Separably injective C*-algebras

Cho-Ho Chu and Lei Li

Abstract. We show that a C*-algebra is a 1-separably injective Banach space if, and only if, it is linearly isometric to the Banach space $C_0(\Omega)$ of complex continuous functions vanishing at infinity on a substonean locally compact Hausdorff space Ω.

Mathematics Subject Classification (2010). Primary 46B20, 54G05, 17C65.

Keywords. Separably injective Banach space. C*-algebra. Substonean space. F-space.

1. Introduction

It is well-known that there are few examples of 1-injective Banach spaces. These are Banach spaces V for which every continuous linear map $T : Y \to V$ on a Banach space Y admits a norm preserving extension to a super space $Z \supset Y$, equivalently, contractive linear maps $T : Y \to V$ extend to contractive ones on Z. Indeed, a Banach space V is 1-injective if and only if it is linearly isometric to the continuous function space $C(\Omega)$ on some stonean space Ω [5, 9]. If the 1-injectivity condition is relaxed to requiring that each continuous linear map $T : Y \to V$ extends to a continuous one on $Z \supset Y$, then it is unclear what constitutes the larger class of Banach spaces, called λ-*injective* or $\mathcal{P}_\lambda$ *spaces*, satisfying this condition. However, one can consider the class of λ-*separably injective* Banach spaces to which only continuous linear maps on *separable* spaces are extendable to *separable* super spaces. Of particular interest is the subclass of 1-*separably injective* Banach spaces to which *contractive* linear maps on separable spaces admit *contractive* extension on separable super spaces. While c_0 is the only λ-separably injective space among infinite dimensional *separable* Banach spaces [14], it has been shown recently in [1] that among *nonseparable* real Banach spaces, there are indeed many interesting examples of λ-separably injective spaces. In particular, the Banach space $C(\Omega, \mathbb{R})$ of real continuous functions on a compact Hausdorff space Ω is 1-separably injective if, and only if, Ω is an F-space. It is natural

The second author is supported by NSF(China) research grants No.11301285 and No.11371201.

to ask if this result also holds for the space $C_0(S)$ of continuous functions vanishing at infinity on a locally compact space S. This case has not been discussed in [1] and in fact, the example of c_0 provides a negative partial answer since $\mathbb{N}$ is an F-space, but c_0 is not 1-separably injective although it is 2-spearably injective.

In this paper, we give a complete answer to the above question and prove, more generally, that a C*-algebra is 1-separably injective if, and only if, it is linearly isometric to the Banach space $C_0(S)$ of complex continuous functions vanishing at infinity on a *substonean* locally compact Hausdorff space S. Particularly, abelian monotone sequentially complete C*-algebras are 1-separably injective. This example may be of interest as the class of monotone complete C*-algebras is closely related to generic dynamics [13].

2. Separably injective Banach spaces

The concept of a separably injective *real* Banach space, considered in [1], can be extended naturally to that for a complex Banach space.

Definition 2.1. A complex (resp. real) Banach space V is said to be 1-*separably injective* if for every complex (resp. real) *separable* Banach space Z and every closed subspace $Y \subset Z$, every bounded linear operator $T : Y \to V$ extends to a bounded linear operator $\widetilde{T} : Z \to V$ with $\|\widetilde{T}\| = \|T\|$.

Given a locally compact Hausdoff space Ω, we will denote by $C_0(\Omega)$ the abelian C*-algebra of complex continuous functions on Ω vanishing at infinity. If Ω is compact, then we omit the subscript 0 and denote by $C(\Omega, \mathbb{R})$ the Banach space of real continuous functions on Ω.

Definition 2.2. Let Ω be a locally compact Hausdorff space. It is called an *F-space* if for each real continuous function f on Ω, there is a real continuous function k on Ω such that $f = k|f|$ (cf. [3, 14.25]). Following [4], we call Ω *substonean* if any two disjoint open σ-compact subsets of Ω have disjoint compact closures.

The compact substonean spaces are exactly the compact *F-spaces*. However, infinite discrete spaces are F-spaces without being substonean. We refer to [8, Example 5] for an example of a substonean space which is not an F-space.

Example 2.3. Let Ω be a compact Hausdorff space. Using the results in [6, 11], it has been shown in [1, Proposition 4.2] that the real continuous function space $C(\Omega, \mathbb{R})$ is 1-separably injective if, and only if, Ω is an F-space. This result remains true if we replace $C(\Omega, \mathbb{R})$ by the complex continuous function space $C(\Omega)$. Indeed, if $C(\Omega, \mathbb{R})$ is 1-separably injective and given a contractive complex linear operator $T : Y \to C(\Omega)$, where Y is a closed subspace of a separable complex Banach space Z, the real part $\operatorname{Re} T : y \in Y_r \mapsto \operatorname{Re} T(y) \in$

$C(\Omega, \mathbb{R})$ extends to a real linear contraction $T_r : Z_r \to C(\Omega, \mathbb{R})$ which, as in the proof of [5, Theorem 2], gives a complex linear contraction

$$z \in Z \mapsto T_r(z) - iT_r(iz) \in C(\Omega)$$

extending T since $T(y) = \mathrm{Re}\, T(y) - i\mathrm{Re}\, T(iy)$ for $y \in Y$. Hence $C(\Omega)$ is 1-separably injective. Conversely, if $C(\Omega)$ is 1-separably injective, it will follow from Theorem 3.5 that $C(\Omega, \mathbb{R})$ is 1-separably injective.

However, as noted earlier, the above result is not valid for the space $C_0(S)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space S. Separable injectivity of $C_0(S)$ has not been considered in [1]. A topological criterion for 1-separable injectivity of $C_0(S)$ follows from Theorem 3.5.

Let V be a complex 1-separably injective Banach space and let $T : Y \to V$ be a bounded linear operator on a closed subspace Y of a complex Banach space X. The arguments in the proof of [1, Proposition 3.5 (a)] for *real* 1-separably injective spaces can be extended to the complex case and one can show that T has a norm preserving extension $\widetilde{T} : X \to V^{**}$. A further application of the arguments in the proof of [10, Theorem 2.1, (9)⇒(1)], which are also valid for complex spaces, gives the following result. The result for real 1-separably injective spaces has been shown in [1].

Lemma 2.4. *Let V be a 1-separably injective complex Banach space. Then the bidual V^{**} is 1-injective.*

3. Separably injective C*-algebras

We characterize 1-separably injective C*-algebras in this section. Let us begin with a simple lemma.

Lemma 3.1. *A 1-separably injective C*-algebra is abelian.*

Proof. Let A be a 1-separably injective C*-algebra. By Lemma 2.4, the bidual A^{**} is 1-injective and hence linearly isometric to a continuous function space $C(\Omega)$ on some stonean space Ω [5]. The linear isometry between the C*-algebras A^{**} and $C(\Omega)$ preserves the Jordan triple product

$$\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a) \qquad (a, b, c \in A^{**})$$

by a well-known result of Kadison [7] (see also [2, Theorem 3.1.7]). Since $C(\Omega)$ is an abelian algebra, we must have, via the isometry between $C(\Omega)$ and A^{**},

$$\{a, b, \{c, \mathbf{1}, \mathbf{1}\}\} = \{a, \{b, c, \mathbf{1}\}, \mathbf{1}\}$$

for $a, b, c \in A^{**}$, where $\mathbf{1}$ denotes the identity in A^{**}. Let p be a projection in A^{**} and $a \in A^{**}$. A simple computation gives

$$\begin{aligned} \frac{1}{2}(pa + ap) &= \{p, p, \{a, \mathbf{1}, \mathbf{1}\}\} = \{p, \{p, a, \mathbf{1}\}, \mathbf{1}\} \\ &= \frac{1}{4}(pa + ap + 2pap) \end{aligned}$$

and $pa + ap = 2pap$, which implies $pa = ap$. Hence A^{**} is abelian since it is generated by projections. In particular, A itself is abelian. □

We have the following result readily.

Proposition 3.2. *Let A be a von Neumann algebra. The following conditions are equivalent.*

(i) *A is 1-separably injective.*
(ii) *A is 1-injective.*

Proof. (i) ⇒ (ii). By the above observation, the unital algebra A is abelian and hence linearly isometric to a continuous function space $C(\Omega)$ on some compact Hausdorff space Ω. Since A has a predual, Ω must be hyperstonean and therefore $C(\Omega)$ is 1-injective by [5].

□

Remark 3.3. The above proposition is false for unital C*-algebras. Indeed, let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$. Then $\beta\mathbb{N}\backslash\mathbb{N}$ is a compact F-space by [3, p. 210]. Hence the C*-algebra $C(\beta\mathbb{N}\backslash\mathbb{N})$ is 1-separably injective (cf. Example 2.3), but not 1-injective since $\beta\mathbb{N}\backslash\mathbb{N}$ is not stonean [3, p. 98].

We now determine the class of 1-separably injective C*-algebras. A useful fact noted in [4, Proposition 1.1] is that a locally compact Hausdorff space S is substonean if, and only if, the following condition holds: given f and g in $C_0(S)$ satisfying $fg = 0$, there are functions $f_1, g_1 \in C_0(S)$ such that $f_1 g_1 = 0$, $f_1 f = f$ and $g_1 g = g$. We will need the following definition introduced in [8].

Definition 3.4. A nonempty subset S_0 of a topological space S is called a *P-set* if it is closed and any G_δ-set containing S_0 is a neighborhood of S_0. A point $p \in S$ is called a *P-point* if $\{p\}$ is P-set in S.

It has been remarked in [8] that a nonempty subspace S_0 of S is a P-set if, and only if, each real continuous function f on S, vanishing on S_0, must vanish on a neighborhood of S_0. If S is a locally compact and noncompact space, then S is substonean if, and only if, the one-point compactification $S \cup \{\infty\}$ is an F-space and ∞ is a P-point in $S \cup \{\infty\}$ (cf. [8, Theorem 1]).

Theorem 3.5. *Let A be a C*-algebra. The following conditions are equivalent.*

(i) *A is 1-separably injective.*
(ii) *A is linearly isometric to the Banach space $C_0(S)$ of complex continuous functions vanishing at infinity on a substonean locally compact Hausdorff space S.*

Proof. (i) ⇒ (ii). By Lemma 3.1, A is abelian and hence linearly isometric to the function space $C_0(S)$ on some locally compact Hausdorff space S. We show that S is substonean.

Given any function $f \in C_0(S)$, we define the *cozero set* of f to be the set $coz(f) = \{x \in S : f(x) \neq 0\}$. Let U and V be two disjoint open σ-compact sets in S. We show that they have disjoint compact closures. It has been observed in [4, p.125] that one can find, via Urysohn's lemma, two functions $f, g \in C_0(S)$ with $0 \le f, g \le 1$ such that $U = coz(f)$ and $V = coz(g)$. We note that $fg = 0$ in this case.

Let $h = \chi_{coz(f)}$ be the characteristic function of $coz(f)$. Let Y be the closed linear span of $\{f^{1/n}, g^{1/n} : n = 1, 2, \ldots\}$ in $C_0(S)$, and let $Z = Y + \mathbb{C}\, h \subset \ell^\infty(S)$. Since $C_0(S)$ is 1-separably injective, the identity map $\iota : Y \to C_0(S)$ admits a norm preserving extension $\tilde{\iota} : Z \to C_0(S)$. Write $k = \tilde{\iota}(h) \in C_0(S)$ and note that $\|k\| \le 1$.

For each $n \in \mathbb{N}$, we have $\|h - 2f^{\frac{1}{n}}\| \le 1$ and therefore $\|k - 2f^{\frac{1}{n}}\| = \|\tilde{\iota}(h) - 2\tilde{\iota}(f^{\frac{1}{n}})\| \le 1$. In particular, $|k(x) - 2f^{\frac{1}{n}}(x)| \le 1$ for all $n \in \mathbb{N}$ which, together with $|k(x)| \le 1$, implies $k(x) = 1$ for $x \in coz(f)$. It follows that $kf = f$.

Since $hg = 0$, we have $\|h + e^{i\theta} g^{\frac{1}{n}}\| \le 1$ for every $\theta \in [0, 2\pi)$ which gives $\|k + e^{i\theta} g^{\frac{1}{n}}\| = \|\tilde{\iota}(h) + \tilde{\iota}(e^{i\theta} g^{\frac{1}{n}})\| \le 1$. Hence for each $x \in coz(g)$, one has $|k(x) + e^{i\theta} g^{\frac{1}{n}}(x)| \le 1$ for all $n \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. This implies $k(x) = 0$ for $x \in coz(g)$ and therefore $kg = 0$.

We have $coz(f) \subset \{x \in S : k(x) = 1\}$ and $coz(g) \subset \{x \in S : k(x) = 0\}$, where $\{x \in S : k(x) = 1\}$ is compact and contained in the open set $\{x \in S : |k(x)| > \frac{3}{4}\}$. Applying Urysohn's lemma to the compact set $\overline{coz(f)} \subset \{x \in S : k(x) = 1\}$, one can find a function $f_1 \in C_0(S)$ such that $0 \le f_1 \le 1$, $f_1 = 1$ on $\overline{coz(f)}$ and $f_1 = 0$ outside $\{x \in S : |k(x)| > \frac{3}{4}\}$.

Considering the function $h' = \chi_{coz(g)}$ with similar arguments, we can find another function $k' \in C_0(S)$ with $\|k'\| \le 1$ such that $coz(g) \subset \{x \in S : k'(x) = 1\}$ and hence $\overline{coz(g)}$ is a compact subset of S. Since $\{x \in S : |k(x)| < \frac{1}{4}\}$ is an open subset of S containing $\overline{coz(g)}$, Urysohn's lemma again yields a function $g_1 \in C_0(S)$ such that $0 \le g_1 \le 1$, $g_1 = 1$ on $\overline{coz(g)}$ and $g_1 = 0$ outside $\{x \in S : |k(x)| < \frac{1}{4}\}$. It follows that $f_1 g_1 = 0$ and $f_1 f = f, g_1 g = g$.

It follows that the closures $\overline{U} = \overline{coz(f)} \subset \{x \in S : f_1(x) \ge 1/2\}$ and $\overline{V} = \overline{coz(g)} \subset \{x \in S : g_1(x) \ge 1/2\}$ are compact and disjoint.

(ii) $\Rightarrow$ (i). Let A be linearly isometric to $C_0(S)$ where S is a substonean locally compact Hausdorff space. We show that $C_0(S)$ is separably injective. This is true if S is compact, as shown in Example 2.3. Let S be noncompact. Since the one-point compactification $S \cup \{\infty\}$ is an F-space, $C(S \cup \{\infty\})$ is 1-separably injective, again by Example 2.3.

Let Y be a closed subspace of a separable Banach space Z and let $T : Y \to C_0(S)$ be a bounded linear operator. We identify $C_0(S)$ with the closed subspace $\{u \in C(S \cup \{\infty\}) : u(\infty) = 0\}$ of $C(S \cup \{\infty\})$. Let $\widetilde{T} : Z \to C(S \cup \{\infty\})$ be a norm preserving extension of T.

Let $\{f_n : n \in \mathbb{N}\}$ be a countable dense subset of $T(Y) \subset C_0(S)$. Since S is substonean, ∞ is a P-point and there exists an open neighbourhood U_n of ∞ such that $f_n = 0$ on U_n.

Moreover, the G_δ-set $\bigcap_n U_n$ contains an open neighbourhood U of ∞. Hence Urysohn's lemma again enables us to choose a function $e \in C_0(S)$ with

$\|e\| = 1$ such that $e = 1$ on $S \backslash U$ and $e(\infty) = 0$. This gives $f_n e = f_n$ for all $n \in \mathbb{N}$.

It can now be seen readily that the linear map $T_e : Z \to C_0(S)$ defined by

$$T_e(z) = \widetilde{T}(z)e \qquad (z \in Z).$$

is a norm preserving extension of T. □

We conclude by mentioning some interesting examples of substonean spaces. For any locally compact, σ-compact Hausdorff space S with Stone-Čech compactification βS, the space $\beta S \backslash S$ is a compact F-space [3, 14.27]. Locally compact substonean spaces include the *Rickart spaces* which are exactly those locally compact spaces S for which $C_0(S)$ is monotone sequentially complete [4]. In particular, abelian monotone sequentially complete C*-algebras are 1-separably injective.

References

[1] A. Avilés, F. Cabello Sánchez, J. M.F. Castillo, M. González and Y. Moreno, *On separably injective Banach spaces*, Adv. Math. **234** (2013) 192–216.

[2] C-H. Chu, *Jordan structures in geometry and analysis*, Cambridge Tracts in Math. **190**, Cambridge Univ. Press, Cambridge, 2012.

[3] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.

[4] K. Grove and G.K. Pedersen, *Sub-Stonean spaces and corona sets*, J. Funct. Anal., **56** (1984), 124–143.

[5] M. Hasumi, *The extension property of complex Banach spaces*, Tohoku Mathematical Journal, **10**(2) (1958), 135–142.

[6] M. Henriksen, *Some remarks on a paper by Aronszajn and Panichpakdi*, Pacific J. Math. **7** (1957) 1619-1621.

[7] R.V. Kadison, *Isometries of operator algebras*, Ann. of Math. **54** (1951) 325-338.

[8] M. Henriksen and R. G. Woods, *F-spaces and substonean spaces: general topology as a tool in functional analysis*, Papers on general topology and related category theory and topological algebra (New York, 1985/1987), 60–68, Ann. New York Acad. Sci., 552, New York Acad. Sci., New York, 1989.

[9] J.L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. **72** (1952) 323-326.

[10] J. Lindenstrauss, *Extension of compact operators*, Memoir Amer. Math. Soc. **48**, Amer. Math. Soc. Providence, 1964.

[11] J. Lindenstrauss, *On the extension of operators with range in a $C(K)$ space*, Proc. Amer. Math. Soc. **15** (1964) 218-225.

[12] G.L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. **133** (1968) 267-280.

[13] K. Saito and J.D. Maitland Wright, *Monotone complete C*-algebras and generic dynamics*, Springer-Verlag, Berlin, 2015.

[14] M. Zippin, *The separable extension problem*, Israel J. Math. **26** (1977) 372-387.

Cho-Ho Chu
School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, UK
e-mail: c.chu@qmul.ac.uk

Lei Li
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China
e-mail: leilee@nankai.edu.cn