the Murnaghan-Nakayama lemma [46], we find that $\chi_R([J])$ is non-zero only if R is a hook rep, and equal to $(-)^k$ for a hook of length k. This constrains the sum to run only over hook representations, and so

$$\langle \prod_{i=1}^{k} (\mathrm{tr} Z^{J_i}) \mathrm{tr} Z^{\dagger J} \rangle = J! \sum_{l=0}^{J-1} \binom{N+l}{J} (-)^{J-1-l} \chi_{H_l}([J_1 \ldots J_k]), \tag{B.16}$$

where H_l denotes the hook representation $[l + 1, 1^{J-1-l}]$. The Murnaghan-Nakayama lemma states that we can knock J_k boxes off this J-box hook rep to get

$$\chi_{H_l}[J_1 \ldots J_k] = \delta(l \geq J_k) \chi_{H_{l-J_k}}([J_1 \ldots J_{k-1}]) + (-)^{J_k+1} \delta(J - l > J_k) \chi_{H_l}([J_1 \ldots J_{k-1}]) \tag{B.17}$$

If we replace the expressions in the binomial coefficient by the general terms M, m, we have

$$\sum_{l=0}^{J-1} (-)^l \binom{M+l}{m} \chi_{H_l}([J_1 \ldots J_k]) = \sum_{l=0}^{J-J_k-1} (-)^{J_k} \chi_{H_l}([J_1 \ldots J_{k-1}]) \left[\binom{M+J_k+l}{m} - \binom{M+l}{m} \right]$$

We can plug this formula in to (B.16) for different values of M and m. We get

$$\langle \prod_{i=1}^{k} (\mathrm{tr} Z^{J_i}) \mathrm{tr} Z^{\dagger J} \rangle = J!(-)^{J-1} \sum_{l=0}^{J-1} (-)^l \chi_{H_l}([J_1 \ldots J_k]) \binom{N+l}{J}$$

$$= J!(-)^{J-J_k-1} \sum_{l=0}^{J-J_k-1} (-)^l \chi_{H_l}([J_1 \ldots J_{k-1}]) \left[\binom{N+J_k+l}{J} - \binom{N+l}{J} \right]$$

$$= J!(-)^{J-J_k-J_{k-1}-1} \sum_{l=0}^{J-J_k-J_{k-1}-1} (-)^l \chi_{H_l}([J_1 \ldots J_{k-2}]) \left[\binom{N+J_k+J_{k-1}+l}{J} - \binom{N+J_k+l}{J} \right.$$

$$\left. - \binom{N+J_{k-1}+l}{J} + \binom{N+l}{J} \right]$$

$$= J!(-)^{J_1-1} \sum_{l=0}^{J_1-1} (-)^l \chi_{H_l}([J_1]) \left[\binom{N+J-J_1+l}{J} - \ldots + (-)^{k-1} \binom{N+l}{J} \right], \tag{B.18}$$

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where we have omitted the intermediate binomials with arguments containing all sums of elements in $\{J_2, J_3 \ldots, J_k\}$. Using $\chi_{H_l}([J_1]) = (-)^{J_1-1-l}$ and

$$\sum_{l=0}^{J_1-1} \binom{M+l}{J} = \binom{M+J_1}{J+1} - \binom{M}{J+1}, \tag{B.19}$$

we can now evaluate the sums to find that

$$\langle \prod_{i=1}^{k} (\mathrm{tr} Z^{J_i}) \mathrm{tr} Z^{\dagger J} \rangle = J! \left[\binom{N+J}{J+1} - \ldots + (-)^k \binom{N}{J+1} \right] \tag{B.20}$$

and restoring the omitted terms, we deduce that

$$\langle \prod_{i=1}^{k} (\mathrm{tr} Z^{J_i}) \mathrm{tr} Z^{\dagger J} \rangle = J! \sum_{t=0}^{k} \sum_{\substack{S \subseteq \{1,\ldots,k\} \\ |S|=t}} (-)^{k-t} \binom{N + \sum_{i \in S} J_i}{J+1}. \tag{B.21}$$

The sum over S is a sum over all the subsets of the k-element set.

B.3   Asymptotics of the $k+1$-point function

In this section we derive the asymptotic form of the $k+1$-point function (B.21). Assuming that $J_i \ll N^{\frac{2}{3}}$ and that Λ is some sum of the J_i, we have from Section 3

$$J! \binom{N+\Lambda}{J+1} \sim \frac{N^{J+1}}{J} \exp\left(\frac{J(2\Lambda - J)}{2N} - \mathcal{O}\left(\frac{J^3}{N^2} \right) \right). \tag{B.22}$$

We can then write (B.21) as

$$\langle \prod_{i=1}^{k} (\mathrm{tr} Z^{J_i}) \mathrm{tr} Z^{\dagger J} \rangle \sim \frac{N^{J+1}}{J} \sum_{S \subseteq \{1\ldots k\}} (-)^{k-|S|} e^{-\frac{J^2}{2N} + \mathcal{O}(J^3/N^2)} e^{\frac{J}{N} \sum_{i \in S} J_i} \tag{B.23}$$

$$\sim \frac{N^{J+1}}{J} e^{-\frac{J^2}{2N} + \mathcal{O}(J^3/N^2)} (-)^k \sum_{S \subseteq \{1\ldots k\}} \prod_{i \in S} \left(-e^{\frac{J}{N} J_i} \right) \tag{B.24}$$

We can evaluate this sum over subsets explicitly by first partitioning the sum into two; one sum over the subsets including the element k, and one over the subsets not including k. We can then apply this for each integer from 1 to k. We have

$$\sum_{S \subseteq \{1\ldots k\}} \prod_{i \in S} \left(-e^{\frac{J}{N} J_i} \right) = \left(-e^{\frac{J}{N} J_k} \right) \sum_{S \subseteq \{1\ldots k-1\}} \prod_{i \in S} \left(-e^{\frac{J}{N} J_i} \right) + 1 \sum_{S \subseteq \{1\ldots k-1\}} \prod_{i \in S} \left(-e^{\frac{J}{N} J_i} \right) \tag{B.25}$$

$$= (-e^{\frac{J}{N} J_1} + 1)(-e^{\frac{J}{N} J_2} + 1) \ldots (-e^{\frac{J}{N} J_k} + 1). \tag{B.26}$$

Taking out a factor of $e^{J^2/N}$ from this product, we have the asymptotic form of the unnormalized correlator,

$$\langle \prod_{i=1}^{k}(\text{tr}Z^{J_i})\text{tr}Z^{\dagger J}\rangle \sim \frac{N^{J+1}}{J}\exp\left(\frac{J^2}{2N}+\mathcal{O}\left(\frac{J^3}{N^2}\right)\right)\prod_{i=1}^{k}\left(1-e^{-\frac{JJ_i}{N}}\right). \tag{B.27}$$

Together with the known asymptotic form of the 2-point function

$$\langle \text{tr}Z^{J_i}\text{tr}Z^{\dagger J_i}\rangle \quad \sim \quad \frac{N^{J_i+1}}{J_i}e^{\frac{J_i^2}{2N}}\left(1-e^{-\frac{J_i^2}{N}}\right), \tag{B.28}$$

we can therefore write the full correlator in the large J, small J^3/N^2 limit,

$$\langle\langle\prod_{i=1}^{k}(\text{tr}Z^{J_i})\text{tr}Z^{\dagger J}\rangle\rangle \sim \sqrt{\frac{J_1\ldots J_k}{JN^{k-1}}}e^{\frac{J^2}{4N}-\frac{J_1^2}{4N}-\cdots-\frac{J_k^2}{4N}}\frac{\prod_{i=1}^{k}(1-e^{-JJ_1/N})}{\sqrt{(1-e^{-J^2/N})\prod_{i=1}^{k}(1-e^{-J_1^2/N})}} \tag{B.29}$$

$$\sim \sqrt{\frac{J_1\ldots J_k}{JN^{k-1}}}\frac{(1-e^{-\frac{JJ_1}{N}})\ldots(1-e^{-\frac{JJ_k}{N}})}{\sqrt{(1-e^{-\frac{J_1^2}{N}})\ldots(1-e^{-\frac{J_k^2}{N}})(1-e^{-\frac{J^2}{N}})}}\exp\left(\sum_{i<j}\frac{J_iJ_j}{2N}+\mathcal{O}\left(\frac{J^3}{N^2}\right)\right). \tag{B.30}$$

When all the angular momenta are equal, $J_1 = J_2 = \ldots = J_k = J$, and J^2/N is large, then this expression becomes

$$\langle\langle(\text{tr}Z^J)^k\text{tr}Z^{\dagger kJ}\rangle\rangle \sim \sqrt{\frac{J^{k-1}}{kN^{k-1}}}\exp\left(\frac{k(k-1)}{4}\frac{J^2}{N}\right). \tag{B.31}$$

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