

Guessing Games on Triangle-free Graphs

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Abstract

The guessing game introduced by Riis[15] is a variant of the “guessing your own hats” game and can be played on any simple directed graph G on n vertices. For each digraph G , it is proved that there exists a unique guessing number $\text{gn}(G)$ associated to the guessing game played on G . When we consider the directed edge to be bidirected, in other words, the graph G is undirected, Christofides and Markström [6] introduced a method to bound the value of the guessing number from below using the fractional clique number $\kappa_f(G)$. In particular they showed $\text{gn}(G) \geq |V(G)| - \kappa_f(G)$. Moreover, it is pointed out that equality holds in this bound if the underlying undirected graph G falls into one of the following categories: perfect graphs, cycle graphs or their complement. In this paper, we show that there are triangle-free graphs that have guessing numbers which do not meet the fractional clique cover bound. In particular, the famous triangle-free Higman–Sims graph has guessing number at least 77 and at most 78, while the bound given by fractional clique cover is 50.

1 Introduction

The motivation of developing guessing games [15] comes from the study of a specific class of problems in network coding [1], namely multiple unicast network coding. A multiple unicast network is a communication network in which each sender has a unique receiver that wishes to obtain messages from it. Such a network can be represented by a directed acyclic graph where senders, receivers and routers are vertices in the graph and channels are directed edges between vertices. Furthermore, we simplify the problem by require that each channel only allows one message to pass through it at a time. If we merge each vertex representing a sender with a vertex which represent its corresponding receiver in the directed acyclic graph, we will obtain an auxiliary digraph in which we no longer have the distinction between senders, receivers, or routers. We can define a guessing game to play on this auxiliary digraph; the rules of our guessing game will allow us to translate strategies on the auxiliary graph into coding functions on the simplified multiple unicast network and *vice versa*. The precise definition of guessing number will be given in Section 2, and a measure of the performance of the optimal strategy for a guessing game.

It is worth noting that guessing games were the main ingredients in Riis’ proof of the invalidity of two conjectures raised by Valiant [17] in circuit complexity in which he asked about the optimal Boolean circuit for a Boolean function. (See [15].)

Our paper deals with the guessing game played on a special type of directed graph in which each directed edge is bidirected, i.e. our graphs are undirected. In particular, we show that there are triangle-free undirected graphs where the guessing numbers of these graphs can not be computed using the fractional clique cover method developed by Christofides and Markström in [6]. This also give counterexamples to their conjecture about the optimal guessing strategy based on fractional clique cover in [6]. (The first counter example to Christofides and Markström’s conjecture was illustrated in [2] but the graph is not triangle-free.)

Our paper is organised as follows. Firstly, we introduce the rules of guessing games played on undirected graphs in Section 2. Then in Section 3 we prove the existence of the asymptotic guessing numbers. In Section 4 the fractional clique cover strategy from [6] is formally defined. Our main results appear in Section 5. Sections 2, 3, 4 already appeared in [9], [6], and [2] but we reproduce them here in order to make this paper self-contained.

2 Definitions

An *undirected graph* $G = (V, E)$, or *graph* for short, consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *undirected edges*. An undirected edge $e \in E(G)$, or *edge* for short, is an unordered pair (u, v) of vertices, which we also denote by uv or vu , with u and v are elements of $V(G)$. We say vertices u and v are *adjacent* if uv is an edge. Similarly, we say two edges are adjacent if they share a common vertex. Given a graph G , we will denote its adjacency

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matrix by A . We also denote I for the identity matrix, and it will be clear from the context that I and A have the same order.

The guessing game is defined to be played on *simple graphs*, i.e. graphs not containing *loops* (edges of the form uu for u a vertex) or *multiple edges* (two or more edges with the same vertices). Thus, two edges are adjacent if and only if they share exactly one common vertex.

Given a graph G and a vertex $v \in V(G)$, the *neighbourhood* of v is $\Gamma(v) = \{u : uv \in E(G)\}$.

An important class of guessing strategies introduced in Section 4 involve *cliques*, i.e. subgraphs of G in which every pair of vertices are joined by an undirected edge.

Given two graphs G and H , the *tensor product* or *categorical product* $G \times H$ of G and H is a graph with vertex set $V(G \times H)$ which is the Cartesian product of $V(G)$ and $V(H)$; and an edge $e = ((u, v), (u', v')) \in E(G \times H)$ if uu' is an edge in G and vv' is an edge in H . A special case where H is K_t^o a complete graph of order t with a loop at each vertex, i.e. $|V(H)| = t$ and $E(H)$ is the set of all unordered pairs (u, v) with $u, v \in V(H)$, we will call $G \times H$ the *t-uniform blowup* of G , and denote as $G(t)$. We can also see the t -uniform blowup as a graph formed by replacing each vertex v in G with a class of t vertices v_1, \dots, v_t with $u_i v_j \in E(G(t))$ if and only if $uv \in E(G)$.

Given a graph $G = (V, E)$ of order n , and a positive integer s greater than 1, we play a *guessing game* (G, s) as follows:

A (G, s) game consists of n players with each player corresponds to one of the vertices of G . Throughout this paper, we use vertex $v \in V(G)$ to indicate the player who is assigned to it. Each player is informed about its corresponding vertex, its neighbourhood $\Gamma(v) = \{u : uv \in E(G)\}$, and the natural number s . The players can use this information to decide a strategy beforehand, but all communication are forbidden as soon as the game is started.

Once the game starts each player $v \in V(G)$ is assigned a value a_v from *alphabet set* $A_s = \{0, 1, \dots, s-1\}$ uniformly and independently at random. The value a_v is hidden from the player v . Instead, each player v is provided a list containing its neighbourhood $\Gamma(v)$ and the value assigned to each one of them correspondingly. The player is required to deduce its assigned value using just this information. Each player must announce its guessing value only once. A game is won if every player deduce the assigned value correctly, and the game is lost if otherwise. We are interested in the question about the maximal probability of winning when we play a guessing game (G, s) .

To illustrate the nature of this problem, let us play guessing games (C_5, s) with $s = 2, 3, 4$ and C_5 is the cycle graph of order 5. This is example 3.2 in [6]. Naively each player should guess randomly as non of the provided information directly relate to its assigned value, hence the winning probability of this strategy is s^{-5} . We name this naive strategy Random. The interesting property of guessing game is that we are almost always able to outperform the Random strategy.

For $s = 2$, a possible strategy for the game $(C_5, 2)$ is as follows: each player v guesses 0 if all of its neighbours are assigned 1, and guesses 1 for the rest cases. The game is won if the assigned values to players belong to one of the following cases $\{11010, 10110, 10101, 01101, 01011\}$. Hence, the winning probability with this strategy is $\frac{5}{2^5} = \frac{5}{32}$, and indeed this is the highest possible winning probability for the guessing game $(C_5, 2)$ [6].

For $s = 3$, using computer search, Christofides and Markström [6] able to show that the best possible winning probability for $(C_5, 3)$ is $\frac{12}{3^5} = \frac{12}{243}$ with a complicated guessing strategy which is highly non-symmetric in the sense that each vertex uses a distinct guessing function from the other.

For $s = 4$, it is firstly showed in [14] an optimal strategy involves what is so called fractional clique cover strategy which we will introduce in Section 4. The winning probability corresponds to this strategy is $\frac{2 \cdot 4^2}{4^5} = 4^{-2.5}$.

In examples above, guessing strategies are *pure strategies*, i.e. each player deduces its guessing value by using a deterministic function with inputs are the values of its neighbours. We can also play *mixed strategy* in which the players randomly select a strategy to follow from a set of pure strategies. The winning probability of a mixed strategy is the expected winning probability computed by averaging the winning probability of each chosen pure strategies with the according probabilities that they are selected. Hence, playing with mixed strategy will gain us no advantages when computing maximal winning probabilities as the maximal winning probability with mixed strategy can not surpass the winning probability provided by best pure strategies. Therefore we only concern with pure strategies throughout this paper.

Let (G, s) be a guessing game played on the graph G and with alphabet set $A_s = \{0, 1, \dots, s-1\}$ with s is a positive integer greater than 1. Each *strategy for player v* is a function $f_v : A_s^{|\Gamma(v)|} \rightarrow A_s$ which takes the possible values of the neighbours of v and maps them to the guessing value of v . A *strategy \mathcal{F}* for (G, s) is a $|V(G)|$ -tuple of deterministic functions $(f_v)_{v \in V(G)}$ where f_v corresponds to a strategy for player v . We denote $\text{Win}(G, s, \mathcal{F})$ for the event that the game (G, s) is won by using strategy \mathcal{F} . We are interested in constructing a strategy \mathcal{F} which maximises $\mathbf{P}[\text{Win}(G, s, \mathcal{F})]$.

For each strategy \mathcal{F} , we denote $\text{gn}(G, s, \mathcal{F})$ for the value $|V(G)| + \log_s \mathbf{P}[\text{Win}(G, s, \mathcal{F})]$ which is the *guessing number with respected to \mathcal{F}* . We define the guessing number $\text{gn}(G, s)$ to be

$$\text{gn}(G, s) = |V(G)| + \log_s \left(\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s, \mathcal{F})] \right).$$

We can see that:

$$\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s, \mathcal{F})] = \frac{s^{\text{gn}(G, s)}}{s^{|V(G)|}}.$$

The guessing number $\text{gn}(G, s)$ is a measure of how much better an optimal strategy outperforms random strategy Random when playing (G, s) .

3 The asymptotic guessing number

In our examples of guessing games $(C_5, s)_{s=2,3,4}$, the guessing numbers $\text{gn}(C_5, s)$ depend on s , and in this case the sequence $\{\text{gn}(C_5, s)\}_{s=2,3,\dots}$ is not a monotone sequence of s . In general case for guessing games (G, s) , it is extremely difficult to determine the exact value of $\text{gn}(G, s)$ for each value of s . Therefore, we are rather interested in evaluating the value of $\text{gn}(G, s)$ when s tends to infinity, and we call it the *asymptotic guessing number* $\text{gn}(G)$:

$$\text{gn}(G) = \lim_{s \rightarrow \infty} \text{gn}(G, s).$$

It is proved in [6] and [9] that this limit exists. The following arguments are due to Christofides and Markström [6]. Their strategy is to prove that the sequence $\{\text{gn}(G, s)\}_{s=\{2,3,\dots\}}$ for general graph G is an almost monotonically increasing sequence with respect to the size s of the alphabet A_s , and this sequence is bounded above by the obvious bound $|V(G)|$ (in fact in their paper [6], the upperbound of this sequence is $|V(G)| - \alpha(G)$ where $\alpha(G)$ is the independent number of graph G).

We start with the following lemma

Lemma 3.1. *Let G be an undirected graph, and s, t are integers with $s \geq 2$ and $t \geq 1$. We have*

$$t \text{gn}(G, s) \leq \text{gn}(G(t), s) \tag{1}$$

$$\text{gn}(G(t), s) = t \text{gn}(G, s^t) \tag{2}$$

Proof. The graph $G(t)$ contains t vertex disjoint copies of G . Given a strategy \mathcal{F} played on (G, s) , we construct a strategy $\mathcal{F}(t)$ of $(G(t), s)$ by playing \mathcal{F} on each of the t disjoint copies of G in $G(t)$. This gives us

$$\left(\max_{\mathcal{F}} \mathbf{P}[\text{Win}(s, \mathcal{F})] \right)^t \leq \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G(t), s, \mathcal{F})]$$

and the inequality (1) follows immediately.

To prove (2) holds, we show that there is a one-to-one correspondence between strategies played on $(G(t), s)$ and strategies played on (G, s^t) satisfies that given a strategy $\mathcal{F}(t)$ played on $(G(t), s)$ its corresponding strategy \mathcal{F} played on (G, s^t) gives the same winning probabilities. Hence, we have

$$\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G(t), s, \mathcal{F})] = \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s^t, \mathcal{F})].$$

and the result follows from the definition of guessing number.

We note that each member a of the alphabet of size s^t can be uniquely represented as a t -tuple $\{a_1, \dots, a_t\}$ with a_i s are in base s .

Given a strategy \mathcal{F} on (G, s^t) , we construct a corresponding strategy $\mathcal{F}(t)$ to be played on $(G(t), s)$ as follow:

For each player v in (G, s^t) , its guessing function is $f_v : A_{s^t}^{|\Gamma(v)|} \rightarrow A_{s^t}$. The map f_v induces a unique corresponding map $f_{[v_1, \dots, v_t]} : (A_s^t)^{|\Gamma(v)|} \rightarrow A_s^t$ due to the one-to-one correspondence between elements of A_{s^t} and A_s^t . Therefore, if a player v in (G, s^t) follows a strategy f_v , the vertex class of t players $[v_1, \dots, v_t]$ in $(G(t), s)$ simulate playing as v by agreeing to use $f_{[v_1, \dots, v_t]}$ as guessing function for each member in the class. The output of $f_{[v_1, \dots, v_t]}$ can be considered as the guess of each member v_i about the overall value assigned to the whole class $[v_1, \dots, v_t]$. This guessing function is well defined since each v_i received precisely the same input. Moreover, the guessing output produced by each member v_i of the class will be the same. Each member can decomposed the value of the guess for the vertex class into t values from A_s and uses this information as the individual guesses for each one of them. Hence we have converted a guessing strategy \mathcal{F} of (G, s^t) to a guessing strategy $\mathcal{F}(t)$ of $(G(t), s)$. We can also see that

$$\mathbf{P}[\text{Win}(G, s^t, \mathcal{F})] = \mathbf{P}[\text{Win}(G(t), s, \mathcal{F}(t))].$$

Clearly the map $\mathcal{F} \mapsto \mathcal{F}(t)$ defined above is an injection from the set of all guessing strategies can be played on (G, s^t) to the set of all guessing strategies can be played on $(G(t), s)$. We use a similar argument to show that given a guessing strategy $\mathcal{F}(t)$ to be played on $(G(t), s)$ there is a corresponding guessing strategy \mathcal{F} of (G, s^t) .

The unique corresponding between $f_v : A_{s^t}^{|\Gamma(v)|} \rightarrow A_{s^t}$ and $f_{[v_1, \dots, v_t]} : (A_s^t)^{|\Gamma(v)|} \rightarrow A_s^t$ described above allow each player v in (G, s^t) to pretend to be t players in the class $[v_1, \dots, v_t]$ in $(G(t), s)$. The strategy $\mathcal{F}(t)$ of $(G(t), s)$ can then be used and the guesses for each vertex class $[v_1, \dots, v_t]$ can be reconstructed into a guess for the original player v in (G, s^t) . This completes the proof. \square

Using these results about the guessing number of the t -uniform blowup of graphs we now showing that the guessing number is almost monotonically increasing with respect to the size of the alphabet.

Lemma 3.2. *Given a graph G , positive integer s , and real number $\epsilon > 0$, there exists $t_0(G, s, \epsilon) > 0$ such that for all integers $t \geq t_0$*

$$\text{gn}(G, t) \geq \text{gn}(G, s) - \epsilon.$$

Proof. To prove the statement holds, it is sufficient to show that

$$\text{gn}(G, t) \geq \frac{\lfloor \log_s t \rfloor}{\log_s t} \text{gn}(G, s) \quad (3)$$

holds for all $t \geq s$ since the right hand side of (3) tends to $\text{gn}(G, s)$ as t increases.

Let $k = \lfloor \log_s t \rfloor$. On the set of all guessing strategies of (G, t) , we consider only strategies $\mathcal{F} = (f_v)_{v \in V(G)}$ such that f_v is a map $A_t^{\Gamma(v)} \rightarrow \{0, 1, \dots, s^k - 1\}$ for every $v \in V(G)$. The maps $f_v s$ are well defined as $t \geq s^k$. We have

$$\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, t, \mathcal{F})] \geq \mathbf{P}[a_v < s^k \text{ for all } v \in V(G)] \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s^k, \mathcal{F})].$$

Hence

$$\frac{t^{\text{gn}(G, t)}}{t^{|V(G)|}} \geq \left(\frac{s^k}{t}\right)^{|V(G)|} \frac{s^{k \text{gn}(G, s^k)}}{s^{k|V(G)|}}. \quad (4)$$

Rearrange (4) we have

$$\text{gn}(G, t) \geq \frac{k}{\log_s t} \text{gn}(G, s^k). \quad (5)$$

Lemma 3.1 shows that $\text{gn}(G, s^k) \geq \text{gn}(G, s)$ which together with (5) completes the proof of (3). \square

Theorem 3.3. *For any graph G , $\text{gn}(G) = \lim_{s \rightarrow \infty} \text{gn}(G, s)$ exists.*

Proof. The sequence $\{\max_{s \leq n} \text{gn}(G, s)\}_{n=\{2,3,\dots\}}$ is an increasing sequence, and by definition $\text{gn}(G, s) \leq |V(G)|$ for all s , therefore the limit $\lim_{n \rightarrow \infty} \max_{s \leq n} \text{gn}(G, s) =: \text{gn}(G)$ exists. We note that $\text{gn}(G, s) \leq \text{gn}(G)$ for all s . It is sufficient to show that $\text{gn}(G, s)$ converges to $\text{gn}(G)$ from below.

Given $\epsilon > 0$ there exists $s_0(\epsilon)$ such that $\text{gn}(G, s_0(\epsilon)) \geq \text{gn}(G) - \epsilon$ (by the definition of $\text{gn}(G)$). Lemma 3.2 proves that there exists $t_0(\epsilon)$ satisfies that for all $t \geq t_0(\epsilon)$, $\text{gn}(G, t) \geq \text{gn}(G, s_0(\epsilon)) - \epsilon$ which implies $\text{gn}(G, t) \geq \text{gn}(G) - 2\epsilon$ proving we have convergence. \square

Remark 3.4. *A consequence of Lemma 3.2 is that for any s the guessing number $\text{gn}(G, s)$ is a lower bound for $\text{gn}(G)$. By definition of guessing number $\text{gn}(G, s)$, we have*

$$\text{gn}(G, s) \geq |V(G)| + \log_s \mathbf{P}[\text{Win}(G, s, \mathcal{F})].$$

for any strategy \mathcal{F} on (G, s) . Therefore, any strategy on any alphabet size provides us a lowerbound for the asymptotic guessing number.

4 Lower bounds using the fractional clique cover

The remark in the previous section tells us that we can provide a lower bound for the asymptotic guessing number using any guessing strategy on any alphabet size. Christofides and Markström [6] used this fact to provide a simple lowerbound for guessing number $\text{gn}(G)$ by constructing a general guessing strategy for graphs G called the fractional clique cover strategy.

Given a graph G , we denote $K(G)$ for the set of all cliques in G , and denote $K(G, v)$ for the set of all cliques in G containing vertex v . A *fractional clique cover* of G is a weighting $w : K(G) \rightarrow [0, 1]$ such that

$$\sum_{k \in K(G, v)} w(k) \geq 1.$$

for all $v \in V(G)$. We denote $\kappa_f(G)$ for the minimum value of $\sum_{k \in K(G)} w(k)$ over all choices of fractional clique covers w . (Note for any graph G , we have the identity: $\kappa_f(G) = \chi_f(G^c)$ where $\chi_f(G^c)$ is the fractional chromatic number of the graph's complement.)

We say a fractional clique cover is *regular* if its weighting $w : K(G) \rightarrow [0, 1]$ satisfies

$$\sum_{k \in K(G, v)} w(k) = 1.$$

for all $v \in V(G)$.

We will only consider fractional clique cover which is regular from now on as it is more convenient for our purpose of constructing guessing game strategies. In fact, even though we only focus on this smaller class of fractional clique cover, we do not lose any information about the value of $\kappa_f(G)$ as it can be proved that

$$\kappa_f(G) = \min_{\substack{k \in K(G) \\ w \text{ regular}}} w(k)$$

To prove the above identity holds, we need to show that

$$\kappa_f(G) \leq \min_{\substack{k \in K(G) \\ w \text{ regular}}} w(k)$$

and

$$\kappa_f(G) \geq \min_{\substack{k \in K(G) \\ w \text{ regular}}} w(k)$$

The first inequality comes from the definition of $\kappa_f(G)$. To show the second inequality holds, we prove that given an optimal fractional clique cover w we can make it into a regular fractional cover w_r with a property that $\sum_{k \in K(G,v)} w(k) = \sum_{k \in K(G,v)} w_r(k)$. This is done by moving weights from larger cliques to smaller ones.

Let $k \in K(G, v)$ be a clique containing v . We denote $k' = k \setminus \{v\}$ be the subclique obtained by removing vertex v . If we reduce the weight $w(k)$ and increase the weight $w(k')$ by the same amount then the sum $\sum_{k \in K(G,v)} w(k)$ is reduced but all other sums remain constant. Hence, the result follows.

The result of Christofides and Markström states the following:

Theorem 4.1. *If G is an undirected graph then*

$$\text{gn}(G) \geq |V(G)| - \kappa_f(G).$$

and for some positive integer $s \geq 2$, there is a guessing strategy \mathcal{F} on (G, s) such that

$$|V(G)| - \kappa_f(G) = |V(G)| + \log_s \mathbf{P}[\text{Win}(G, s, \mathcal{F})].$$

Remark 4.2. *In [6] it was proved that the above lower bound is actually an equality for various families of undirected graphs including perfect graphs, odd cycles and complements of odd cycles. This leads to a conjecture that the inequality is actually an equality, but this conjecture was proved to be false in [3]. The counter-example is a graph on 10 vertices which contains a large amount of cliques of size 3.*

5 Triangle-free graphs with large guessing number

A natural question is that if we forbid the appearance of triangle in undirected graph, then is the fractional clique cover the best guessing strategy for our undirected graphs? In other words,

Conjecture 5.1. *If G is an undirected triangle-free graph then*

$$\text{gn}(G) = |V(G)| - \kappa_f(G).$$

A useful bound on $\kappa_f(G)$ which we will make use of is given by the following lemma.

Lemma 5.2. *For any undirected graph G*

$$\kappa_f(G) \geq \frac{|V(G)|}{\omega(G)},$$

where $\omega(G)$ is the number of vertices in a maximum clique in G .

Proof. Let w be an optimal regular fractional clique cover. Since $\sum_{k \in K(G,v)} w(k) = 1$ holds for all $v \in V(G)$, summing both sides over v gives us,

$$\sum_{k \in K(G)} w(k)|V(k)| = |V(G)|,$$

where $|V(k)|$ is the number of vertices in clique k . The result trivially follows from observing

$$\sum_{k \in K(G)} w(k)|V(k)| \leq \sum_{k \in K(G)} w(k)\omega(G) = \kappa_f(G)\omega(G). \quad \square$$

Corollary 5.3. *For triangle-free graph G , the $\kappa_f(G) \geq |V(G)|/2$.*

We will show in this section that there are triangle-free graphs satisfies that the asymptotic guessing $\text{gn}(G)$ of G is greater than $|V(G)|/2$ and combining with the Corollary 5.3 we will prove that the answer to the Conjecture 5.1 is negative. Before illustrating our results, we need to introduce the following definition:

Definition 5.4. Given graph $G = (V(G), E(G))$ of order n , we say a square matrix M of order n with entries selected from a finite field \mathbf{F}_q of q elements with rows and columns indexed by vertices $i \in V(G)$ represents G over the finite field \mathbf{F}_q if the diagonal entries of M are not 0 and the entry m_{ij} is not 0 if ij is an edge in $E(G)$.

Clearly from the above definition, we can see that such a matrix M represents a guessing strategy for our guessing game $(G, |\mathbf{F}_q|)$. Moreover, the guessing number provided by the guessing strategy M equals the order of M minus the rank of M over finite field \mathbf{F}_q , i.e.

$$\text{gn}(G, q, M) = n - \text{rk}_{\mathbf{F}_q}(M) \quad (6)$$

At this point, it is clear that in order to disprove Conjecture 5.1 we will construct a triangle-free graph that has a representation matrix M with $\text{rk}_{\mathbf{F}_q}(M) < |V(G)|/2$ over some finite field \mathbf{F}_q .

Definition 5.5. A Steiner system $S(t, k, n)$ is a family of k -element subsets of $\{1, 2, \dots, n\} =: [n]$ with the property that each t -element subset of $[n]$ contained in exactly one element of $S(t, k, n)$. Elements of $S(t, k, n)$ are called blocks, and elements of $[n]$ are referred to as points.

For more information about Steiner systems, and the particular system used here, we refer to [4, Chapter 1]. The following proposition plays a crucial role in our construction:

Proposition 5.6. The Steiner system $S(3, 6, 22)$ has the following properties:

- (a) $S(3, 6, 22)$ contains 77 blocks.
- (b) Any two blocks in $S(3, 6, 22)$ intersect in zero or two points.
- (c) No three blocks in $S(3, 6, 22)$ are disjoint.

Proof. (a) We simply count the number of blocks containing a fixed set of points. Fix two points i, j in $[n]$, there are 20 choices of the third point $k \in [n] \setminus \{i, j\}$ to form a group of 3-points. By definition each 3-points of $[n]$ belongs to exactly one block, hence there are at most 20 blocks containing both two fixed points i and j .

Let x and y be two blocks containing both i and j . We have $x \cap y = \{i, j\}$ and there are 4 other points in x which is not i or j , so there are actual $20/4 = 5$ blocks that contain both i and j .

Now we fix one point i in $[n]$. There are 21 pairs of $[n]$ containing i and if x is a block that contains i then it also contains 5 pairs of $[n]$ containing i , hence each point i of $[n]$ belongs to $21 * 5/5 = 21$ blocks.

We repeat our argument for zero point of $[n]$ and we derive that there are $22 * 21/6 = 77$ blocks of $S(3, 6, 22)$. Moreover, any two blocks x and y in $S(3, 6, 22)$ intersect either in two points or an empty set.

(b) We see in the first part that each point in $S(3, 6, 22)$ belongs to 21 different blocks. If we fix a point p , then there are 21 points $q \neq p$, and these points belong to 21 blocks that containing p . These 21 blocks on 21 points form a projective plane where each block is a line in this plane. Hence any two blocks that contain a fixed point p must also contain another point q . This proves that any two blocks in $S(3, 6, 22)$ either intersect in zero or two points.

(c) We fix a block $B \in S(3, 6, 22)$. There are 16 points that are not in B , and there are 16 blocks that have empty intersection with B . From (b) we know that any two blocks must intersect in zero or two points, this makes the 16 points and 16 blocks a symmetric balanced incomplete block design (BIBD) $(16, 6, 2)$. It follows from the property of symmetric BIBD that any two blocks intersect in 2 points. \square

Theorem 5.7. There exists an explicit family \mathcal{G} of undirected triangle-free graphs G_i on N_i vertices where $\text{gn}(G_i) \geq \frac{77}{100}N_i$.

Proof. It suffices to prove the statement for case $N = 100$ and denote such a graph G . We can take elements G_k in the family of \mathcal{G} of triangle-free graphs to be $G^{\otimes k}$ which are the tensor product of G with itself k times for $k \geq 1$.

We define the vertex set of the graph G to be 22 points plus 77 blocks of the Steiner system $S(3, 6, 22)$ plus an extra point $\{\infty\}$. We define an edge between two vertices u and v if one of the following conditions is satisfied:

- u is $\{\infty\}$, and v is a point.
- u is a point and v is a block which contains u as an element.
- u and v are blocks of $S(3, 6, 22)$ and $u \cap v = \emptyset$.

According to the previous proposition, the graph obtained from our construction is triangle-free. It remains to show that there is a matrix representing G with rank less than 50 over some finite field \mathbf{F}_q .

The chosen matrix is $A + I$ where A is the adjacency matrix of G and I is the identity matrix of order 100. We have the rank of the matrix $A + I$ is 23 over finite field \mathbf{F}_3 (see the next lemma).

In this graph, the size of the maximal independent set is at least 22, hence the guessing number of this graph is at most 78. \square

Proposition 5.8. *If A is the adjacency matrix of the Higman–Sims graph, then the rank of $A + I$ over \mathbf{F}_3 is 23.*

Proof. Let r_v be the row of $B = A + I$ corresponding to vertex v . We write the vertex set as $\{\infty\} \cup X \cup Y$, where X and Y are the neighbours and non-neighbours of ∞ . Consider the 22 vectors r_x for $x \in X$. Since the graph is triangle-free, the restriction of r_x to the coordinates in X has a one in position x and zeros elsewhere; so these 22 vectors are linearly independent. Take the 23rd vector to be the all-1 vector j . Note that j is not in the span of the first 22. For if it were, it would have to be their sum (looking at the restriction to X). But the sum of the r_x has coordinate $22 \equiv 1 \pmod{3}$ at ∞ , 1 at each point of X , and $6 \equiv 0 \pmod{3}$ at each point of Y ; that is, it is r_∞ . So our 23 vectors are linearly independent. Also, they are all contained in the row space of B . (This is clear for the r_x ; also the sum of all the vectors r_v is $2j$, since all column sums of B are $23 \equiv 2 \pmod{3}$, so j is also in the row space.)

We claim that they span the row space. It is clear that their span contains all r_x for $x \in X$, and we just showed that it contains r_∞ . Take a vertex $y \in Y$. Consider the sum of the vectors r_x for the 16 vertices $x \in X$ which are not joined to y . This has coordinate $16 \equiv 1 \pmod{3}$ at ∞ , 0 at points of X joined to y , and 1 at points of X not joined to y . The coordinate at y is zero. If y' is joined to y , then the six neighbours of y' in X are a subset of the 16 points not joined to y , so the coefficient at y' is $6 \equiv 0 \pmod{3}$. If y' is not joined to y , then y' is joined to two neighbours of y in X and to four non-neighbours, so the coefficient at y' is $4 \equiv 1 \pmod{3}$. Thus the sum of our sixteen vectors is $j - r_y$, showing that r_y lies in the span of our 23 chosen vectors.

Notice incidentally that $B^2 = 2B$, so that the minimum polynomial of B is the product of distinct linear factors, so B is diagonalisable. \square

The constructed graph is in fact the Higman–Sims graph [10], which is a strongly regular triangle-free graph with parameters $(100, 22, 6)$. The Higman–Sims graph was first introduced by Dale Mesner in his 1956 PhD thesis [13]; see [12] for a historical account.

We also found other triangle-free graphs which have guessing number larger than the lower bound given by fractional clique cover. See [4, Chapter 8] for further details about these graphs. The arguments for the ranks of various matrices are similar to those for the Higman–Sims graph above – the eigenvalues of all these graphs are tabulated on page 109 of [4].

Proposition 5.9. *The following triangle-free graphs on n vertices have their guessing number larger than $n/2$:*

- (a) *The Clebsch graph on 16 vertices has $10 \leq \text{gn}(G) \leq 11$.*
- (b) *The Hoffman–Singleton graph on 50 vertices has $29 \leq \text{gn}(G) \leq 35$.*
- (c) *The Gewirtz graph on 56 vertices $36 \leq \text{gn}(G) \leq 40$.*
- (d) *The M_{22} graph on 77 vertices $54 \leq \text{gn}(G)$.*
- (e) *The Higman–Sims graph on 100 vertices has $77 \leq \text{gn}(G) \leq 78$.*

Proof. The upperbound for the guessing number derived for these graphs is $\text{gn}(G) \leq |V(G)| - \alpha(G)$ where $\alpha(G)$ is the independent number [15]. The independent numbers of the Clebsch graph, the Hoffman–Singleton graph, and the Gewirtz graph are 5, 15, and 16, respectively [19].

(a) The Clebsch graph is a triangle-free strongly regular graph with parameters $(16, 5, 2)$ which is constructed as follows: We start with a finite set $S = \{1, 2, 3, 4, 5\}$. The set V contains all subsets of size 1, and 2 of S , and V also contains an extra single point set $\{*\}$. We form the $(16, 5, 2)$ graph with vertex set V and an edge between two vertices u and v if one of the following conditions is satisfied:

- u is $\{*\}$, and v is a subset of S with cardinality 1.
- u is a subset of S with cardinality 1, v is a subset of S with cardinality 2, and u is a subset of v .
- u and v are subsets of S with cardinality 2, and u intersect v is empty.

We have the rank of the matrix $A + I$ is 6 over finite field \mathbf{F}_2 .

(b) The Hoffman–Singleton graph which is triangle-free strongly regular with parameters $(50, 7, 1)$ has one way of construction as follows: We take five 5-cycles C_h and their complements C_h^c , and we join vertex j of C_h to vertex $hi + j \pmod{5}$ of C_h^c . This construction is due to Conway. The rank of the matrix $A + 3I$ over finite field \mathbf{F}_5 is 21.

(c) The Gewirtz graph with parameters $(56, 10, 2)$ can be constructed from the $S(3, 6, 22)$ by fixing an element and let the vertices be the 56 blocks not containing that element. Two vertices are adjacent if the intersection of their corresponding blocks is empty. The rank of the matrix $A + I$ over finite field \mathbf{F}_3 is 20.

(d) The triangle-free strongly regular graph $M22$ with parameters $(77, 16, 4)$ which can be constructed by let the 77 blocks of $S(3, 6, 22)$ be the vertices of the graph, and an edge uv between two vertices u and v if u and v are disjoint as blocks. The rank of $A + I$ over finite field \mathbf{F}_3 is 23.

(e) Theorem 5.7. □

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