

# Quantum MHV Diagrams

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## Abstract

Over the past two years, the use of on-shell techniques has deepened our understanding of the  $S$ -matrix of gauge theories and led to the calculation of many new scattering amplitudes. In these notes we review a particular on-shell method developed recently, the quantum MHV diagrams, and discuss applications to one-loop amplitudes. Furthermore, we briefly discuss the application of  $D$ -dimensional generalised unitarity to the calculation of scattering amplitudes in non-supersymmetric Yang-Mills.

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# 1 Introduction

Recently, remarkable progress has been made in understanding the structure of the  $S$ -matrix of four-dimensional gauge theories. This progress was prompted by Witten’s proposal [1] of a new duality between  $\mathcal{N}=4$  supersymmetric Yang-Mills (SYM) and the topological open string B-model with target space the Calabi-Yau supermanifold  $\mathbf{CP}^{3|4}$ , a supersymmetric version of Penrose’s twistor space. In contrast to usual dualities, this novel duality relates two weakly-coupled theories, and as such it can in principle be tested by explicit computations on both sides.

One of the striking results of Witten’s analysis is the understanding of the remarkable simplicity of scattering amplitudes in Yang-Mills and gravity – simplicity which is completely obscure in computations using textbook techniques – in terms of the geometry of twistor space. More precisely, Witten [1] observed that tree-level scattering amplitudes, when Fourier transformed to twistor space, localise on algebraic curves in twistor space. For the simplest case of the maximally helicity violating (MHV) amplitude, the curve is just a (complex) line.

The simple geometrical structure in twistor space of the amplitudes was also the root of further important developments of new efficient tools to calculate amplitudes. In [2], Cachazo, Svrček and Witten (CSW) proposed a novel perturbative expansion for amplitudes in YM, where the MHV amplitudes are lifted to vertices, and joined by scalar propagators in order to form amplitudes with an increasing number of negative helicity gluons. At tree level, numerous successful applications of the MHV diagram method have been carried out so far [3]–[10]. An elegant proof of the method at tree level was presented in [11] based on the analyticity properties of the scattering amplitudes. Later, this method was shown to be applicable to one-loop amplitudes in supersymmetric [12–14] and non-supersymmetric theories [15], where a new infinite series of amplitudes in pure YM was calculated. In a different development, the new twistor ideas were merged with earlier applications of unitarity [16, 17] and generalised unitarity [18], and led to highly efficient techniques to calculate one-loop amplitudes in  $\mathcal{N} = 4$  SYM [19, 20] and in  $\mathcal{N} = 1$  SYM [21, 22].

There are many important reasons for the interest in new, more powerful techniques to calculate scattering amplitudes. Besides improving our theoretical understanding of gauge theories at the perturbative level the most important reason is the need for higher precision in our theoretical predictions. The advent of the Large Hadron Collider requires the knowledge of perturbative QCD backgrounds at unprecedented precision to distinguish “old” physics from the sought for “new” physics, and there exist long wish-

lists of processes that are yet to be computed. Traditional methods using Feynman rules are rather inefficient since they hide the simplicity of scattering amplitudes, intermediate expressions tend to be larger than the final formulae and one has to face the problem of the factorial growth of the number of diagrams which hampers the use of brute force methods. Therefore, techniques that directly lead to the simple final answers are desirable. In the following we want describe some of the novel “twistor string inspired” techniques, focusing on methods relevant for one-loop amplitudes.

## 2 Colour Decomposition and Spinor Helicity Formalism

Here we describe two ingredients that are essential in order to make manifest the simplicity of scattering amplitudes: the colour decomposition, and the spinor helicity formalism. We will later see how new twistor-inspired techniques merge fruitfully with these tools.

At tree level, Yang-Mills interactions are planar, hence an amplitude can be written as a sum over single-trace structures times partial or colour-stripped amplitudes,

$$\mathcal{A}_n^{\text{tree}}(\{p_i, \epsilon_i, a_i\}) = \sum_{\sigma \in S_n/\mathbf{Z}_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) \quad (1)$$

$$A_n(\sigma(p_1, \epsilon_1), \dots, \sigma(p_n, \epsilon_n)) .$$

Partial amplitudes do not carry any colour structure. They receive contributions only from Feynman diagrams with a fixed cyclic ordering of the external lines and have a simpler analytic structure than the full amplitude. At loop level, also multi-trace structures appear, which are subleading in the  $1/N$  expansion. However, for the one-loop gluon scattering amplitudes there exist a simple (linear) relation between the planar and non-planar terms and, therefore, we have to consider only planar ones.

The spinor helicity formalism, is largely responsible for the existence of very compact formulas of tree and loop amplitudes in massless theories. In four dimensions the Lorentz group is  $\mathbf{SL}(2, \mathbf{C})$  and a Lorentz vector is equivalent to a bi-spinor, i.e. the four-momentum can be written as a  $2 \times 2$  matrix  $p_{a\dot{a}} = p_\mu \sigma_{a\dot{a}}^\mu$ ,  $a, \dot{a} = 1, 2$ , where  $a$  and  $\dot{a}$  are left and right handed spinor indices, respectively. If  $p_\mu$  describes the momentum of a massless, on-shell particle, then  $p^2 = \det p_{a\dot{a}} = 0$  and the matrix  $p_{a\dot{a}}$  can be written as a product of a left and a right-handed spinor

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} . \quad (2)$$

In real Minkowski space the left and right-handed spinors are related by complex conjugation  $\tilde{\lambda} = \pm \bar{\lambda}$ . It is useful to introduce the following Lorentz invariant brackets:  $\langle ij \rangle = \epsilon^{ab} \lambda_a^i \lambda_b^j$  and  $[ji] = \epsilon^{\dot{a}\dot{b}} \tilde{\lambda}_{\dot{a}}^i \tilde{\lambda}_{\dot{b}}^j$ , in terms of which dot products of on-shell four-momenta can be rewritten as  $2k_i \cdot k_j = \langle ij \rangle [ji]$ .

The simplest non-vanishing tree-level gluon scattering amplitudes have exactly two negative helicity gluons, denoted by  $i$  and  $j$ , and otherwise only positive helicity gluons. Since amplitudes with zero or one negative helicity gluon vanish, these amplitudes are called maximally helicity violating (MHV). Up to a trivial momentum conservation factor they take the following form

$$A_{\text{MHV}}(\lambda_i) = ig^{n-2} \frac{\langle ij \rangle}{\langle 12 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle}. \quad (3)$$

The most remarkable fact about this formula, besides its simplicity, is that it is holomorphic (no  $\tilde{\lambda}$ s appear), which has important consequences when the amplitude is transformed to twistor space.

### 3 Twistor Space and MHV Diagrams

In a nutshell, twistor space is obtained from the spinor variables  $(\lambda_a, \tilde{\lambda}_{\dot{a}})$  by performing a “half Fourier transform” (FT), i.e. a FT of one of the spinor variables, say  $\tilde{\lambda}_{\dot{a}} \rightarrow \mu_{\dot{a}}$ . Twistor space (for complexified Minkowski space) is a four-dimensional complex space  $(\lambda_a, \mu_{\dot{a}})$ . However, amplitudes turn out to be homogeneous functions of the twistor variables and it is more natural to think about projective twistor space  $(\lambda_a, \mu_{\dot{a}}) \sim (t\lambda_a, t\mu_{\dot{a}})$  with  $t$  a non-zero complex number.

In twistor theory a central rôle is played by the incidence relation

$$\mu_{\dot{a}} + x_{a\dot{a}} \lambda^a = 0, \quad (4)$$

which describes the correspondence between Minkowski space and twistor space. In particular if we fix a point in Minkowski space  $x_{a\dot{a}}$  this leads to two complex equations that describe a line in projective twistor space. As mentioned earlier the MHV tree amplitudes are holomorphic, except for the overall momentum conservation factor  $\delta^4(\sum \lambda_i \tilde{\lambda}_i)$ , and therefore their transformation to twistor space is particularly simple:

$$\begin{aligned} \tilde{A}_{\text{MHV}}(\lambda, \mu) &\sim A_{\text{MHV}}(\lambda) \int \prod_i d\tilde{\lambda}_i e^{i\mu_i \tilde{\lambda}_i} e^{ix\lambda_i \tilde{\lambda}_i} \\ &\sim A_{\text{MHV}}(\lambda) \prod_i \delta(\mu_i + x\lambda_i). \end{aligned} \quad (5)$$

Notice the appearance in (5) of the incidence relation (4), which implies that MHV amplitudes are supported on a line in twistor space. For more complicated amplitudes the localisation properties are almost as simple, in particular an amplitude with  $Q$  negative helicity gluons localises on sets of  $Q - 1$  intersecting lines. These simple geometrical structures in twistor space

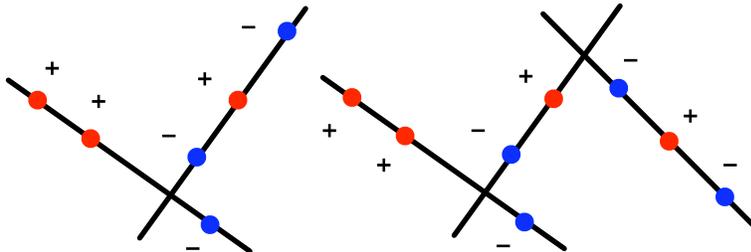


Figure 1: Twistor space localisation of tree amplitudes with  $Q = 3$  and  $Q = 4$

of the amplitudes led Cachazo, Svrček and Witten (CSW) to propose [2] a novel perturbative expansion for tree-level amplitudes in Yang-Mills using MHV amplitudes as effective vertices. It is indeed natural to think of an MHV amplitude as a local interaction, since the line in twistor space on which a MHV amplitude localises corresponds to a point in Minkowski space via the incidence relation.

In the MHV diagrammatic method of [2], MHV vertices are connected by scalar propagators  $1/P^2$  and all diagrams with a fixed cyclic ordering of external lines have to be summed. The crucial point is clearly to define an off-shell continuation of MHV amplitudes. CSW proposed to associate internal (off-shell) legs with momentum  $P$  the spinor  $\lambda_{Pa} := P_{a\dot{a}}\eta^{\dot{a}}$ , where  $\eta$  denotes an arbitrary reference spinor. We can assign a spinor to every internal momentum and insert this spinor in the Parke-Taylor formula (3) to define the off-shell MHV vertex. It can be shown [2] that after summing all diagrams the dependence on the reference spinor drops out. Proofs of the equivalence of MHV diagrams and usual Feynman diagrams at tree level have been presented in [11, 23–25]. Interestingly, MHV rules for tree-level gravity have also been derived in [26].

## 4 From Trees to Loops

The success of the MHV method at tree level brings up the question if this can be extended to the quantum level, i.e. to loop amplitudes. The original prognosis from twistor string theory was negative because of the presence of

unwanted conformal supergravity modes that spoil the duality with Yang-Mills at loop level. Remarkably, we will find perfect agreement between loop MHV diagrams (such as in Figure 2) and the results obtained with more standard methods.

For simplicity let us focus on the simplest one-loop amplitudes, the MHV one-loop amplitudes in  $\mathcal{N} = 4$  super Yang-Mills. These amplitudes were

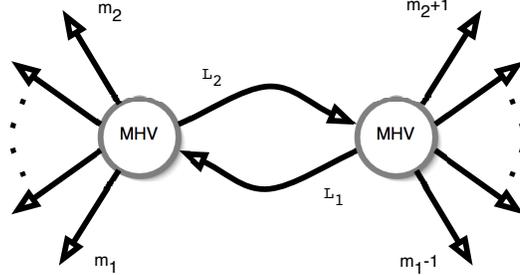


Figure 2: MHV diagrams for the MHV one-loop amplitudes in  $\mathcal{N} = 4$  SYM

computed in [16] using the four-dimensional cut-constructibility approach, which utilises the fact that one-loop amplitudes in supersymmetric theories can be reconstructed from their discontinuities. The result is surprisingly simple and can be expressed in terms of the so-called *2-mass easy box functions*  $F^{2me}(s, t, P^2, Q^2)$  as

$$A_{\text{MHV}}^{1\text{-loop}} = A_{\text{MHV}}^{\text{tree}} \times \sum_{p,q} F^{2me}(s, t, P^2, Q^2). \quad (6)$$

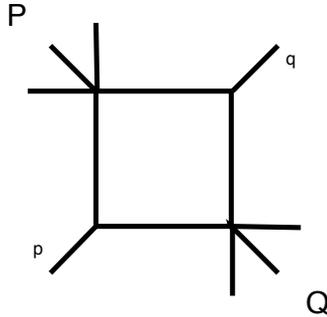


Figure 3: A graphical representation of the 2-mass easy box function

It turns out that the MHV diagrammatic calculation [12] agree perfectly with the result of [16]. A few remarks are in order here:

- In the calculation it is crucial to decompose the loop momentum  $L$  in an on-shell part  $l$  plus a part proportional to the reference null-momentum  $\eta$ :  $L = l + z\eta$ .
- This decomposition naturally leads to *dispersion integrals*  $\int dz/z(\dots)$ , which do not need subtractions.
- This calculation provides a check of the MHV method, e.g. the proof of covariance ( $\eta$ -independence) is highly non-trivial.
- This calculation incorporates a large number of conventional Feynman diagrams.

This approach readily applies to non-MHV amplitudes and theories with less supersymmetry. In [13, 14] the method was applied to the case of MHV one-loop amplitudes in  $\mathcal{N} = 1$  SYM and complete agreement with a calculation using the cut-constructibility approach [17] was found. Finally, in [15] the first new result from MHV diagrams at one-loop was obtained: the cut-containing part of the MHV one-loop amplitudes in pure Yang-Mills.

Having discussed these important checks of the MHV method, in the next Section we will argue that generic one-loop scattering amplitudes can be equivalently computed with MHV diagrams.

## 5 From Loops to Trees

In order to prove that MHV diagrams produce the correct result for scattering amplitudes at the quantum level, one has to

1. Prove the covariance of the result;
2. Prove that all physical singularities of the scattering amplitudes (soft, collinear and multiparticle) are correctly reproduced.

Indeed, if  $A_{\text{MHV}}^{(n)}$  is the result of the calculation of an  $n$ -point scattering amplitude based on MHV diagrams, and  $A_{\text{F}}^{(n)}$  is the correct result (obtained using Feynman diagrams), the difference  $A_{\text{MHV}}^{(n)} - A_{\text{F}}^{(n)}$  must be a polynomial which, by dimensional analysis, has dimension  $4 - n$ . But such a polynomial cannot exist except for  $n = 4$ . This case can be studied separately, and we have already discussed the agreement with the known results for theories with  $\mathcal{N} = 1, 2, 4$  supersymmetry.

In this section we will report on the calculation of generic one-loop amplitudes with MHV diagrams [27]. Firstly, we will prove the covariance of the result. This proof relies on two basic ingredients: the locality of the MHV vertices, and a beautiful result by Feynman known as the Feynman

Tree Theorem [28]. This theorem relates the contribution of a loop amplitude to those of amplitudes obtained by opening up the loop in all possible ways; strikingly, this theorem allows one to calculate loops from on-shell trees. Our next goal will be to show that soft and collinear singularities are correctly reproduced by MHV diagrams at one loop. Thus, for a complete proof of the MHV method at one loop, it would only remain to show the agreement of multiparticle singularities.

## 5.1 The Feynman Tree Theorem and the Proof of Covariance

We begin by briefly reviewing Feynman's Tree Theorem. This result is based on the decomposition of the Feynman propagator into a retarded (or advanced) propagator and a term which has support on shell. For instance,

$$\Delta_F(P) = \Delta_R(P) + 2\pi\delta(P^2 - m^2)\theta(-P_0) . \quad (7)$$

Suppose we wish to calculate a certain one-loop diagram  $\mathcal{L}$ , and let  $\mathcal{L}_R$  be the quantity obtained from  $\mathcal{L}$  by replacing all Feynman propagators by retarded propagators.<sup>2</sup> Clearly  $\mathcal{L}_R = 0$ , as there are no closed timelike curves in Minkowski space. This equation can fruitfully be used to work out the desired loop amplitude. Indeed, by writing the retarded propagator as a sum of Feynman propagator plus an on-shell supported term as dictated by (7), one finds that

$$\mathcal{L} = \mathcal{L}_{1\text{-cut}} + \mathcal{L}_{2\text{-cut}} + \mathcal{L}_{3\text{-cut}} + \mathcal{L}_{4\text{-cut}} . \quad (8)$$

Here  $\mathcal{L}_{p\text{-cut}}$  is the sum of all the terms obtained by summing all possible diagrams obtained by replacing  $p$  propagators in the loop with delta functions. Each delta function cuts open an internal loop leg, and therefore a term with  $p$  delta functions computes a  $p$ -particle cut in a kinematical channel determined by the cut propagators (whose momentum is set on shell by the delta functions).

Feynman's Tree Theorem (8) states that a one-loop diagram can be expressed as a sum over all possible cuts of the loop diagram. The process of cutting puts internal lines on shell; the remaining phase space integrations have still to be performed, but these are generically easier than the original loop integration. Thus the Feynman Tree Theorem implies that one-loop scattering amplitudes can be determined from on-shell data alone. Interestingly, one can iterate this procedure and apply it to higher loop diagrams.

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<sup>2</sup>The same argument works for advanced propagators.

Now we apply the Feynman Tree Theorem to MHV diagrams. This possibility is guaranteed by the local character in Minkowski space of an MHV interaction vertex; thus a closed loop MHV diagram where propagators are replaced by retarded or advanced propagators vanishes, and one arrives at an equation identical to (8). Specifically, in [27] we used Feynman’s Tree Theorem in order to prove that one-loop amplitudes calculated with MHV diagrams are covariant. More precisely, we were able to show that the sum of all possible  $p$ -cut MHV diagrams is separately covariant. The remaining Lorentz-invariant phase space integrations are also invariant, hence the full amplitude – given by summing over  $p$ -trees with  $p = 1, \dots, 4$  as indicated by (8) – is also covariant.

A sketch of the proof of covariance for the case of one-loop amplitudes with the MHV helicity configuration is as follows. The one-loop MHV diagrams contributing to an  $n$ -point MHV amplitude are presented in Figure 2. In Figure 4 we show the one-particle and two-particle cut diagrams which



Figure 4: One-particle and two-particle MHV diagrams contributing to the one-loop MHV scattering amplitude.

are generated in the application of the Feynman Tree Theorem.<sup>3</sup> We start by focussing on one-particle cut diagrams. These one-particle cut-diagrams are nothing but tree-level diagrams, which are then integrated using a Lorentz invariant phase space measure. We now make the following important observation: these tree (one-cut) diagrams would precisely sum to a tree-level next-to-MHV (NMHV) amplitude with  $n+2$  external legs (which would then be covariant as shown in [2]), *if* we also include the set of diagrams where the two legs into which the cut propagator is broken are allowed to be at the same MHV vertex. Such diagrams are obviously never generated by cutting a loop leg in MHV diagrams of the type depicted in Figure 2. These “missing” diagrams are drawn in Figure 5. MHV rules tell us, before any phase space integration is performed, that the sum of one-particle cut diagrams

<sup>3</sup>In our notation we only draw vertices and propagators (or cut-propagators) connecting them. It will be understood that we have to distribute the external gluons among the MHV vertices in all possible ways compatible with cyclic ordering, and the requirement that the two vertices must have the helicity configuration of an MHV amplitude. Moreover we will have to sum over all possible helicity assignments of the internal legs and, where required, over all possible particle species which can run in the loop.

of Figures 4 and 5 generates a NMHV amplitude with  $n + 2$  external legs. Since the phase space measure is Lorentz invariant, it follows that the sum of one-particle cut diagrams, including the missing diagrams, is covariant.

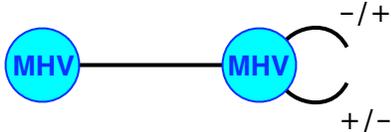


Figure 5: In this Figure we represent “missing diagrams”, mentioned in the text.

To complete the proof of covariance we have to justify the inclusion of these missing diagrams. Here we will present an explanation which relies on supersymmetry.<sup>4</sup> The diagrams where two adjacent and opposite helicity legs from the same MHV vertex are sewn together vanish when summed over particle species in a supersymmetric theory. Individual diagrams before summing over particle species diverge because of the collinearity of the momenta of the two legs, but the sum over particle species vanishes even before integration. So we discover that we could have actually included these diagrams from the start, since their contribution is zero. Hence one-particle cut diagrams of MHV one-loop amplitudes generate phase space integrals of tree-level NMHV amplitudes, and are, therefore, covariant.

Next we look at two-particle cut diagrams. These split the one-loop diagram of Figure 2 into two disconnected pieces (see the second diagram in Figure 4). These are two MHV amplitudes, because the two internal legs are put on shell by the Feynman cuts. Therefore, no  $\eta$ -dependence is produced by these two-particle cut diagrams.

Summarising, we have shown that Feynman one-particle and two-particle cut diagrams are separately covariant. By Feynman’s Tree Theorem (8) we conclude that the physical one-loop MHV amplitude is covariant too.

The main lines of the proof of covariance we have discussed in the simple example of an MHV amplitude are easily generalised more complicated amplitudes. In particular, we notice that:

1. In a one-loop MHV diagram with  $v$  vertices and  $n$  external particles (contributing to an  $N^{v-2}$ MHV amplitude), the top-cut is necessarily the  $v$ -particle cut. This will always be  $\eta$ -independent by construction. Notice that this top cut will generically vanish if  $v > 4$ .

<sup>4</sup>An alternative proof, not relying on supersymmetry, can be found in [27].

2. All  $p$ -particle cuts which are generated by the application of Feynman's Tree Theorem split each one-loop MHV diagram into  $p$  disconnected pieces when  $p > 1$ . In all such cases we see that amplitudes are produced on all sides of the cut propagators when the sum over all MHV diagrams is taken.

3. The case of a one-particle cut is special since it generates a connected tree diagram. Similarly to the example discussed earlier, one realises that by adding missing diagrams the one-cut diagrams group into  $N^{v-1}$ MHV amplitudes with  $n + 2$  external legs (which are of course covariant).

4. To see amplitudes appearing on all sides of the cuts, one has to sum over all one-loop MHV diagrams.

In this way one can show the covariance of the result of a MHV diagram calculation for amplitudes with arbitrary helicities.

## 5.2 Collinear limits

In this section we address the issue of reproducing the correct singularities of the scattering amplitudes from MHV diagrams. We begin with collinear limits. At tree level, collinear limits were studied in [2], and shown to be in agreement with expectations from field theory. Consider now a one-loop scattering amplitude  $\mathcal{A}_n^{1\text{-loop}}$ . When the massless legs  $a$  and  $b$  become collinear, the amplitude factorises as [16, 17, 29, 30]

$$\begin{aligned} \mathcal{A}_n^{1\text{-loop}}(1, \dots, a^{\lambda_a}, b^{\lambda_b}, \dots, n) &\xrightarrow{a\parallel b} \\ &\sum_{\sigma} \left[ \text{Split}_{-\sigma}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}_{n-1}^{1\text{-loop}}(1, \dots, (a+b)^{\sigma}, \dots, n) \right. \\ &\quad \left. + \text{Split}_{-\sigma}^{1\text{-loop}}(a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}_{n-1}^{\text{tree}}(1, \dots, (a+b)^{\sigma}, \dots, n) \right]. \end{aligned} \quad (9)$$

$\text{Split}^{\text{tree}}$  are the gluon tree-level splitting functions, whose explicit forms can be found e.g. in [31].  $\text{Split}^{1\text{-loop}}$  is a supersymmetric one-loop splitting function. In [32] and [33] explicit formulae for this one-loop splitting function, valid to all orders in the dimensional regularisation parameter  $\epsilon$ , were found. The result of [33] is:

$$\text{Split}_{-\sigma}^{1\text{-loop}}(a^{\lambda_a}, b^{\lambda_b}) = \text{Split}_{-\sigma}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) r(z), \quad (10)$$

where, to all orders in  $\epsilon$ ,  $r(z) =$

$$\frac{c_{\Gamma}}{\epsilon^2} \left( \frac{-s_{ab}}{\mu^2} \right)^{-\epsilon} \left[ 1 - {}_2F_1 \left( 1, -\epsilon, 1 - \epsilon, \frac{z-1}{z} \right) - {}_2F_1 \left( 1, -\epsilon, 1 - \epsilon, \frac{z}{z-1} \right) \right] \quad (11)$$

and

$$c_\Gamma = \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1 - 2\epsilon)}. \quad (12)$$

Here the parameter  $z$  is introduced via the relations  $k_a := zk_P$ ,  $k_b := (1 - z)k_P$ , where  $k_P^2 \rightarrow 0$  in the collinear limit. Notice that in (9) we sum over the two possible helicities  $\sigma = \pm$ .

In [27] we were able to reproduce (9) from a calculation based on one-loop MHV diagrams; in particular we were able to re-derive the all-orders in  $\epsilon$  expressions (10) and (11). Similarly to the tree-level, the different collinear limits at one-loop arise from different MHV diagrams.

As an example, consider the  $++ \rightarrow +$  collinear limit. Consider first the diagrams where the two legs becoming collinear,  $a$  and  $b$ , are either a proper subset of the legs attached to a single MHV vertex, or belong to a four-point MHV vertex  $\mathcal{A}_{4,\text{MHV}}(a, b, l_2, -l_1)$  but the loop legs  $L_1$  and  $L_2$  are connected to different MHV vertices.<sup>5</sup> In this case we call  $s_{ab}$  a *non-singular* channel. Summing over all MHV diagrams where  $s_{ab}$  is a non-singular channel, one immediately sees that a contribution identical to the first term in (9) is generated.

Next we consider singular-channel diagrams (a prototypical one is shown in Figure 6), i.e. diagrams where the legs  $a$  and  $b$  belong to a four-point MHV vertex and the two remaining loop legs are attached to the same MHV vertex. The diagram shown in Figure 6 has been calculated in [12], and in [27] we

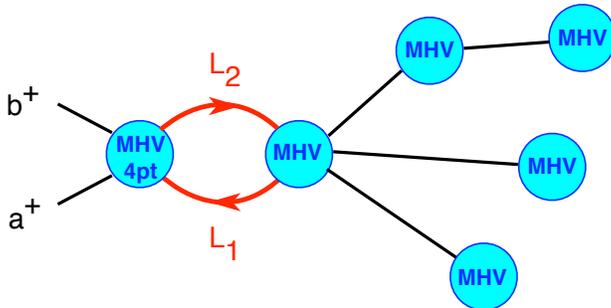


Figure 6: A schematic example of a one-loop MHV diagram contributing to a generic non-MHV one-loop amplitude where  $s_{ab}$  is a singular channel. In the collinear limit  $a \parallel b$ , diagrams of this type generate the second term on the right hand side of (9).

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<sup>5</sup>Thus, even if  $L_1$  and  $L_2$  become null and collinear, nothing special happens to the sum of tree MHV diagrams on the right hand side of this four-point MHV vertex, precisely because  $L_1$  and  $L_2$  are not part of the same MHV vertex.

showed that it precisely accounts for the second term in (9).

A similar analysis can be carried out for the other collinear limits, as well as for the soft limits. In all cases we found complete agreement with the known expressions for the all-orders in  $\epsilon$  collinear and soft functions [27].

## 6 Generalised Unitarity

One-loop scattering amplitudes in supersymmetric gauge theories are linear combinations of scalar box, tensor triangle and tensor bubble integral functions with coefficients that are rational functions of the momenta and spinor variables. One wonders if these coefficients can be determined directly without performing any loop integration. The answer turns out to be yes and the main tool are unitarity (2-particle) cuts and generalised (3-particle and 4-particle) cuts [34, 35]. In an  $n$ -particle cut of a loop diagram,  $n$  propagators are replaced by  $\delta$ -functions which (partially) localise the loop integration and produce a sufficient number of linear equations to fix all coefficients. The method using 2-particle cuts was introduced [16, 17], where it was used to calculate the first infinite series of one-loop amplitudes in SYM, and later generalised to include 3-particle cuts in [18, 20]. Finally, in [19] 4-particle cuts were shown to be an efficient tool to find coefficients of box functions. In particular they reduce the calculation of all one-loop amplitudes in  $\mathcal{N} = 4$  SYM to a simple algebraic exercise.

In non-supersymmetric theories loop amplitudes contain additional rational terms, which do not have cuts in 4 dimensions. This problem can be elegantly solved by working in  $D = 4 - 2\epsilon$  dimensions, since then also rational terms develop cuts and become amenable to unitarity techniques [36]. In [37] generalised unitarity techniques in  $D$  dimensions were shown to be a powerful tool to calculate complete one-loop amplitudes in non-supersymmetric theories like QCD. The price to pay is that we have to evaluate cuts in  $D$  dimensions; in practice this means that we have to make the internal particles massive with uniform mass  $\mu$  and integrate over  $\mu$  [36].

We now illustrate this technique for the simplest one-loop amplitude in pure YM which has four positive helicity gluons  $\langle 1^+ 2^+ 3^+ 4^+ \rangle$ . It turns out that in this case we only have to consider the 4-particle cut depicted in Figure 7; no other cuts are necessary. The 4-particle cut gives  $\mu^4 \frac{[12][34]}{(12)(34)}$  where  $\mu$  is the mass of the internal particle. This implies that the amplitude is proportional to a scalar box integral with  $\mu^4$  inserted in the integral  $I_4[\mu^4] = -\epsilon(1 - \epsilon)I_4^{D=8-2\epsilon} = -\frac{1}{6} + \mathcal{O}(\epsilon)$ . Hence, the amplitude is purely rational, and

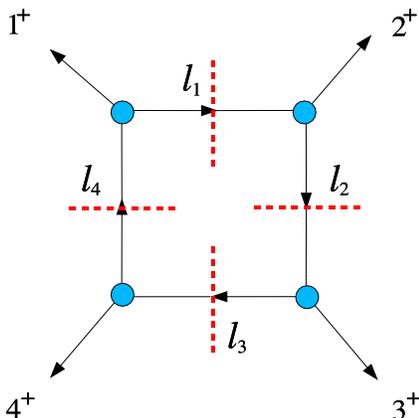


Figure 7: The 4-particle cut of the one-loop  $\langle 1^+2^+3^+4^+ \rangle$  amplitude in Yang-Mills

we find

$$A_4(1^+, 2^+, 3^+, 4^+) = -\frac{1}{6} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}. \quad (13)$$

In [37] this method was applied successfully to the remaining 4-point amplitudes  $\langle 1^+2^+3^+4^- \rangle$ ,  $\langle 1^-2^-3^+4^+ \rangle$  and  $\langle 1^-2^+3^-4^+ \rangle$ , and to the particular 5-point amplitude  $\langle 1^+2^+3^+4^+5^+ \rangle$ . Interestingly, it turned out that for the amplitudes with one or more negative helicity gluons, only 3-particle and 4-particle cuts were needed in order to determine the amplitude. This method can be applied directly to more general amplitudes and the 5- and 6-point amplitudes are currently under investigation.

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