

New Modular Hopf Algebras related to rational $k \widehat{sl}(2)$

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We show that the Hopf link invariants for an appropriate set of finite dimensional representations of $U_q SL(2)$ are identical, up to overall normalisation, to the modular S matrix of Kac and Wakimoto for rational $k \widehat{sl}(2)$ representations. We use this observation to construct new modular Hopf algebras, for any root of unity $q = e^{-i\pi m/r}$, obtained by taking appropriate quotients of $U_q SL(2)$, that give rise to 3-manifold invariants according to the approach of Reshetikin and Turaev. The phase factor correcting for the ‘framing anomaly’ in these invariants is equal to $e^{-\frac{i\pi}{4}(\frac{3k}{k+2})}$, an analytic continuation of the anomaly at integer k . As expected, the Verlinde formula gives fusion rule multiplicities in agreement with the modular Hopf algebras. This leads to a proposal, for $(k+2) = r/m$ rational with an odd denominator, for a set of $\widehat{sl}(2)$ representations obtained by dropping some of the highest weight representations in the Kac-Wakimoto set and replacing them with lowest weight representations. For this set of representations the Verlinde formula gives non-negative integer fusion rule multiplicities. We discuss the consistency of the truncation to highest and lowest weight representations in conformal field theory.

1. Introduction

Integer k $SU(2)$ WZW models of CFT have led to the discovery of some intriguing relations between $\widehat{sl(2)}$ and $U_qSL(2)$. One notable relation is the fact that the modular S matrix for the characters of the integrable representations is proportional to the Hopf link invariant computed from knot theory, when we make a certain map between representations of the two algebras. Further, the 6j symbols of the quantum group are related to the braiding matrices in WZW conformal field theory[1][2]. Since Kac and Wakimoto [3] found a finite set of representations of $\widehat{sl(2)}$ that close under modular transformations, it has been expected that there might be a conformal field theory based on these representations. Some properties of the characters and modular S matrix have been studied in [4] [5] [6]. How far do the connections between affine algebra and quantum group go through at rational k ? What do they tell us about an eventual CFT at rational k and how much of the connections between 2D CFTs and 3D gauge theories goes through at rational k ?

We find that it is possible to define a map between quantum group representations and affine algebra representations such that the relation between modular S matrix and Hopf link invariant is preserved. We find that new quotients of $U_qSL(2)$ can be defined, which are modular Hopf algebras. These modular Hopf algebras allow the construction of 3-manifold invariants which are characterised by a framing anomaly which indicates that they might be related to a conformal field theory with $sl(2)$ current algebra symmetry. The framing anomaly also has a well defined large k limit, which suggests that they might be related to a 3D gauge theory. We show that the Verlinde formula gives fusion rule multiplicities in agreement with the tensor products of the quantum group.

For odd m , $k + 2 = r/m$, we can select out a set of highest weight representations of $\widehat{sl(2)}$ and their charge conjugates, whose characters form a representation of the modular group, so that the Verlinde formula gives non negative integer fusion rule multiplicities.

Whereas the classical tensor products of highest weight with lowest weight representations of $sl(2)$ can contain irreducible continuous series type of representations, we show that the quantum fusion of $\widehat{sl(2)}$ modules, one of which contains a certain null vector, excludes the irreducible C-series. For $m = 3$, any r , this null vector condition suffices to show that the entire set of highest and lowest weight representations proposed decouples from the irreducible continuous series.

2. Definitions and notations

In most of the considerations in this paper we will be concerned with fusion rules, which follow from the null vector structure of the modules. We will not be too concerned with unitarity and our discussion of decoupling in section 10 will apply whether one uses commutation relations in the $\widehat{su(2)}$ form or the $\widehat{su(1,1)}$ form.

2.1. Definitions for $\widehat{sl(2)}$

The commutation relations of $\widehat{su(2)}$ are the following

$$\begin{aligned}
 [J_m^0, J_n^\pm] &= \pm J_{m+n}^\pm \\
 [J_m^0, J_n^0] &= \frac{mk}{2} \delta_{m,-n} \\
 [J_m^+, J_n^-] &= 2J_{m+n}^0 + mk\delta_{m,-n} \\
 [k, J_n^a] &= 0,
 \end{aligned} \tag{2.1}$$

where a takes values $+, -, 0$. For unitary $\widehat{su(2)}$ modules there is an inner product satisfying the condition $(J_n^+)^\dagger = J_{-n}^-$, $(J_n^0)^\dagger = J_{-n}^0$. The commutation relations of $\widehat{su(1,1)}$ are [7]

$$\begin{aligned}
 [J_m^0, J_n^\pm] &= \pm J_{m+n}^\pm \\
 [J_m^0, J_n^0] &= -\frac{mk}{2} \delta_{m,-n} \\
 [J_m^+, J_n^-] &= -2J_{m+n}^0 + mk\delta_{m,-n}, \\
 [k, J_n^a] &= 0,
 \end{aligned} \tag{2.2}$$

Unitary modules have an inner product consistent with $(J_n^+)^\dagger = J_{-n}^-$, $(J_n^0)^\dagger = J_{-n}^0$. Notice that taking k to $-k$, and J_n^+ to $-J_n^+$, converts the first set of commutation relations to the second. Therefore for every highest weight module of compact type, there is a map to a highest weight module of the DPL type. For every compact module that has a null vector this map gives us a DPL module that has a null vector. And a formula for a null vector in the compact module can, by the simple substitution give a formula for a null vector in the DPL module. For example, $(J_{-1}^+)^R$ is a null vector in a highest weight module of compact type, at any k , when the highest weight j is given by $2j + 1 = -R + k + 2$, and when $(J_{-1}^+)^R|j >$ is a null vector in a DPL highest weight module with highest weight j given by $2j + 1 = -R - k + 2$. Another example is the following [8]. For any k , a highest

weight module of highest weight j given by $j = -t/2, t = k + 2$ has a null vector given by the following formula

$$\chi = [J_0^- J_{-1}^+ J_0^- - t(J_0^- J_{-1}^0 + J_0^- J_{-1}^0) - t^2 J_{-1}^-] |j\rangle. \quad (2.3)$$

The same expression, with $t = -k + 2$ and J_{-1}^+ replaced by $-J_{-1}^+$ gives a null vector in a DPL type module. This map does not necessarily give us a positive norm state if we start with a positive norm state, so if we require unitarity then these modules are very different. We will work with $\widehat{su(2)}$ notation in the rest of this paper and since we will not be dealing with issues of unitarity, our conclusions apply equally well if we had used $\widehat{su(1,1)}$ notation.

2.2. Definitions and conventions for $U_q sl(2)$

Recall the definition of $U_q SL(2)$, which we will also call U_q

$$\begin{aligned} [H, X_{\pm}] &= \pm X_{\pm} \\ [X_+, X_-] &= \frac{q^{2H} - q^{-2H}}{q - q^{-1}}. \end{aligned} \quad (2.4)$$

The coproduct Δ , antipode S and counit ϵ are :

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H \\ \Delta(X_{\pm}) &= X_{\pm} \otimes q^H + q^{-H} \otimes X_{\pm} \\ S(H) &= -H \\ S(X_+) &= -qX_+ \\ S(X_-) &= -q^{-1}X_- \\ \epsilon(H) &= 0 \\ \epsilon(X_{\pm}) &= 0. \end{aligned} \quad (2.5)$$

One could also choose for the coproduct $\Delta' = \sigma \circ \Delta$, where

$$\sigma(a \otimes b) = b \otimes a, \quad \forall a, b \in U_q$$

At generic q , the universal R matrix [9] satisfying $R\Delta = \Delta'R$ is given by

$$\begin{aligned} R &= q^{2H \otimes H} \sum_{n=0}^{\infty} q^{-1/2n(n-1)} \frac{(1 - q^{-2})}{[n]!} q^{n(H \otimes 1 - 1 \otimes H)} (X_+)^n \otimes (X_-)^n \\ [n] &= \frac{q^n - q^{-n}}{q - q^{-1}} \\ [0]! &= 1 \\ [n]! &= [n][n-1] \dots [1] \end{aligned} \quad (2.6)$$

At roots of unity, $q = e^{\frac{-i\pi m}{r}}$ we can define a quotient of $U_q sl(2)$, [10][11][12] by imposing the relations

$$(X_+)^r = 0, (X_-)^r = 0.$$

Then the sum in the R matrix truncates at $n = r$ [11]

$$R = q^{2H \otimes H} \sum_{n=0}^{r-1} q^{-1/2n(n-1)} \frac{(1 - q^{-2})}{[n]!} q^{n(H \otimes 1 - 1 \otimes H)} (X_+)^n \otimes (X_-)^n. \quad (2.7)$$

Notice that if we write K for q^H and K^{-1} for q^{-H} we can still write down the coproducts, antipode and counit, for the smaller algebra generated by

$$X_+, X_-, K, K^{-1},$$

instead of H, X_+, X_- . This algebra, called U_t , is discussed by [12]. If we call $e = q^H X_+$, $f = q^{-H} X_-$ we can also write these down in terms of

$$e, f, K^2, K^{-2}$$

[13]. For any two representations V_1 and V_2 of these smaller algebras, the operator $R \in \text{END}(V_1 \otimes V_2)$ intertwines the two coproducts. So with these generators we have all the structures of a quasitriangular Hopf algebra. We will find it necessary to consider other algebras smaller than that generated by H, X_+, X_- but larger than that generated by

$$e, f, K^2, K^{-2}.$$

Indeed we define for

$$q = e^{-i\pi m/r},$$

the algebra generated by $K^{2/m}, K^{-2/m}, e, f$ calling it $U[m]$, equipped with all the Hopf algebra structures inherited from U_q , for example

$$K^{2/m} e K^{-2/m} = q^{2H/m} q^H X_+ q^{2H/m} = q^2 q^H X_+ = qe. \quad (2.8)$$

2.3. reducibility of Verma modules of $\widehat{sl(2)}$ and $U_q Sl(2)$

We begin by drawing attention to a close relation between the existence of null vectors in $sl(2)$ affine algebra Verma modules and the reducibility of quantum group Verma

modules for any $k \in \mathfrak{R}$. A highest weight representation of $\widehat{sl(2)}$ with highest weight j_i has null vectors when j_i takes one of the values parametrized by two integers r_i and s_i

$$\begin{aligned} 2j_i + 1 = r_i - s_i(k + 2) \quad \text{with} \quad r_i \in \mathbb{N}; \quad s_i \in \mathbb{N} \cup \{0\} \\ \text{or} \quad -r_i \in \mathbb{N}; \quad -s_i \in \mathbb{N}. \end{aligned} \tag{2.9}$$

The null vector occurs at level $r_i s_i$ and has J_0^0 eigenvalue $j_i - r_i$. This is a consequence of the Kac-Kazhdan determinant formula [14]. For generic k a Verma module has one null vector. For k rational there are degenerate representations with an infinite number of null vectors.

Using the formula

$$[X_+, X_-^n] = [2H + n - 1][n]X_-^{n-1}, \tag{2.10}$$

which can be proved by induction, and writing $q = e^{-i\pi/\nu}$, $\nu \in \mathfrak{R}$ we find that a U_q highest weight module with highest weight j_i has a null vector with weight $j_i - r_i$ for

$$2j_i + 1 = r_i - s_i \nu, \text{ for } r_i \in Z_+, \text{ and } s_i \in Z. \tag{2.11}$$

The dimension of the module is r_i , and its quantum dimension $tr(q^{2H})$ is $e^{i\pi s_i}[r_i]$.

At generic ν there is one null vector in the Verma module with highest weight j_i of the form given above. Quotienting out the Verma module by the submodule generated by the null vector, we obtain a finite dimensional irreducible module of U_q . If we write ν as $(k + 2)$ this formula is identical to (2.9), except for the ranges of r_i and s_i . For s_i non-negative we can define a map from the finite dimensional irreducible quantum group module with highest weight j_i to the $\widehat{sl(2)}$ irreducible module with highest weight j_i (we are making here the natural identification $H \rightarrow J_0^0$). If the affine algebra module has a null vector specified by (r_i, s_i) with $r_i, s_i \in Z_-$ then we will associate with that highest weight representation an irreducible quantum group module of dimension $|r_i|$ and highest weight $j_i + |r_i|$. At rational k , we focus on highest weights that are in the Kac Wakimoto set [3]. They have the form

$$2j_i + 1 = r_i - s_i(r/m), \quad r_i = 1, 2 \cdots (r - 1), \quad s_i = 0, 1, 2 \cdots (m - 1). \tag{2.12}$$

j_i can also be written as $2j_i + 1 = (r_i - r) - (s_i - m)(r/m)$. The module has two generating null vectors and we could associate a quantum group module to it in two ways. Because of the symmetries of S , either of these maps will have the property that the Hopf link

invariant is proportional to the modular transformation matrix. For concreteness we will pick the map induced by the expression (2.12).

In the next section we will calculate the regular isotopy invariant of the Hopf link at rational k and show that the map we have defined above takes the Hopf link invariant to the modular S matrix of Kac and Wakimoto up to an overall factor of $\sqrt{(2/rm)}$. This generalises the correspondence which is well known for positive integer k and which was exploited in [15] to construct 3 manifold invariants from three dimensional Chern Simons field theory, which are closely related to modular functors defined in [1]. Subsequently, Reshetikin and Turaev gave a construction of 3D topological field theories in the sense of Atiyah [16] based on modular Hopf algebras.

3. Computation of the S matrix

We will compute the regular isotopy invariant of the Hopf link using the R matrix in the form written down by Keller [11] for example. The same argument can be carried through with the expression of [12].

$$S_{Hopf}(j_1, j_2) = (tr_{j_1} \otimes tr_{j_2})(q^{2H} \otimes q^{2H})(\check{R})^2 \quad (3.1)$$

Here j_1 and j_2 are highest weights of quantum group representations V_1 and V_2 .

$$2j_1 + 1 = r_1 - s_1(r/m).$$

$$2j_2 + 1 = r_2 - s_2(r/m).$$

\check{R} is equal PR where P is the permutation operator. It can be proved that as an operator acting on the space V_1 the

$$(1 \otimes tr)(1 \otimes q^{2H})(\check{R})^2 \quad (3.2)$$

is proportional to the identity. This is proved in [9] using Schur's lemma, the cyclicity of the trace and the fact that

$$\check{R}\Delta(a) = \Delta(a)\check{R}, \forall a \in U_qSL(2)$$

So we act on the state $|j_1 > \otimes |m_2 >$, which is annihilated by all but the first term in (2.7), to get

$$\begin{aligned}
& (1 \otimes tr)(1 \otimes q^{2H})PRPR_{j_1 m_2}^{j_1 m_2} |j_1 \rangle \otimes |m_2 \rangle \\
&= (1 \otimes tr)(1 \otimes q^{2H}) \sum_{n=0}^{r-1} R_{m_2 j_1}^{m_2+n j_1-n} R_{j_1 m_2}^{j_1 m_2} |j_1 - n \rangle \otimes |m_2 + n \rangle \\
&= \sum_{m_2=j_2-r_2+1}^{j_2} q^{2m_2} R_{m_2 j_1}^{m_2 j_1} R_{j_1 m_2}^{j_1 m_2} \\
&= \sum_{m_2=j_2-r_2+1}^{j_2} q^{2m_2} q^{4m_2 j_1} \\
&= e^{-\frac{2i\pi m}{r}(j_2+1-r_2)(2j_1+1)} \frac{(1 - e^{-\frac{2i\pi m}{r}r_2(2j_1+1)})}{(1 - e^{-\frac{2i\pi m}{r}(2j_1+1)})} \\
&= e^{i\pi(s_1 r_2 + s_2 r_1 - s_1 s_2 r/m)} \frac{\sin(\pi m r_1 r_2 / r)}{\sin(\pi m r_1 / r) e^{i\pi s_1}}.
\end{aligned} \tag{3.3}$$

The Hopf link invariant is then obtained by multiplying this with

$$tr_1(q^{2H}) = e^{i\pi s_1} \sin(\pi m r_1 / r)$$

to get

$$S_{Hopf}(j_1, j_2) = \frac{1}{\sin(\pi m / r)} e^{i\pi(s_1 r_2 + s_2 r_1 - s_1 s_2 r/m)} \sin(\pi m r_1 r_2 / r). \tag{3.4}$$

Now this is proportional to the Kac-Wakimoto modular S matrix, if we make the identifications described in the previous section

$$S_{Hopf}(j_1, j_2) = \frac{\sqrt{(rm/2)}}{\sin(\pi m / r)} S_{KW}(j_1, j_2). \tag{3.5}$$

For $m = 1$ this equation is well known.

4. Quantum group tensor products and the different algebras

We recall some facts about the tensor products of U_q discussed in detail in [11]. The quotient U_t is discussed in [12]. The complete set of highest weights j_i of irreducible representations of U_q with non zero q dimension is given by

$$2j_i + 1 = r_i - s_i r / m, \quad 1 \leq r_i \leq r - 1, \quad s_i \in Z.$$

The tensor products of these also contain representations of zero q -dimension, I_z^l with $z \in Z$ and $1 \leq l \leq r$. The representation with highest weight characterised by the pair

(r_i, s_i) will be denoted by $\langle r_i, s_i \rangle$. Keller proves that the representations of zero q dimension, $\{I_z^l | 1 \leq l \leq r, z \in Z\}$ generate an ideal I_z in the ring of representations R_z of U_q . This allows the definition of an *associative and commutative tensor product* $\hat{\otimes}$ on R_z/I_z . With this tensor product we have

$$\langle r_1, s_1 \rangle \hat{\otimes} \langle r_2, s_2 \rangle = \bigoplus_{r_3=|r_1-r_2|+1, +3, \dots}^{\min(r_1+r_2-1, 2r-r_1-r_2-1)} \langle r_3, s_1 + s_2 \rangle \quad (4.1)$$

This set of representations has a *distinguished identity element* $\langle 1, 0 \rangle$ satisfying

$$\langle r_i, s_i \rangle \hat{\otimes} \langle 1, 0 \rangle = \langle r_i, s_i \rangle . \quad (4.2)$$

For each representation V_i , also called $\langle r_i, s_i \rangle$, there is a unique $V_{\bar{i}}$ or $\langle r_{\bar{i}}, s_{\bar{i}} \rangle$ with $r_{\bar{i}} = r_i$ and $s_{\bar{i}} = -s_i$ with the property that the tensor product of the two contains the identity representation. $V_{\bar{i}}$ is called the dual of V_i . Note that each representation has a *unique dual*, and

$$V_{(\bar{i})} = V_i, \quad \text{for all } i.$$

One easily checks that the dual of a representation of U_q with highest weight j_i has lowest weight $-j_i$. The set of irreducible highest weight representations of non zero q dimension of the algebra H, X_+, X_- is infinite because s_i runs over all integers. We will define quotients of this algebra for which the set of non-isomorphic representations is finite.

The following property of the representations $\langle r_i, s_i \rangle$ allows us to deduce the properties of the other algebras we are going to deal with, U_t and $U[m]$ for $q = e^{-i\pi m/r}$. It is easy to prove [11]

$$\langle r_i, s_i \rangle \cong \langle r_i, 0 \rangle \otimes \langle 1, s_i \rangle \quad (4.3)$$

Now clearly $\langle r_i, 0 \rangle$ with distinct r_i are non isomorphic when regarded as representations of any of the algebras U_q, U_t or $U[m]$, because they have dimension r_i . The only state in $\langle 1, s_i \rangle$ is annihilated by X_+ and X_- , and $K^{2/m}$ has eigenvalue $e^{(\frac{i\pi s_i}{m})}$, while K has eigenvalue $e^{(\frac{i\pi s_i}{2})}$. This means that the residues of s_i modulo $(2m)$ in case of $U[m]$ and the residues mod (4) in the case of U_t suffice to determine the representations as representations of the respective algebras, up to isomorphism. We can thus deduce the tensor products of representations for these different algebras from (4.1). The s_3 on the right hand side gets replaced by $s_3 \pmod{(2m)}$ for $U[m]$. The correspondence with the description of the representations of U_t given in [12] is made by noting that their fourth root of unity α can be identified with $e^{i\pi s_1/2}$.

We see that for $U[m]$, U_t or U_q there is a commutative and associative tensor product, an identity representation and each representation has a unique dual. It is also possible to write down the isomorphisms between V_i and \check{V}_i with the properties required by the first two axioms of [12], and to follow through the proof of the third and fourth axioms, by using the above correspondence between U_t , U_q and $U[m]$.

5. Modular Hopf Algebra

We will now show that, for each (m, r) , we can pick a set of representations to make $U[m]$ a modular Hopf algebra as defined in [12]. Reshetikin and Turaev give a general construction of three manifold invariants and topological field theories in the sense of Atiyah [16] from any modular Hopf algebra (MHA). They also give examples of MHA based on U_t for any odd m , but in their modular Hopf algebras at odd m the framing anomaly does not have a well defined large k limit, except for $m = 1$. We will use the algebra $U[m]$ and our set of representations will be different from theirs, and will actually correspond (under the map we have described in section 1) to a set of representations closely related to the Kac-Wakimoto representations, a relation which will be discussed in more detail in the section 9. The main interest of our new examples of modular Hopf algebras is that the phase factor correcting for the framing anomaly will be $e^{-\frac{i\pi}{4} \frac{3k}{k+2}}$ just as for positive integer k . At large k , this becomes $e^{-\frac{i\pi}{4}(3)}$. With a well defined large k limit of the framing anomaly, it can be hoped that these invariants could be understood as being related to 3 dimensional Chern-Simons-Witten gauge theory. In the positive integer k case the large k behaviour of the framing anomaly was part of the evidence for the relation between the Reshetikin-Turaev construction and the Witten invariants [12], more detailed evidence is discussed in [17] and references therein. We will choose sets of quantum group representations closed under fusion and satisfying the genus zero axioms [1]. And then we will check some genus one equations which will prove that $U[m]$ is a modular Hopf algebra. We will prove

Theorem 1:

$$(S^2)_{ij} = \frac{rm}{2 \sin^2(\pi m/r)} \delta_{i\check{j}} \quad (5.1)$$

If we use the normalisation from affine algebra this is equivalent to :

$$(\tilde{S}^2)_{i,j} = \delta_{i\check{j}} \quad (5.2)$$

We will also prove

Theorem 2:

$$\text{If } \sum_k v_k S_{k,l} d_k = v_l^{-1} \dim_q(V_l) \quad (5.3a)$$

then

$$d_l = \sqrt{\frac{2}{rm}} e^{-\frac{i\pi}{4} \frac{3k}{(k+2)}} \sin(\pi m/r) \dim_q(V_l) \quad (5.3b)$$

which implies

$$C = \sum_i v_i^{-1} \dim_q(V_i) d_i = e^{-\frac{i\pi}{2} \frac{3k}{(k+2)}} \neq 0 \quad (5.3c).$$

The v_i that appears above is the value of an element v of U_q in the representation of highest weight j_i and is equal to $q^{-2j_i(j_i+1)}$ [2][12]. In the construction of 3-manifold invariants in [12] the d_l are weights which enter the weighted averages of framed link invariants, that are equal to 3-manifold invariants. The equation (5.3), together with the precise form of the averages [12], guarantees that these averages are invariant under Kirby moves. These moves relate different links for which surgery gives the same three manifold [18].

Using \tilde{S} the (5.3a) reads :

$$(\tilde{S})_{0l} = \sum_k T_0 \tilde{S}_{0k} T_k \tilde{S}_{kl} T_l, \quad (5.4)$$

a special case of $(\tilde{S}T)^3 = C$, C here being the charge conjugation matrix mapping a representation to its dual (distinct from the C in (5.3)). The modular transformation matrix on affine algebra characters $T_{jl} = T_j \delta_{jl} = e^{2\pi i [\frac{j(j+1)}{(k+2)} - \frac{c}{24}]}$, $c = 3k/(k+2)$. We note here the following properties of S ,

$$\text{in general } S_{ij} \neq S_{i\check{j}} \quad (5.5a)$$

$$\text{but } S_{i\check{j}} = S_{ij} = S_{ji} \quad (5.5b)$$

$$\text{and in particular } S_{0\check{j}} = S_{0j} = S_{j0} \quad (5.5c).$$

Note that from the quantum group point of view the part of the T matrix involving the classical casimir has a natural meaning but c is only determined after selecting an appropriate set of representations and solving for the weights.

We will discuss even m and odd m separately.

6. ODD m

We take as our set of representations those with

$$\begin{aligned} r_i &\in I(1) = \{1, 2, \dots, (r-1)\} \\ s_i &\in I(2) = \{0, \pm 2, \pm 4, \dots, \pm(m-1)\}. \end{aligned} \tag{6.1}$$

These are all non isomorphic and have non zero q dimension. The equation (4.1) and the moding by $2m$ discussed before implies that this set closes under the tensor product. The indecomposables have zero q dimension and we can define a tensor product which is commutative and associative. The first four axioms of [12] are satisfied. In the language of modular tensor categories described in [19] [1], these guarantee that the genus zero axioms are satisfied.

6.1. computation of S^2

We have for the square of the S matrix

$$\begin{aligned} (S^2)_{j_1 j_3} &= \frac{1}{\sin^2(\pi m/r)} \sum_{r_2, s_2} e^{i\pi[r_2(s_1+s_3)+r_2(s_1+s_3)]} e^{-i\pi[s_2 r(s_1+s_3)/m]} \\ &\quad \sin(\pi m r_1 r_2/r) \sin(\pi m r_2 r_3/r) \\ &= \frac{1}{\sin^2(\pi m/r)} \sum_{s_2} e^{-i\pi s_2 r(s_1+s_3)/m} \sum_{r_2} \sin(\pi m r_1 r_2/r) \sin(\pi m r_2 r_3/r). \end{aligned} \tag{6.2}$$

The s_2 sum can be written as

$$e^{2i\pi(m-1)r/m} \frac{[1 - e^{-2i\pi r(s_1+s_3)}]}{[1 - e^{-2i\pi r(s_1+s_3)/m}]}.$$

Note that the numerator is always zero whereas the denominator is only non zero if (s_1+s_3) is zero or a multiple of m . Since s_i are all even, no two of them can sum up to $\pm m$ which is odd, and any higher multiple is not possible because of the range. So the s_2 sum is only non zero if $s_1 = -s_3$ and then the sum is m . Now the summand in the r_2 sum is symmetric under change of sign of r_2 and it is zero when $r_2 = 0$ or r . So we can write it as

$$1/2 \sum_{r_2=(-r+1), (-r+2), \dots, r} \sin(\pi m r_1 r_2/r) \sin(\pi m r_2 r_3/r)$$

After writing in this form and doing the four geometric series we see that the sums will vanish unless $\pm(r_1 \pm r_3) = 0, \pm 2r, \dots$. Now the ranges of r_1 and r_3 guarantee that the only possibility is $r_1 = r_3$, when the sum is $r/2$. This proves equation (5.1).

6.2. computation of d_i

Using (5.1) we see that (5.3a) implies that

$$\begin{aligned}
& q^{-2j_{\check{k}}(j_{\check{k}}+1)} d(\check{r}_k, \check{s}_k) r m / 2 \\
&= \sum_{r_i \in I(1)} \sum_{s_i \in I(2)} q^{2j_i(j_i+1)} \sin(\pi m r_i / r) \sin(\pi m r_i r_k / r) e^{i\pi[r_i s_k + r_k s_i - s_i s_k r / m]} \\
&= \sum_{r_i \in I(1)} \sum_{s_i \in I(2)} e^{-i\pi m / (2r)[r_i^2 - 1 + s_i^2 r^2 / m^2]} e^{-i\pi[s_i s_k r / m]} \sin(\pi m r_i / r) \sin(\pi m r_i r_k / r) \\
&= \sum_{s_i \in I(2)} e^{-i\pi r / (2m)(s_i^2 + 2s_i s_k)} \\
&\quad \sum_{r_i \in I(1)} \sin(\pi m r_i / r) \sin(\pi m r_i r_k / r) e^{-i\pi m / 2r(r_i^2 - 1)}
\end{aligned} \tag{6.3}$$

If we write $s'_i = s_i/2$ in the s_i sum. Note that s'_i runs over all residues modulo m , and the exponential only depends on residue class modulo m , so we can shift s'_i without changing the sum. So the s_i sum is equal to

$$e^{i\pi(r/2m)s_k^2} \sum_{s'_i(m)} e^{-i\pi(2r/m)s_i'^2}, \tag{6.4}$$

where we have adopted the notation $s'_i(m)$ for a sum over s'_i running over a complete set of residues modulo (m) . The result for the r_i sum can be extracted from [12]

$$\begin{aligned}
& \sum_{r_i=1}^{r-1} \sin(\pi m r_i / r) \sin(\pi m r_i r_k / r) e^{-i\pi m / 2r(r_i^2 - 1)} \\
&= \frac{e^{3i\pi m / 2r - i\pi/2}}{2} e^{\frac{i\pi m}{2r}(r_k^2 - 1)} \sin(\pi m r_k / r) \sum_{k=0}^{2r-1} e^{-i\pi k^2 m / (2r)}
\end{aligned} \tag{6.5}$$

A short manipulation then shows that

$$d(r_k, -s_k) = \frac{1}{(rm)} e^{-i\pi/2 + 3i\pi m / (2r)} \sin(\pi m r_k / r) \sum_{s'_i(m)} e^{-i\pi(2r/m)s_i'^2} \sum_{k(2r)} e^{-i\pi k^2 m / (2r)} \tag{6.6}$$

This product of Gaussian sums equals another Gaussian sum

$$\sum_{h(2rm)} e^{-i\pi h^2 / (2mr)} = \sqrt{2rm} e^{-i\pi/4}.$$

This can be proved by imitating the steps in the proof of a similar multiplicative formula for Gauss sums given in [20]. The result then is

$$d(r_k, s_k) = d(\check{r}_k, \check{s}_k) = \sqrt{\frac{2}{rm}} e^{-\frac{3i\pi}{4} \frac{(r-2m)}{r}} \sin(\pi m r_k / r) \tag{6.7}$$

This proves theorem 2.

7. EVEN m

The set of irreducible representations of $U[m]$ we choose, have highest weights j_i characterised by

$$\begin{aligned} r_i &\in I(1) = \{1, 3, 5 \dots r - 2\} \\ \text{and } s_i &\in I(2) = \{0, \pm 1, \pm 2, \dots \pm (m - 1), m\} \end{aligned} \quad (7.1)$$

The set of representations closes under the tensor product. We prove that with this choice of representations we have theorems 1 and 2 .

$$\begin{aligned} (S^2)_{j_1 j_3} &= \frac{1}{\sin^2(\pi m/r)} \sum_{r_2, s_2} e^{i\pi[r_2(s_1+s_3)+s_2(r_1+r_3)]} e^{-i\pi[s_2 r/m(s_1+s_3)]} \\ &\quad \sin(\pi m r_1 r_2/r) \sin(\pi m r_2 r_3/r) \\ &= \frac{1}{\sin^2(\pi m/r)} \sum_{s_2 \in I(2)} e^{-i\pi s_2 r/m(s_1+s_3)} \sum_{r_2 \in I(1)} e^{i\pi r_2(s_1+s_3)} \\ &\quad \sin(\pi m r_1 r_2/r) \sin(\pi m r_2 r_3/r) \end{aligned} \quad (7.2)$$

We used $e^{i\pi s_2(r_1+r_3)} = 1$ because r_1 and r_3 are both odd. The s_2 sum can be written as

$$e^{2i\pi(m-1)r/m} \frac{[1 - e^{-2i\pi r(s_1+s_3)}]}{[1 - e^{-i\pi r(s_1+s_3)/m}]}$$

Now the numerator is zero but the denominator is also zero if $(s_1 + s_3)$ is $0 \pmod{2m}$. Now from the range of s_i it is clear that 0 is the only possibility for any $s_1 \neq m$, and for $s_1 = m$, $s_3 = m$ is the only possibility. So the s_2 sum is equal to $2m\delta(s_1, -s_3 \pmod{2m})$. Then the s_1, s_3 dependent phase drops out of r_2 sum.

$$\begin{aligned} &\sum_{r_2=1,3,\dots,r-2} \sin(\pi m r_1 r_2/r) \sin(\pi m r_2 r_3/r) \\ &= 1/2 \sum_{r'_2=0,\pm 1,\pm 3,\dots,\pm(r-2)} \sin(\pi m r_1 r_2/r) \sin(\pi m r_2 r_3/r) \end{aligned} \quad (7.3)$$

Again we expand the sines to get geometric sums of the form

$$\sum_{r_2=0,\pm 1,\pm 3,\dots,\pm(r-2)} e^{i\pi m N r_2/r}$$

proportional to

$$\frac{1 - e^{2i\pi m N}}{1 - e^{2i\pi r_2 m N/r}}$$

where N can be $\pm(r_1 \pm r_3)$. The sum is non vanishing only if $N = 0 \pmod{r}$. Now because of the range of r_i negative multiples and positive multiples higher than the first are clearly excluded. Also r is odd because we required $(r, m) = 1$. $r_1 + r_3$ cannot be zero because they are both positive and it cannot be r because they are both odd and must add to an even number. For the same reason $r_1 - r_3$ cannot be r so the only non zero contribution arise from $r_1 = r_3$, so that the r_2 sum is $\frac{r}{4}\delta(r_1, r_3)$. This proves (5.1) for even m .

Now we consider the determination of the weights. Using (5.1) we see that (5.3a) implies that

$$\begin{aligned}
& q^{-2j_{\bar{k}}(j_{\bar{k}}+1)} d(\check{r}_k, \check{s}_k) r m / 2 \\
&= \sum_{r_i \in I(1)} \sum_{s_i \in I(2)} q^{2j_i(j_i+1)} \sin(\pi m r_i / r) \sin(\pi m r_i r_k / r) \\
&\quad e^{i\pi[r_i s_k + r_k s_i - s_i s_k r / m]} e^{i\pi s_i} \\
&= e^{i\pi s_k} \sum_{s_i \in I(2)} e^{-i\pi r / (2m)(s_i^2 + 2s_i(s_k + m))} \\
&\quad \sum_{r_i \in I(1)} \sin(\pi m r_i / r) \sin(\pi m r_i r_k / r) e^{-i\pi m / 2r(r_i^2 - 1)}
\end{aligned} \tag{7.4}$$

where we have used the fact that $e^{i\pi r_i s_k} = e^{i\pi s_k}$ for r_i odd. Now $e^{-i\pi r s_i^2 / (2m)}$ only depends on residue class of $s_i \pmod{2m}$. This allows a shift in the s_i sum to give

$$e^{\frac{i\pi(s_k+m)^2}{2m}} \sum_{s'_i(2m)} e^{-i\pi(r/2m)s_i'^2}$$

The r_i sum can be written as

$$\begin{aligned}
& \sum_{r_i=1,3,\dots,(r-2)} \sin(\pi m r_i / r) \sin(\pi m r_i r_k / r) e^{-i\pi m(r_i^2 - 1)/2r} \\
&= \sum_{r_i=1,3,\dots,(r)} \sin(\pi m r_i / r) \sin(\pi m r_i r_k / r) e^{-i\pi m(r_i^2 - 1)/(2r)} \\
&= \frac{e^{-i\pi/2}}{2} e^{i\pi m[r_k^2 + 2]/(2r)} \sin(\pi m r_k / r) \sum_{r_i=1,3,\dots,2r-1} e^{-i\pi(m/2r)r_i'^2}.
\end{aligned} \tag{7.5}$$

The first equality above is a simple observation that the summand at $r_i = r$ is zero, the second uses some steps very similar to those used in [12] in their computation of the weights. We then obtain the following equation

$$\begin{aligned}
d(r_k, -s_k) &= \frac{e^{-i\pi[1/2 - s_k]}}{r m} e^{+3i\pi(m/2r)} e^{i\pi r m / 2} \sin(\pi m r_k / r) \\
&\quad \sum_{r_i=1,3,\dots}^{2r-1} e^{-i\pi(m/2r)r_i'^2} \sum_{s_i(2m)} e^{-i\pi(r/2m)s_i^2}
\end{aligned} \tag{7.6}$$

r'_i runs over all the odd residues mod $(2r)$ so we add the even number $r + 1$ to it without changing the sum to get

$$\sum_{r_i=1,3..}^{2r-1} e^{-i\pi(2m/r)r_i'^2} = e^{-i\pi mr/2} \sum_{r_i''=0,1..r-1} e^{-i\pi(2m/r)r_i''^2},$$

where we also changed the summation index to $r_i'' = (r_i' + 1)/2$.

$$\frac{e^{i\pi[-1/2+s_k]}}{rm} e^{3i\pi m/2r} \sin(\pi m r_k/r) \sum_{r_i''(r)} e^{-i\pi(2m/r)r_i''^2} \sum_{s_i(2m)} e^{-i\pi(r/2m)s_i^2}$$

The two Gauss sums can again be combined into a Gauss sum over residues mod $(2mr)$ [20], which can be evaluated. The result is then

$$d(r_k, s_k) = d(\check{r}_k, \check{s}_k) = \sqrt{\frac{2}{rm}} e^{\frac{-3i\pi}{4}(r-2m)/r} \sin(\pi m r_k/r) e^{i\pi s_k}, \quad (7.7)$$

proving theorem 2 for the even m case.

8. Fusion rules and Verlinde formula

We show in this section that the fusion rules as computed using the Verlinde formula [21] [22] agree with the tensor product structure of representations of $U[m]$. In particular there are no minus signs in N_{ij}^k . We will use \tilde{S} , with the normalisation natural from the relation to $\widehat{sl}(2)$. We will prove :

$$\begin{aligned} N_{ij}^k &= \sum_n \tilde{S}_{in} \tilde{S}_{jn} \tilde{S}_{kn}^* / \tilde{S}_{0n} \\ &= \Delta(r_i, r_j; r_k) \delta(s_k, s_i + s_j \bmod 2m) \end{aligned} \quad (8.1)$$

where

$$\begin{aligned} \Delta(r_i, r_j; r_k) &= 1 \\ &\text{if } r_k \in \{|r_i - r_j| + 1, \dots, \min(r_i + r_j - 1, 2r - r_i - r_j - 1)\} \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (8.2)$$

This is exactly the fusion of the Hopf algebra $U[m]$. Note that

$$N_i = N_i(sl(2)_{r-2}) \otimes N_i(U(1)_{m/2}), \quad (8.3)$$

where we are using the notation of [19] for the abelian fusion rules.

8.1. ODD m

Writing out the formula we have

$$N_{ij}^k = \sum_{r_n s_n} \tilde{S}(r_i s_i; r_n s_n) \tilde{S}(r_j s_j; r_n s_n) \tilde{S}^*(r_k s_k; r_n s_n) / \tilde{S}(1, 0; r_n s_n) \quad (8.4a)$$

$$= \left(\frac{2}{rm}\right) \sum_{r_n s_n} e^{i\pi s_n(r_i+r_j-r_k-1)} e^{i\pi r_n(s_i+s_j-s_k)} e^{i\pi s_n(s_i+s_j-s_k)r/m} \frac{\sin(\pi m r_i r_n/r) \sin(\pi m r_j r_n/r) \sin(\pi m r_k r_n/r)}{\sin(\pi m r_n/r)} \quad (8.4b)$$

$$= \left(\frac{2}{rm}\right) \sum_{s_n=0, \pm 2, \dots, \pm(m-1)} e^{i\pi s_n(s_i+s_j-s_k)r/m} \sum_{r_n=1, 2, \dots, (r-1)} \sin(\pi m r_i r_n/r) \sin(\pi m r_j r_n/r) \sin(\pi m r_k r_n/r) / \sin(\pi m r_n/r). \quad (8.4c)$$

The first two exponentials in (8.4b) are equal to one because the s_i are chosen to be even. Now the s_n sum is $m\delta(s_i+s_j-s_k, 0 \pmod{2m})$, since $s_i+s_j-s_k$ cannot equal to m which is odd. Using the symmetry under change of sign of r_n , and noting that the summand is zero for $r_n = 0, r$, we rewrite the r_n sum as

$$1/2 \sum_{r_n=0, \pm 1, \pm 2, \dots, \pm(r-1), r} \sin(\pi m r_i r_n/r) \sin(\pi m r_j r_n/r) \sin(\pi m r_k r_n/r) / \sin(\pi m r_n/r).$$

Now r_i is running over a complete set of residues modulo $2r$ and the sum depends only on residue class mod $(2r)$. But if r_n runs over a complete set of residues mod $(2r)$, so does $m r_n$ because $(m, 2r) = 1$. So this r_n sum is the same as at $m = 1$. This sum can be written down because we know the fusion rules for k positive integer. It is $\frac{r}{2}\Delta(r_i, r_j : r_k)$. The claim is then proved for odd m .

8.2. EVEN m

$$\begin{aligned} N_{ij}^k &= \sum_{r_n s_n} \tilde{S}(r_i s_i; r_n s_n) \tilde{S}(r_j s_j; r_n s_n) \tilde{S}^*(r_k s_k; r_n s_n) / \tilde{S}(1, 0; r_n s_n) \\ &= \frac{2}{rm} \sum_{r_n s_n} e^{i\pi s_n(r_i+r_j-r_k-1)} e^{i\pi r_n(s_i+s_j-s_k)} e^{i\pi s_n(s_i+s_j-s_k)r/m} \\ &\quad \sin(\pi m r_i r_n/r) \sin(\pi m r_j r_n/r) \sin(\pi m r_k r_n/r) / \sin(\pi m r_n/r) \end{aligned} \quad (8.5)$$

Now

$$e^{i\pi s_n(r_i+r_j-r_k-1)} = 1$$

because the r_i are all odd. The s_n sum is then seen to equal $2m\delta(s_i+s_j-s_k, 0 \pmod{2m})$. This condition for the product of sums to be non zero guarantees that the exponential appearing in the r_n sum is also equal to 1. Then we are left with

$$\sum_{r_n=1,3,\dots,(r-2)} \sin(\pi m r_i r_n / r) \sin(\pi m r_j r_n / r) \sin(\pi m r_k r_n / r) / \sin(\pi m r_n / r)$$

Using the symmetry under change of sign of r_n , and noting that the summand is zero for $r_n = r$, we rewrite the r_n sum as

$$1/2 \sum_{r_n=\pm 1, \pm 3, \dots, \pm(r-2), r} \sin(\pi m r_i r_n / r) \sin(\pi m r_j r_n / r) \sin(\pi m r_k r_n / r) / \sin(\pi m r_n / r)$$

Now r_i is running over a complete set of odd residues modulo r and the sum depends only on residue class mod (r) . Now if r_n runs over a complete set of residues mod (r) , so does $m r_n$ because $(m, r) = 1$. By the same argument as for odd m case then the sum is $\frac{r}{4} \Delta(r_i, r_j, r_k)$. The claim is then proved for even m .

9. Relation to $\widehat{sl}(2)$

So we have chosen a set of quantum group representations which could consistently describe the braiding and modular properties of a conformal field theory. We chose a set that closely resembled the Kac Wakimoto set, under the map we described between U_q and $\widehat{sl}(2)$. We can use the map now to propose, for odd m , a set of $\widehat{sl}(2)$ representations closely related to the Kac-Wakimoto set. This set contains both highest weight and lowest weight representations, since we want a set where every representation has a dual. We will then prove, directly using relations between the characters of highest weight and lowest weight representations properties that the modular invariant partition functions written in [4] can also be interpreted as partition functions for this alternative spectrum. Then we will discuss some other possibilities for the state space of some CFTs with $sl(2, R)$ current algebra symmetry.

For odd m , we take highest weight representations with highest weights j_i given by

$$j_i(h.w) = \frac{(r_i - 1)}{2} - \frac{s_i}{2}(r/m) \tag{9.1}$$

with $r_i = 1, 2 \dots r - 1$ and $s_i = 0, 2, 4 \dots (m - 1)$. And we take lowest weights

$$j_i(l.w) = -j_i(h.w), \quad \text{for } s_i \neq 0. \quad (9.2)$$

The $s_i = 0$ representations are self dual and are only counted once.

This set has exactly the same number of representations as the Kac-Wakimoto set but some of the highest weights there have been replaced by lowest weights. We have to make sure that this set also has all the characters necessary to give a representation of the modular group. This is guaranteed by the relations between the characters of the highest weight modules that we have dropped and the characters of the lowest weight modules that we have included.

It can be shown using properties of the characters [4] that

$$\chi_{D^-(r_i, s_i)}(\tau, z) = -\chi_{D^-(r-r_i, m-s_i)}(\tau, -z), \quad s_i \neq 0. \quad (9.3)$$

We can also prove the following

Proposition :

$$\chi_{D^-(j)}(\tau, z) = \chi_{D^+(-j)}(\tau, -z). \quad (9.4)$$

where $D^+(j)$ is an irreducible lowest weight representation of lowest weight j and $D^-(j)$ is an irreducible highest weight representation of highest weight j .

Proof. We first prove the analogous result for Verma modules V_j^- and V_{-j}^+ then use the fact that the characters of the irreducible modules are obtained as an alternating sum of characters of Verma modules.

$$\begin{aligned} \chi_{V^-(j)}(\tau, z) &= Tr_{V^-(j)}[e^{2\pi i(\tau L_0 - z J_0^0)}] \\ &= \frac{e^{2\pi i[\frac{\tau j(j+1)}{(k+2)} - zj]}}{1 - e^{2\pi iz}} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1 - e^{2\pi i\tau n})(1 - e^{2\pi i\tau n} e^{2\pi iz})(1 - e^{2\pi i\tau n} e^{-2\pi iz})} \end{aligned} \quad (9.5)$$

We have used the fact that the monomials

$$\prod_{i=1}^{\infty} (J_{-i}^+)^{p_{i,+}} \prod_{i=1}^{\infty} (J_{-i}^0)^{p_{i,0}} \prod_{i=0}^{\infty} (J_{-i}^-)^{p_{i,-}} |j\rangle \quad (9.6)$$

form a basis for the Verma module. This is a consequence of the Poincare-Birkhoff-Witt theorem as explained for example in [23]. For the lowest weight module a basis is given by

$$\prod_{i=1}^{\infty} (J_{-i}^-)^{p_{i,-}} \prod_{i=1}^{\infty} (J_{-i}^0)^{p_{i,0}} \prod_{i=0}^{\infty} (J_{-i}^+)^{p_{i,+}} | -j \rangle \quad (9.7)$$

Clearly then the character is

$$\begin{aligned} \chi_{V+(-j)}(\tau, z) &= \text{Tr}_{V+(-j)} [e^{2\pi i(\tau L_0 - z J_0^0)}] \\ &= \frac{e^{2\pi i[\frac{\tau j(j+1)}{(k+2)} + zj]}}{1 - e^{-2\pi iz}} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1 - e^{2\pi i\tau n})(1 - e^{2\pi i\tau n} e^{2\pi iz})(1 - e^{2\pi i\tau n} e^{-2\pi iz})} \end{aligned} \quad (9.8)$$

The change in sign of the exponent in the denominator comes about because of the change in the range of summation in (9.7) compared to (9.6). This shows that

$$\chi_{V-(j)}(\tau, z) = \chi_{V+(-j)}(\tau, -z) \quad (9.9)$$

The same formula clearly relates characters of highest and lowest weight modules related by change of sign of J_0^0 eigenvalue of the distinguished vector, for any level n . The characters look like the ones above except that the L_0 eigenvalue is shifted by n . But the characters of the irreducible modules are equal to alternating sums of characters of Verma modules [3][24]. The Verma modules in the expression for the character of $D^+(-j)$ are related to those in the expression for the character of $D^-(j)$ by change of sign of the J_0^0 eigenvalue of the distinguished vector. This is guaranteed by an automorphism of the algebra $\widehat{sl}(2)$. Together with (9.9) this completes the proof of (9.4).

In the special case of integrable modules this equation reduces to the statement that an integrable module with highest weight j , which can be regarded as an irreducible quotient of a highest weight module with highest weight j or as an irreducible quotient of a lowest weight module with lowest weight $-j$, is self-dual.

Equations (9.3) and (9.4) imply that

$$\begin{aligned} -\chi_{D^-(j'_i)}(\tau, z) &= \chi_{D^+(-j_i)}(\tau, z) \\ \text{where } 2j_i + 1 &= r_i - s_i(r/m) \\ \text{and } 2j'_i + 1 &= (r - r_i) - (m - s_i)r/m \end{aligned} \quad (9.10)$$

This means that the set of characters in (9.1) and (9.2) differs from the Kac-Wakimoto set by having some characters replaced by their negatives. Clearly then we can write modular invariant partition functions for this spectrum. That the S matrix for this set of representations gives non-negative integer fusion rules follows from the computations in section 8, together with the relation

$$S(r_1, s_1; r - r_2, m - s_2) = -S(r_1, s_1; r_2, -s_2). \quad (9.11)$$

The choices of representations that are consistent with modular invariant partition functions are not unique. In particular if χ_i is replaced by $\chi_i + C_{ij}\chi_j$, the equations $SC = CS$ and $SS^\dagger = 1$ guarantee that we get other modular invariant partition functions where the state space of the theory is modified by replacing the field ϕ_i in the chiral part with the direct sum of ϕ_i with its charge conjugate.

For even m , although we have a modular Hopf algebra our set of representations includes highest weights with $s_i = m$ which do not appear in the Kac Wakimoto set. While the modular S matrix and the characters certainly possess analytic continuations to these values, and the Hopf link invariant can still be computed, we do not know the characters and modular transformation properties for s_i outside the Kac wakimoto set.

10. Decoupling of the irreducible continuous series

Let us consider the set of highest weight representations that has a null vector of the form $(J_{-1}^+)^R$ and deduce the most general consequences for the fusion of these representations arising from setting the null vector to zero. The currents are :

$$J^a(z) = \sum_{n=-\infty}^{\infty} J_n^a z^{-n-1} \quad (10.1)$$

We define primary fields by their operator products [25] with the currents

$$J^+(z)\phi_m^j(w) = \frac{\tau^+\phi_m^j(w)}{(z-w)} + \text{Regular terms.} \quad (10.2)$$

$$J^0(z)\phi_m^j(w) = \frac{\tau^0\phi_m^j(w)}{(z-w)} + \text{Regular terms.} \quad (10.3)$$

$$J^-(z)\phi_m^j(w) = \frac{\tau^-\phi_m^j(w)}{(z-w)} + \text{Regular terms.} \quad (10.4)$$

The τ 's are operators representing the zero mode $sl(2, R)$ subalgebra. The label m , not necessarily integral, is equal to the eigenvalue of the compact generator J_0^0 . It runs in integer steps over $(-\infty, \infty)$ in the case of the continuous series, over a semi-infinite interval for the discrete series type. Choosing to consider irreducible modules means that we should set null vectors to zero. Therefore

$$\langle [(J_{-1}^+)^R \phi_{j_1}^{j_1}(z_1)] \phi_{m_2}^{j_2}(z_2) \phi_{m_3}^{j_3}(z_3) \rangle = 0. \quad (10.5)$$

where the action of J_{-n}^a [26] is defined by,

$$J_{-n}^a \phi_{m_1}^{j_1}(z_1) = \int dw \frac{J^a(w)}{(w - z_1)^n} \phi_{m_1}^{j_1}(z_1), \quad (10.6)$$

the w integral being taken round a small contour around z_1 .

$$\begin{aligned} & \langle \int dw \frac{J^+(w)}{w - z_1} \phi_{j_1}^{j_1}(z_1) \phi_{m_2}^{j_2}(z_2) \phi_{m_3}^{j_3}(z_3) \rangle \\ &= \langle \int \frac{dw}{w - z_1} \phi_{j_1}^{j_1}(z_1) J^+(w) \phi_{m_2}^{j_2}(z_2) \phi_{m_3}^{j_3}(z_3) \rangle \\ & \quad + \langle \int \frac{dw}{w - z_1} \phi_{j_1}^{j_1}(z_1) \phi_{m_2}^{j_2}(z_2) J^+(w) \phi_{m_3}^{j_3}(z_3) \rangle \\ &= \langle \int \frac{dw}{w - z_1} \phi_{j_1}^{j_1}(z_1) \frac{\tau_2^+}{w - z_2} \phi_{m_2}^{j_2}(z_2) \phi_{m_3}^{j_3}(z_3) \rangle \\ & \quad + \langle \int \frac{dw}{w - z_1} \phi_{j_1}^{j_1}(z_1) \phi_{m_2}^{j_2}(z_2) \frac{\tau_3^+}{w - z_3} \phi_{m_3}^{j_3}(z_3) \rangle \\ &= \langle \phi_{j_1}^{j_1}(z_1) \frac{\tau_2^+}{z_1 - z_2} \phi_{m_2}^{j_2}(z_2) \phi_{m_3}^{j_3}(z_3) \rangle \\ & \quad + \langle \phi_{j_1}^{j_1}(z_1) \phi_{m_2}^{j_2}(z_2) \frac{\tau_3^+}{z_1 - z_3} \phi_{m_3}^{j_3}(z_3) \rangle. \end{aligned} \quad (10.7)$$

Iterating this argument R times we obtain a sum of partitions of the τ^+ operators acting at z_2 and z_3 . We can isolate the term where all the τ^+ are acting at z_2 by its singularity $\frac{1}{(z_1 - z_2)^R}$, to get

$$0 = \langle \phi_{j_1}^{j_1}(z_1) (\tau^+)^R \phi_{m_2}^{j_2}(z_2) \phi_{m_3}^{j_3}(z_3) \rangle. \quad (10.8)$$

This means that any $\phi_{m_2}^{j_2}$ which can be written as $(\tau^+)^R \phi_{m_2}^{j_2}$ will vanish in a three point function with $\phi_{j_1}^{j_1}$ if the highest weight module generated by $\phi_{j_1}^{j_1}$ contains the null vector, i.e if $2j_1 + 1 = -R + (k + 2)$. Any state in the modules of irreducible continuous series type is of this form. Now all the states in this module are generated by the action of the generators on this highest weight, so the correlation function of any state in this module

with the $\phi_{m_2}^{j_2}$ is zero. The proof goes as follows. Let Y be an element of the enveloping algebra of $\widehat{sl(2)}$, generated by J_0^- and J_{-1}^+ . Then $\langle [Y\phi_{j_1}^{j_1}(z_1)]\phi_{m_2}^{j_2}(z_2)\phi_{m_3}^{j_3}(z_3) \rangle$ can be equated by the above contour deformation argument to a sum of terms of the form $\langle \phi_{j_1}^{j_1}(z_1)[\hat{Y}_2]\phi_{m_2}^{j_2}(z_2)[\hat{Y}_3]\phi_{m_3}^{j_3}(z_3) \rangle$ multiplied by some powers of $(z_1 - z_2)$ and $(z_1 - z_3)$, where \hat{Y} is some element generated by J_0^+ and J_0^- . By the above argument these all vanish. Again using the action of the generators and contour deformation shows that all the states in the irreducible continuous series module labelled by j_2 do decouple in correlation functions from the states in the j_1 module.

The same null vector for $j_1 = 0$, k positive integer, was used in [26] using the insertion of the identity operator to show that all the correlation functions of fields transforming according to nonintegrable representations vanish. If we are looking at other values of k , then the R cannot be a positive integer and still satisfy $2j_1 + 1 = -R + k + 2$ for j_1 equal to zero. So the correlation functions for infinite dimensional $sl(2)$ representations at the base are not necessarily zero.

The simple null vector used above also further constrains the fusion of highest weight representations. If ϕ^{j_2} belongs to a highest weight representation then any three point function including it and ϕ^{j_1} will vanish. Correlation functions of ϕ^{j_1} with lowest weight representations are not forced to be zero. This is probably related to the ‘trivialisation’ of the OPE discussed by Dotsenko in [27]. We expect that the argument for the decoupling of the irreducible continuous series representations can be generalised to any k , for any highest or lowest weight module containing any null vector.

The condition $2j_1 + 1 = -R + (k + 2) = -R + r/m$ can be rewritten as

$$2j_1 + 1 = (r - R) - (m - 1)r/m. \quad (10.9)$$

For $R < r$ this belongs to the set of highest weights in the Kac Wakimoto set. For $m = 3$ $s_i = 2 = m - 1$ is the only non zero value of s_i . These, we have proved, decouple from the irreducible continuous series. This means that for $k + 2 = r/3$ for any r , the simple null vector argument guarantees that the entire set we are proposing decouples from the irreducible continuous series.

11. CONCLUSIONS

The maps we have used between affine algebra modules and quantum groups may appear a little stranger than the known maps familiar at positive integer k where affine algebra modules with a finite base are mapped to quantum group modules with the same dimension as the base. We note that Bauer and Sochen [23] in their construction of null vectors in highest weight modules define an action of the subalgebra generated by J_{-1}^+, J_0^- on a finite set of vectors starting from the highest weight and leading to the null vector. For r_i positive we pick the action of $J_{s_i}^+$, and by use of the commutation relations define an action of $J_{-s_i}^-$ and $(J_0^0 - s_i k/2)$. This gives an r_i dimensional representation of $Sl(2)$ starting at the highest weight j_i . The $U_q Sl(2)$ we associate with each generating null vector of a highest weight Verma module is perhaps usefully thought of as a deformation of such an action of $Sl(2)$.

We have found no natural explanation from the quantum group point of view of the choice of $U[m]$. We arrived at it by exploiting the correspondence of S matrices, but it would be interesting if this choice of a quotient of U_q could be understood directly from the quantum group.

It will be interesting to understand the relation of the 3-Manifold invariants to 3D gauge theories. In fact, after completing this paper, we received a paper of C. Imbimbo [28] where the fusion rules in equation (8.3) are arrived at from $SL(2, R)$ Chern-Simons theory. Another problem is to find how to realise the fusion rules and braiding matrices in conformal field theory.

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