

# Graph Guessing Games and non-Shannon Information Inequalities

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## Abstract

Guessing games for directed graphs were introduced by Riis [9] for studying multiple unicast network coding problems. In a guessing game, the players toss generalised dice and can see some of the other outcomes depending on the structure of an underlying digraph. They later guess simultaneously the outcome of their own die. Their objective is to find a strategy which maximises the probability that they all guess correctly. The performance of the optimal strategy for a graph is measured by the guessing number of the digraph.

In [3], Christofides and Markström studied guessing numbers of undirected graphs and defined a strategy which they conjectured to be optimal. One of the main results of this paper is a disproof of this conjecture.

The main tool so far for computing guessing numbers of graphs is information theoretic inequalities. The other main result of the paper is that Shannon's information inequalities, which work particularly well for a wide range of graph classes, are not sufficient for computing the guessing number.

Finally we pose a few more interesting questions some of which we can answer and some which we leave as open problems.

## 1 Introduction

Consider the following 2-player cooperative game: Two players toss a coin with each player seeing the outcome of the coin toss of the other player (but not their own). Then, they simultaneously guess the outcome of their own coin toss. The

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players win the game if they both guess correctly. Of course, if they both guess at random, then the probability of winning is  $1/4$ . It turns out that the players can use the extra information they have in order to improve the probability of success. For example, if they agree beforehand to follow the strategy ‘guess what you see’ then the probability of success increases to  $1/2$ . We can generalise this game (see Section 2) to guessing games with multiple players in which each player sees the outcome of the coin tosses (or more generally of many-sided dice throws) of other players, according to an underlying digraph.

These guessing games [9, 10] emerged from studying network coding problems [1] where the network is multiple unicast, i.e. where each sender has precisely one corresponding receiver who wishes to obtain the sender’s message, and a constraint that only one message can be sent through each channel at a time. A multiple unicast can be represented by a directed acyclic graph with  $n$  inputs/outputs and  $m$  intermediate nodes. By merging the vertices which represent the senders with their corresponding receiver vertices we can create an auxiliary directed graph which has the nice property that there is no longer any distinction between router, sender, or receiver vertices. Due to the way guessing games are defined, coding functions on the original network can be translated into strategies for the guessing game on the auxiliary graph and vice versa. The performance of the optimal strategy for a guessing game is measured by the guessing number which we will define precisely in Section 2.

One of the first applications of guessing games was the disproval in [9] of two conjectures raised by Valiant [11] in circuit complexity in which he asked about the optimal Boolean circuit for a Boolean function.

In this paper we provide a counterexample to a conjecture of Christofides and Markström given in [3] which states that the optimal strategy for the guessing game of an undirected graph is based on the fractional clique cover number of the graph. (See Section 2 for more details.) Additionally, we will show that the guessing number for undirected graphs cannot be determined by considering only the Shannon information inequalities as explained in Section 5. We will also make and investigate the Superman conjecture which suggests that the (asymptotic) guessing number of an undirected graph does not increase when a directed edge is added. Finally we will provide a possible example of a directed graph whose guessing number changes when its edges are reversed.

The outline of our paper is as follows. In Section 2 we introduce the formal language of guessing games. Section 3 is concerned with the asymptotic behaviour of guessing numbers. In Section 4 we formally define the fractional clique cover strategy from [3] which provides a feasible computational method for calculating lower bounds of guessing numbers for undirected graphs. In Section 5 we introduce a method for calculating upper bounds of guessing numbers by making use of entropic arguments. Our main results appear in Section 6. We then discuss some of the technical details of the computer searches we carried out in Section 7. We conclude with some open problems in Section 8.

## 2 Definitions

A *directed graph*, or *digraph* for short, is a pair  $G = (V(G), E(G))$ , where  $V(G)$  is the set of *vertices* of  $G$  and  $E(G)$  is a set of ordered pairs of vertices of  $G$  called the *directed edges* of  $G$ . Given a directed edge  $e = (u, v)$ , which we also denote by  $uv$ , we call  $u$  the *tail* and  $v$  the *head* of  $e$  and say that  $e$  goes from  $u$  to  $v$ .

For the purposes of guessing games we will assume throughout that our digraphs are loopless, i.e. they contain no edges of the form  $uu$  for  $u \in V(G)$ . Once we define the guessing game it will be easily seen that the probability of winning on a digraph  $G$  is equal to the probability of winning on the subgraph of  $G$  obtained by removing all vertices with loops.

Given a digraph  $G$  and a vertex  $v \in V(G)$ , the *in-neighbourhood* of  $v$  is  $\Gamma^-(v) = \{u : uv \in E(G)\}$  and the *out-neighbourhood* of  $v$  is  $\Gamma^+(v) = \{u : vu \in E(G)\}$ .

In this paper our main results will primarily be on *undirected graphs* which are naturally treated as a special type of digraph  $G$  where  $uv \in E(G)$  if and only if  $vu \in E(G)$ . We call the pair of directed edges  $uv$  and  $vu$ , the *undirected edge*  $uv$ . A major role in our guessing strategies will be played by *cliques* i.e. subgraphs in which every pair of vertices are joined by an undirected edge.

Given a digraph  $G$  and an integer  $t \geq 1$ , the *t-uniform blowup* of  $G$  which we will write as  $G(t)$  is a digraph formed by replacing each vertex  $v$  in  $G$  with a class of  $t$  vertices  $v_1, \dots, v_t$  with  $u_i v_j \in E(G(t))$  if and only if  $uv \in E(G)$ .

A *guessing game*  $(G, s)$  is a game played on a digraph  $G$  and the alphabet  $A_s = \{0, 1, \dots, s-1\}$ . There are  $|V(G)|$  players working as a team. Each player corresponds to one of the vertices of the digraph. Throughout the article we will be freely speaking about the player  $v$  instead of the player corresponding to the vertex  $v \in V(G)$ . The players know the digraph  $G$ , the natural number  $s$ , and are told to which of the vertices they correspond to. They may discuss and agree upon a strategy using this information before the game begins, but no communication between players is allowed after the game starts.

Once the game begins, each player  $v \in V(G)$  is assigned a value  $a_v$  from  $A_s$  uniformly and independently at random. The players do not have access to their own values but can see some of the values assigned to the other players according to the digraph  $G$ . To be more precise, once the values have been assigned each player is given a list of the players in its in-neighbourhood with their corresponding values. Using just this information each player must guess their own value. If all players guess correctly they will all win, but if just one player guesses incorrectly they will all lose. The objective of the players is to maximise their probability of winning.

As an example we consider the guessing game  $(K_n, s)$ , where  $K_n$  is the *complete (undirected) graph* of order  $n$ , i.e.  $|V(K_n)| = n$  and  $E(K_n) = \{uv : u, v \in V(G), u \neq v\}$ . Naively we may think that since each player receives no information about their own value that each player may as well guess randomly, meaning that the probability they win is  $s^{-n}$ . This however is not optimal. Certainly the

probability that any given player guesses correctly is  $1/s$ , but Riis [9] noticed that by discussing their strategies beforehand the players can in fact coordinate the moments where they guess correctly, and therefore increase their chance of winning. For example before the game begins they can agree that they will all play under the assumption that

$$\sum_{v \in V(K_n)} a_v \equiv 0 \pmod{s}. \quad (1)$$

Player  $u$  can see all the values except its own, and assuming (1) is true it knows that

$$a_u \equiv - \sum_{\substack{v \in V(K_n) \\ v \neq u}} a_v \pmod{s}.$$

Consequently player  $u$  will guess that its value is  $-\sum_{v \in V(K_n), v \neq u} a_v \pmod{s}$ . Hence if (1) is true every player will guess correctly and if (1) is false every player will guess incorrectly. So the probability they all guess correctly is simply the probability that (1) is true which is  $1/s$ . This is clearly optimal as, irrespective of the strategy, the probability that a single player guesses correctly is  $1/s$  and so we can not hope to do better.

We note that the optimal strategy given in the example was a *pure strategy* i.e. there is no randomness involved in the guess each player makes given the values it sees. The alternative is a *mixed strategy* in which the players randomly choose a strategy to play from a set of pure strategies. The winning probability of the mixed strategy is the average of the winning probabilities of the pure strategies weighted according to the probabilities that they are chosen. This however is at most the maximum of the winning probabilities of the pure strategies, and so we gain no advantage by playing a mixed strategy. As such throughout this paper we will only ever consider pure strategies.

Given a guessing game  $(G, s)$ , for  $v \in V(G)$  a *strategy for player  $v$*  is formally a function  $f_v : A_s^{|\Gamma^-(v)|} \rightarrow A_s$  which maps the values of the in-neighbours of  $v$  to an elements of  $A_s$ , which will be the guess of  $v$ . A *strategy  $\mathcal{F}$*  for a guessing game is a sequence of such functions  $(f_v)_{v \in V(G)}$  where  $f_v$  is a strategy for player  $v$ . We denote by  $\text{Win}(G, s, \mathcal{F})$  the event that all the players guess correctly when playing  $(G, s)$  with strategy  $\mathcal{F}$ . The players' objective is to find a strategy  $\mathcal{F}$  that maximises  $\mathbf{P}[\text{Win}(G, s, \mathcal{F})]$ .

Rather than trying to find  $\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s, \mathcal{F})]$  we will instead work with the *guessing number*  $\text{gn}(G, s)$  which we define as

$$\text{gn}(G, s) = |V(G)| + \log_s \left( \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s, \mathcal{F})] \right).$$

Although this looks like a cumbersome property to work with we can think of it as a measure of how much better the optimal strategy is over the strategy of just

making random guesses, as

$$\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s, \mathcal{F})] = \frac{s^{\text{gn}(G,s)}}{s^{|V(G)|}}.$$

Later we will look at information entropy inequalities as a way of analyzing the guessing game and in this context the definition of the guessing number will appear more natural.

### 3 The asymptotic guessing number

Note that the guessing number of the example  $(K_n, s)$  we discussed earlier is represented by  $\text{gn}(K_n, s) = n - 1$  which does not depend on  $s$ . In general  $\text{gn}(G, s)$  will depend on  $s$  and it is often extremely difficult to determine the guessing number exactly. Consequently we will instead concentrate our efforts on evaluating the *asymptotic guessing number*  $\text{gn}(G)$  which we define to be the limit of  $\text{gn}(G, s)$  as  $s$  tends to infinity. To prove the limit exists we first need to consider the guessing number on the blowup of  $G$ .

**Lemma 3.1.** *Given a digraph  $G$ , and integers  $s, t \geq 1$ ,*

$$\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G(t), s, \mathcal{F})] \geq \left( \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s, \mathcal{F})] \right)^t$$

*or equivalently  $\text{gn}(G(t), s) \geq t \text{gn}(G, s)$ .*

*Proof.* The digraph  $G(t)$  can be split into  $t$  vertex disjoint copies of  $G$ . We can construct a strategy for  $(G(t), s)$  by playing the optimal strategy of  $(G, s)$  on each of the  $t$  copies of  $G$  in  $G(t)$ . The result follows immediately.  $\square$

**Lemma 3.2.** *Given a digraph  $G$ , and integers  $s, t \geq 1$ ,*

$$\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G(t), s, \mathcal{F})] = \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s^t, \mathcal{F})]$$

*or equivalently  $\text{gn}(G(t), s) = t \text{gn}(G, s^t)$ .*

*Proof.* First we will show that the optimal probability of winning on  $(G, s^t)$  is at least that of  $(G(t), s)$ . This follows simply from the fact that the members of the alphabet of size  $s^t$ , can be represented as  $t$  digit numbers in base  $s$ . Hence given a strategy on  $(G(t), s)$ , a corresponding strategy can be played on  $(G, s^t)$  by each player pretending to be  $t$  players: More precisely, if player  $v$  gets assigned value  $a \in A_{s^t}$ , he writes it as  $a_{t-1} \cdots a_1 a_0$  in base  $s$  and pretends to be  $t$  players, say  $v_0, v_1, \dots, v_{t-1}$ , where player  $v_i$ , for  $0 \leq i \leq t - 1$ , gets assigned value  $a_i \in A_s$ . Furthermore, if player  $v$  sees the outcome of player  $u$ , then he can construct the values assigned to the new players  $u_0, u_1, \dots, u_{t-1}$ . So these new fictitious players can play the  $(G(t), s)$  game using an optimal strategy. But if the fictitious players

can win the  $(G(t), s)$  game then the original players can win the  $(G, s^t)$  game as we can reconstruct the value of  $a$  from the values of  $a_0, a_1, \dots, a_{t-1}$ .

A similar argument can be used to show that the optimal probability of winning on  $(G, s^t)$  is at most that of  $(G(t), s)$ . We will show that for every strategy on  $(G, s^t)$  there is a corresponding strategy on  $(G(t), s)$ . Every vertex class of  $t$  players can simulate playing as one fictitious player by its members agreeing to use the same strategy. The  $t$  values assigned to the players in the vertex class can be combined to give an overall value for the vertex class. The strategy on  $(G, s^t)$  can then be played allowing the members of the vertex class to make a guess for the overall value assigned to the vertex class. This guess will be the same for each member as they all agreed to use the same strategy and have access to precisely the same information. Once the guess for the vertex class is made its value can be decomposed into  $t$  values from  $A_s$  which can be used as the individual guesses for each of its members.  $\square$

Using these results about blowups of digraphs we can show that in some sense the guessing number is almost monotonically increasing with respect to the size of the alphabet.

**Lemma 3.3.** *Given any digraph  $G$ , positive integer  $s$ , and real number  $\varepsilon > 0$ , there exists  $t_0(G, s, \varepsilon) > 0$  such that for all integers  $t \geq t_0$*

$$\text{gn}(G, t) \geq \text{gn}(G, s) - \varepsilon.$$

*Proof.* We will prove the result by showing that

$$\text{gn}(G, t) \geq \frac{\lfloor \log_s t \rfloor}{\log_s t} \text{gn}(G, s) \tag{2}$$

holds for all  $t \geq s$ . This will be sufficient since as  $t$  increases the right hand side of (2) tends to  $\text{gn}(G, s)$ .

We will prove (2) by constructing a strategy for  $(G, t)$ . Let  $k = \lfloor \log_s t \rfloor$  and note that  $s^k$  is at most  $t$ . By considering only strategies in which every player is restricted to guess a value in  $\{0, 1, \dots, s^k - 1\}$  we get

$$\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, t, \mathcal{F})] \geq \mathbf{P}[a_v < s^k \text{ for all } v \in V(G)] \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s^k, \mathcal{F})].$$

Hence

$$\frac{t^{\text{gn}(G, t)}}{t^{|V(G)|}} \geq \left(\frac{s^k}{t}\right)^{|V(G)|} \frac{s^{k \text{gn}(G, s^k)}}{s^{k|V(G)|}}$$

which rearranges to

$$\text{gn}(G, t) \geq \frac{k}{\log_s t} \text{gn}(G, s^k). \tag{3}$$

From Lemmas 3.1 and 3.2 we can show  $\text{gn}(G, s^k) \geq \text{gn}(G, s)$  which together with (3) completes the proof of (2).  $\square$

**Theorem 3.4.** *For any digraph  $G$ ,  $\text{gn}(G) = \lim_{s \rightarrow \infty} \text{gn}(G, s)$  exists.*

*Proof.* By definition  $\text{gn}(G, s) \leq |V(G)|$  for all  $s$ , and  $\max_{s \leq n} \text{gn}(G, s)$  is an increasing sequence with respect to  $n$ , therefore its limit exists which we will call  $\ell$ . Since  $\text{gn}(G, s) \leq \ell$  for all  $s$  it will be enough to show that  $\text{gn}(G, s)$  converges to  $\ell$  from below.

By the definition of  $\ell$ , given  $\varepsilon > 0$  there exists  $s_0(\varepsilon)$  such that  $\text{gn}(G, s_0(\varepsilon)) \geq \ell - \varepsilon$ . From Lemma 3.3 we know that there exists  $t_0(\varepsilon)$  such that for all  $t \geq t_0(\varepsilon)$ ,  $\text{gn}(G, t) \geq \text{gn}(G, s_0(\varepsilon)) - \varepsilon$  which implies  $\text{gn}(G, t) \geq \ell - 2\varepsilon$  proving we have convergence.  $\square$

Before we move on to the next section it is worth mentioning that for any  $s$  the guessing number  $\text{gn}(G, s)$  is a lower bound for  $\text{gn}(G)$ . This follows immediately from Lemma 3.3. Furthermore for any strategy  $\mathcal{F}$  on  $(G, s)$  we have

$$\text{gn}(G, s) \geq |V(G)| + \log_s \mathbf{P}[\text{Win}(G, s, \mathcal{F})].$$

Consequently we can lower bound the asymptotic guessing number by considering any strategy on any alphabet size.

## 4 Lower bounds using the fractional clique cover

In this section we will describe a strategy specifically for undirected graphs. As shown in the previous section this can be used to provide a lower bound for the asymptotic guessing number. Christofides and Markström [3] conjectured that this bound always equals the asymptotic guessing number.

In Section 2 we saw that when an undirected graph is complete an optimal strategy is for each player to play assuming the sum of all the values is congruent to 0 mod  $s$  (where  $s$  is the alphabet size). We call this the *complete graph strategy*. We can generalise this strategy to undirected graphs which are not complete. We simply decompose the undirected graph into vertex disjoint cliques and then let the players play the complete graph strategy on each of the cliques. If we are playing on an alphabet of size  $s$  and we decompose the graph into  $t$  cliques, then on each clique the probability of winning is  $s^{-1}$  and so the probability of winning the guessing game, which is equal to the probability of winning in each of the cliques, is  $s^{-t}$ . Clearly the probability of winning is higher if we choose to decompose the graph into as few cliques as possible. The smallest number of cliques that we can decompose a graph into is called the *minimum clique cover number* of  $G$  and we will represent it by  $\kappa(G)$ . In this notation we have

$$\text{gn}(G) \geq \text{gn}(G, s) \geq |V(G)| - \kappa(G).$$

It is worth mentioning that finding the minimum clique cover number of a graph is equivalent to finding the chromatic number of the graph's complement. As

such it is difficult to determine this number in the sense that the computation of the chromatic number of a graph is an NP-complete problem [8].

We can improve this bound further by considering blowups of  $G$ . From Lemma 3.2 we know that  $\text{gn}(G, s^t) = \text{gn}(G(t), s)/t$ , hence by the clique cover strategy on  $G(t)$  we get a lower bound of  $|V(G)| - \kappa(G(t))/t$ . The question is now to determine  $\min_t \kappa(G(t))/t$ . We do this by looking at the fractional clique cover of  $G$ .

Let  $K(G)$  be the set of all cliques in  $G$ , and let  $K(G, v)$  be the set of all cliques containing vertex  $v$ . A *fractional clique cover* of  $G$  is a weighting  $w : K(G) \rightarrow [0, 1]$  such that for all  $v \in V(G)$

$$\sum_{k \in K(G, v)} w(k) \geq 1.$$

The minimum value of  $\sum_{k \in K(G)} w(k)$  over all choices of fractional clique covers  $w$  is known as *the fractional clique cover number* which we will denote by  $\kappa_f(G)$ . (Although we do not define it here, we point out that the fractional clique cover number of a graph is equal to the fractional chromatic number of its complement.)

For the purposes of guessing game strategies it will be more convenient to instead consider a special type of fractional clique cover called the regular fractional clique cover. A *regular fractional clique cover* of  $G$  is a weighting  $w : K(G) \rightarrow [0, 1]$  such that for all  $v \in V(G)$

$$\sum_{k \in K(G, v)} w(k) = 1.$$

The minimum value of  $\sum_{k \in K(G)} w(k)$  over all choices of regular fractional clique covers  $w$  can be shown to be equal to the fractional clique cover number  $\kappa_f(G)$ . To see this, observe firstly that since all regular fractional clique covers are fractional clique covers the minimum value of  $\sum_{k \in K(G)} w(k)$  over all choices of regular fractional clique covers  $w$  is at least  $\kappa_f(G)$ . Finally, to show it is at most  $\kappa_f(G)$  we simply observe that the optimal fractional clique cover can be made into a regular fractional cover by moving weights from larger cliques to smaller cliques. In particular, given a vertex  $v$  for which  $\sum_{k \in K(G, v)} w(k) > 1$  we pick a clique  $k_1 \in K(G, v)$  with  $w(k_1) > 0$  and proceed as follows: We change the weight of  $k_1$  from  $w(k_1)$  to

$$w'(k_1) = \max \left\{ 0, 1 - \sum_{\substack{k \in K(G, v) \\ k \neq k_1}} w(k) \right\} < w(k_1)$$

We also change the weight of the clique  $k'_1 = k_1 \setminus \{v\}$  from  $w(k'_1)$  to  $w'(k'_1) = w(k'_1) + w(k_1) - w'(k_1)$ . We leave the weight of all other vertices the same. In this way, the total sum of weights over all cliques remains the same, the total sum of weights over all cliques containing a given vertex  $v' \neq v$  also remains the same, but the total sum of weights over all cliques containing  $v$  is reduced. This

process has to terminate because whenever we change the weight of  $k_1$  it will either become equal to 0 or the total sum of weight of all cliques containing  $v$  will become equal to 1.

Clearly  $\kappa_f(G)$  and an optimal regular fractional clique cover  $w$  can be determined by linear programming. Since all the coefficients of the constraints and objective function are integers,  $w(k)$  will be rational for all  $k \in K(G)$  as will  $\kappa_f(G)$ . If we let  $d$  be the common denominator of all the weights, then  $dw(k)$  for  $k \in K(G)$  describes a clique cover of  $G(d)$ . In particular it decomposes  $G(d)$  into  $d\kappa_f(G)$  cliques, proving a lower bound of

$$\text{gn}(G) \geq |V(G)| - \kappa_f(G). \quad (4)$$

We claim that

$$\min_t \frac{\kappa(G(t))}{t} \geq \kappa_f(G)$$

and therefore we cannot hope to use regular fractional clique cover strategies to improve (4). To prove our claim we begin by observing that for all  $t$  we have  $\kappa(G(t)) \geq t\kappa_f(G)$ . This is immediate as a minimal clique cover is a special type of fractional clique cover, namely one where all weights are 0 or 1. Hence it is enough to show that

$$\kappa_f(G(t)) = t\kappa_f(G).$$

This can be proved simply from observing that an optimal weighting of  $K(G(t))$  can always be transformed into another optimal weighting which is symmetric with respect to vertices in the same vertex class. This can be done just by moving the weights between cliques. Therefore determining  $\kappa_f(G(t))$  is equivalent to determining  $\kappa_f(G)$  but with the constraints  $\sum_{k \in K(G,v)} w(k) = t$  rather than 1. The result  $\kappa_f(G(t)) = t\kappa_f(G)$  is a simple consequence of this.

A useful bound on  $\kappa_f(G)$  which we will make use of later is given by the following lemma.

**Lemma 4.1.** *For any undirected graph  $G$*

$$\kappa_f(G) \geq \frac{|V(G)|}{\omega(G)},$$

where  $\omega(G)$  is the number of vertices in a maximum clique in  $G$ .

*Proof.* Let  $w$  be an optimal regular fractional clique cover. Since

$$\sum_{k \in K(G,v)} w(k) = 1$$

holds for all  $v \in V(G)$ , summing both sides over  $v$  gives us,

$$\sum_{k \in K(G)} w(k)|V(k)| = |V(G)|,$$

where  $|V(k)|$  is the number of vertices in clique  $k$ . The result trivially follows from observing

$$\sum_{k \in K(G)} w(k)|V(k)| \leq \sum_{k \in K(G)} w(k)\omega(G) = \kappa_f(G)\omega(G). \quad \square$$

The result of Christofides and Markström [3] states the following:

**Theorem 4.2.** *If  $G$  is an undirected graph then*

$$\text{gn}(G) \geq |V(G)| - \kappa_f(G).$$

In [3] it was proved that the above lower bound is actually an equality for various families of undirected graphs including perfect graphs, odd cycles and complements of odd cycles. This led Christofides and Markström [3] to conjecture that we always have equality.

**Conjecture 4.3.** *If  $G$  is an undirected graph then*

$$\text{gn}(G) = |V(G)| - \kappa_f(G).$$

To prove or disprove such a claim we require a way of upper bounding  $\text{gn}(G)$ . This is the purpose of the next section.

## 5 Upper bounds using entropy

Recall that it is sufficient to only consider pure strategies on guessing games. Hence given a strategy  $\mathcal{F}$  on a guessing game  $(G, s)$  we can explicitly determine  $\mathcal{S}(\mathcal{F})$  the set of all assignment tuples  $(a_v)_{v \in V(G)}$  that result in the players winning given they are playing strategy  $\mathcal{F}$ . In this context the players' objective is to choose a strategy that maximizes  $|\mathcal{S}(\mathcal{F})|$ . We have

$$\begin{aligned} \text{gn}(G, s) &= |V(G)| + \log_s \left( \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s, \mathcal{F})] \right) \\ &= |V(G)| + \max_{\mathcal{F}} \log_s \frac{|\mathcal{S}(\mathcal{F})|}{s^{|V(G)|}} \\ &= \max_{\mathcal{F}} \log_s |\mathcal{S}(\mathcal{F})|. \end{aligned}$$

Consider the probability space on the set of all assignment tuples  $A_s^{|V(G)|}$  with the members in  $\mathcal{S}(\mathcal{F})$  occurring with uniform probability and all other assignments occurring with 0 probability. For each  $v \in V(G)$  we define the discrete random variable  $X_v$  on this probability space to be the value assigned to vertex  $v$ . The  $s$ -entropy of a discrete random variable  $X$  with outcomes  $x_1, x_2, \dots, x_n$  is defined as

$$H_s(X) = - \sum_{i=1}^n \mathbf{P}[X = x_i] \log_s \mathbf{P}[X = x_i],$$

where we take  $0 \log_s 0$  to be 0 for consistency. Note that traditionally entropy is defined using base 2 logarithms, however it will be more convenient for us to work with base  $s$  logarithms. We will usually write  $H(X)$  instead of  $H_s(x)$ . We will mention here all basic results concerning entropy that we are going to use. For more information, we refer the reader to [5].

Given a set of random variables  $Y_1, \dots, Y_n$  with sets of outcomes  $\text{Im}(Y_1), \dots, \text{Im}(Y_n)$  respectively, the *joint entropy*  $H(Y_1, \dots, Y_n)$  is defined as

$$- \sum_{y_1 \in \text{Im}(Y_1)} \cdots \sum_{y_n \in \text{Im}(Y_n)} \mathbf{P}[Y_1 = y_1, \dots, Y_n = y_n] \log_s \mathbf{P}[Y_1 = y_1, \dots, Y_n = y_n].$$

Given a set of random variables  $Y = \{Y_1, \dots, Y_n\}$  we will also use the notation  $H(Y)$  to represent the joint entropy  $H(Y_1, \dots, Y_n)$ . Furthermore for sets of random variables  $Y$  and  $Z$  we will use the notation  $H(Y, Z)$  as shorthand for  $H(Y \cup Z)$ . For completeness we also define  $H(\emptyset) = 0$ .

Observe that under these definitions, the joint entropy of the set of variables  $X_G = \{X_v : v \in V(G)\}$  is

$$\begin{aligned} H(X_G) &= - \sum_{(a_v) \in \mathcal{S}(\mathcal{F})} \frac{1}{|\mathcal{S}(\mathcal{F})|} \log_s \left( \frac{1}{|\mathcal{S}(\mathcal{F})|} \right) \\ &= \log_s |\mathcal{S}(\mathcal{F})| \end{aligned}$$

Therefore by upper bounding  $H(X_G)$  for all choices of  $\mathcal{F}$  we can upper bound  $\text{gn}(G, s)$ .

We begin by stating some inequalities that most hold regardless of  $\mathcal{F}$ .

**Theorem 5.1.** *Given  $X, Y, Z \subset X_G$ ,*

1.  $H(X) \geq 0$ .
2.  $H(X) \leq |X|$ .
3. *Shannon's information inequality:*

$$H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z) \geq 0.$$

4. *Suppose  $A, B \subset V(G)$  with  $\Gamma^-(u) \subset B$  for all  $u \in A$ . Let  $X = \{X_v : v \in A\}$  and  $Y = \{X_v : v \in B\}$ . Then*

$$H(X, Y) = H(Y).$$

*Proof.*

1. Property 1 follows immediately from the definition of entropy.

2. Property 2 follows from first observing that  $H(X) = \mathbf{E}[\log_s(1/\mathbf{P}[X])]$ . Since the function  $x \mapsto \log_s(x)$  is concave, by Jensen's inequality we get that

$$H(X) \leq \log_s \mathbf{E}[1/\mathbf{P}[X]] = \log_s |\text{Im}(X)|$$

where  $\text{Im}(X)$  is the set of outcomes for  $X$ . Since  $\text{Im}(X) = A_s^{|X|}$  we have the desired inequality  $H(X) \leq |X|$ .

3. Property 3 again follows from Jensen's inequality. First we observe that

$$H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z) = \mathbf{E}_{X,Y,Z} \left[ -\log_s \left( \frac{\mathbf{P}[X, Z]\mathbf{P}[Y, Z]}{\mathbf{P}[X, Y, Z]\mathbf{P}[Z]} \right) \right].$$

By an application of Jensen's inequality this is at least

$$-\log_s \left( \mathbf{E}_{X,Y,Z} \left[ \frac{\mathbf{P}[X, Z]\mathbf{P}[Y, Z]}{\mathbf{P}[X, Y, Z]\mathbf{P}[Z]} \right] \right) = -\log_s \left( \sum_{X,Y,Z} \frac{\mathbf{P}[X, Z]\mathbf{P}[Y, Z]}{\mathbf{P}[Z]} \right) = 0.$$

4. Property 4 is a simple consequence of the fact that the values assigned to the vertices in  $A$  are completely determined by the values assigned to the vertices in  $B$ . Since  $\mathbf{P}[X, Y]$  is either 0 or  $\mathbf{P}[Y]$ , the result is trivially attained by considering the definition of  $H(X, Y)$  and summing over the variables in  $X$ .  $\square$

From Theorem 5.1 we can form a linear program to upper bound  $H(X_G)$ . In particular the linear program consists of  $2^{|V(G)|}$  variables corresponding to the values of  $H(X)$  for each  $X \subset X_G$ . The variables are constrained by the linear inequalities given in Theorem 5.1 and the objective is to maximize the value of the variable corresponding to  $H(X_G)$ . We call the result of the optimization *the Shannon bound* of  $G$  and denote it by  $\text{Sh}(G)$ .

Note that  $\text{Sh}(G)$  can be calculated without making any explicit use of  $\mathcal{F}$  or  $s$ . Hence it is not only an upper bound on  $\text{gn}(G, s)$  but also on  $\text{gn}(G)$ .

More recently information entropy inequalities that cannot be derived from linear combinations of Shannon's inequality (Property 3 in Theorem 5.1) have been discovered. The first such inequality was found by Zhang and Yeung [12]. The *Zhang-Yeung inequality* states that

$$\begin{aligned} -2H(A) - 2H(B) - H(C) + 3H(A, B) + 3H(A, C) + H(A, D) + \\ 3H(B, C) + H(B, D) - H(C, D) - 4H(A, B, C) - H(A, B, D) \geq 0 \end{aligned}$$

for sets of random variables  $A, B, C, D$ . By setting  $A = X \cup Z$ ,  $B = Z$ ,  $C = Y \cup Z$ ,  $D = Z$ , the Zhang-Yeung inequality reduces to Shannon's inequality. By replacing the Shannon inequality constraints with those given by the Zhang-Yeung inequality we can potentially get a better upper bound from the linear program.

However, we pay for this potentially better bound by a significant increase in the running time of the linear program. We will call the bound on  $\text{gn}(G)$  obtained by use of the Zhang-Yeung inequality *the Zhang-Yeung bound* and denote it by  $\text{ZY}(G)$ .

In fact there are known to be infinite families of non-Shannon inequalities even on 4 variables. We cannot hope to add infinite constraints to the linear program so instead we will consider the 214 inequalities given by Dougherty, Freiling, and Zeger [7, Section VIII]. We will refer to the resulting bound as *the Dougherty-Freiling-Zeger bound* and denote it by  $\text{DFZ}(G)$ . It is perhaps worth mentioning for those interested that the 214 Dougherty-Freiling-Zeger inequalities imply the Zhang-Yeung inequality (simply sum inequalities 56 and 90) and therefore they also imply Shannon's inequality.

The final bound we will consider is *the Ingleton bound* which we will denote by  $\text{Ingl}(G)$ . This is obtained when we replace the Shannon inequality constraints with the *Ingleton inequality*

$$-H(A) - H(B) + H(A, B) + H(A, C) + H(A, D) + H(B, C) + H(B, D) - H(C, D) - H(A, B, C) - H(A, B, D) \geq 0.$$

The Ingleton inequality provides the outer-bound of the inner-cone of linearly representable entropy vectors [4]. By setting  $A = Z, C = Y$  and  $B = D = X \cup Z$ , the Ingleton inequality reduces to Shannon's inequality.

If each player's strategy can be expressed as a linear combination of the values it sees, then the Ingleton inequality will hold. Therefore the inequality holds for a strategy on  $(G, s^t)$  that can be represented as a linear strategy on  $(G(t), s)$  (as described in the proof of Lemma 3.2). As such, the Ingleton bound gives us an upper bound when we restrict ourselves to strategies which are linear on the digits of the values. An important such strategy is the fractional clique cover strategy [3] which leads to the proof of Theorem 4.2.

In searching for a counterexample to Conjecture 4.3 we carried out an exhaustive search on all undirected graphs with at most 9 vertices. We compared the lower bound given by the fractional clique cover with the upper bound given by the Shannon bound and in all cases the two bounds matched. The bounds were calculated using floating point arithmetic and so we do not claim this search to be rigorous, however it suggested the following conjecture.

**Conjecture 5.2.** *If  $G$  is an undirected graph then  $\text{gn}(G) = \text{Sh}(G)$ .*

## 6 Main results

In this section we present our new results, most notably that both Conjectures 4.3 and 5.2 are false. Counterexamples were found by searching through all undirected graphs on 10 vertices or less. For speed purposes, the search was done using floating point arithmetic and as such there may be counterexamples that

were missed due to rounding errors. (Although this is highly unlikely, we do not claim that it is impossible.) Despite this, we feel that it is still remarkable that of the roughly 12 million graphs that were checked we only found 2 graphs whose lower and upper bounds (given by the fractional clique cover, and Shannon bound respectively) did not match: the graph  $R$  given in Figure 1, and the graph  $R^-$  which is identical to  $R$  but with the undirected edge between vertices 9 and 10 removed.

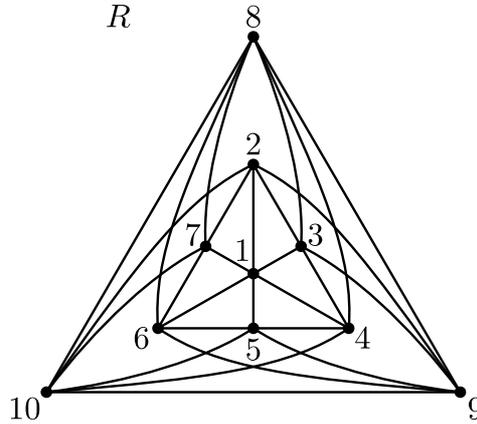


Figure 1: The undirected graph  $R$ .

The graph  $R$  is particularly extraordinary as we will see that with a few simple modifications we can create graphs which answer a few other open problems.

We begin our analysis of  $R$  and  $R^-$  by determining their fractional clique cover number.

**Lemma 6.1.** *We have  $\kappa_f(R) = \kappa_f(R^-) = 10/3$ .*

*Proof.* By Lemma 4.1 we know that  $\kappa_f(R)$  and  $\kappa_f(R^-)$  are bounded below by  $10/3$ . To show they can actually attain  $10/3$  we need to construct explicit regular fractional clique covers whose weights add up to  $10/3$ .

For  $R^-$  we give a weight of  $1/3$  to the cliques  $\{1, 2, 3\}$ ,  $\{1, 4, 5\}$ ,  $\{1, 6, 7\}$ ,  $\{2, 3, 9\}$ ,  $\{2, 7, 10\}$ ,  $\{3, 8, 9\}$ ,  $\{4, 5, 10\}$ ,  $\{4, 8, 10\}$ ,  $\{5, 6, 9\}$ ,  $\{6, 7, 8\}$ , and a weight of 0 to all other cliques. Note that this is also an optimal regular fractional clique cover for  $R$ .  $\square$

**Theorem 6.2.** *We have*

1.  $\text{Sh}(R^-) = 114/17 = 6.705\dots$
2.  $\text{ZY}(R^-) = 1212/181 = 6.696\dots$
3.  $\text{DFZ}(R^-) = 59767/8929 = 6.693\dots$
4.  $\text{Ingl}(R^-) = 20/3 = 6.666\dots$

From Lemma 6.1 and Theorem 6.2 we know that

$$20/3 \leq \text{gn}(R^-) \leq 59767/8929,$$

and although we could not determine the asymptotic guessing number exactly it does show that it does not equal the Shannon bound, disproving Conjecture 5.2. Given that the Shannon bound is not sharp we might be tempted to conjecture that the asymptotic guessing number is the same as the Zhang-Yeung bound, but Theorem 6.2 also shows this to be false. Interestingly the Ingleton bound does match the lower bound, showing that if we restrict ourselves to only considering linear strategies on blowups we can do no better than the fractional clique cover strategy.

It remains an open question as to whether a non-linear strategy on  $R^-$  can do better than  $20/3$  or whether by considering the right set of entropy inequalities we can push the upper bound down to  $20/3$ .

*Proof of Theorem 6.2.* Calculating the upper bounds involves solving rather large linear programs. Hence the proofs are too long to reproduce here and it is unfeasible for them to be checked by humans. Data files verifying our claims can be provided upon request. We stress that although the results were verified using a computer that no floating point data types were used during the verification. Consequently no rounding errors could occur in the calculations making the results completely rigorous.  $\square$

Although  $R$  is a counterexample to Conjecture 4.3 its optimal strategy is somewhat complicated. So instead we will disprove the conjecture by showing a related graph which we will call  $R_c$  is a counterexample. The undirected graph  $R_c$  is constructed from  $R$  by *cloning* 3 of its vertices. (Cloning 3 vertices is equivalent to creating a blowup of  $R$  with 2 vertices in 3 of the vertex classes and just 1 vertex in the other classes.) The vertices we clone are 8, 9, 10, and we label the resulting new vertices  $8'$ ,  $9'$ , and  $10'$  respectively.

**Theorem 6.3.** *We have  $\text{gn}(R_c) = 9$  while the fractional clique cover bound of  $R_c$  is  $26/3 < 9$ . In particular,  $R_c$  provides a counterexample to Conjecture 4.3.*

*Proof.* To prove that the fractional clique cover bound is  $26/3$  it is enough to show that  $\kappa_f(R_c) = 13 - 26/3 = 13/3$ . Lemma 4.1 tells us  $\kappa_f(R_c) \geq 13/3$ . It is also easy to show  $\kappa_f(R_c) \leq 13/3$  as it trivially follows from extending the regular fractional clique cover given in the proof of Lemma 6.1 by giving a weight of 1 to the clique  $\{8', 9', 10'\}$ .

The Shannon bound of  $R_c$  is 9 proving  $\text{gn}(R_c) \leq 9$ . We do not provide the details of the Shannon bound proof as it is too long to present here, however data files containing the proof are available upon request.

All that remains is to prove  $\text{gn}(R_c) \geq 9$ . Even though this proof was discovered partly using a computer it can be easily verified by humans. In particular, the

main conclusion of this theorem, that  $R_c$  is a counterexample to Conjecture 4.3, can be verified without the need of any computing power.

Recall that in Section 3 we showed that the asymptotic guessing number can be lower bounded by considering any strategy on any alphabet size. We will take our alphabet size  $s$  to be 3. Our strategy involves all players agreeing to play assuming the following four conditions hold on the assigned values

$$a_1 + a_2 + 2a_3 + a_4 + 2a_5 + a_6 + 2a_7 \equiv 0 \pmod{3}, \quad (5)$$

$$a_2 + a_5 + a_8 + a_{8'} + a_9 + a_{10'} \equiv 0 \pmod{3}, \quad (6)$$

$$a_3 + a_6 + a_8 + a_{9'} + a_{10} + a_{10'} \equiv 0 \pmod{3}, \quad (7)$$

$$a_4 + a_7 + a_{8'} + a_9 + a_{9'} + a_{10} \equiv 0 \pmod{3}. \quad (8)$$

Note that the terms in (5) consist of  $a_1$ , and values which player 1 can see. Hence (5) naturally gives us a strategy for player 1, i.e. that player 1 should guess  $-a_2 - 2a_3 - a_4 - 2a_5 - a_6 - 2a_7 \pmod{3}$ . Similarly strategies for players 8, 8', 9, 9', 10, and 10' can be achieved by rearranging conditions (7), (8), (6), (7), (8) and (6) respectively. A strategy for player 2 can be obtained by taking a linear combination of the conditions. In particular if we sum (7), (8), twice (5), and twice (6) we get

$$2a_1 + a_2 + 2a_3 + 2a_7 + 2a_{9'} + 2a_{10} \equiv 0 \pmod{3},$$

which consists of  $a_2$  and values which player 2 can see, allowing us to construct a strategy for player 2. We leave it to the reader to verify that by taking the following linear combinations we obtain strategies for players 3, 4, 5, 6, and 7:

- For player 3, we sum (5),(6),(8) and twice (7).
- For player 4, we sum (6),(7), twice (5) and twice (8).
- For player 5, we sum (5),(7),(8) and twice (6).
- For player 6, we sum (6),(8), twice (5) and twice (7).
- For player 7, we sum (5),(6),(7) and twice (8).

The probability that all players guess correctly under this strategy is  $3^{-4}$ , i.e. the probability that (5), (6), (7), (8) all hold. (It is not difficult to check that the conditions are linearly independent.) Consequently

$$\text{gn}(R_c) \geq |V(R_c)| + \log_3 \mathbf{P}[\text{Win}(R_c, 3, \mathcal{F})] = 9$$

as desired. □

For completeness we give the asymptotic guessing number of  $R$  and note that it does not match the fractional clique cover bound of  $20/3$  as claimed.

**Theorem 6.4.** *We have  $\text{gn}(R) = 27/4$ .*

*Proof.* The Shannon bound of  $R$  is  $27/4$  (data files can be provided upon request).

To show  $\text{gn}(R) \geq 27/4$  we will show  $\text{gn}(R, 81) \geq 27/4$ . By Lemma 3.2 this can be achieved if we can construct a strategy on the guessing game  $(R(4), 3)$  which has a probability of winning  $3^{-13}$  (which implies  $\text{gn}(R(4), 3) \geq 27$ ). To describe such a strategy let us label the vertices of  $R(4)$  such that the four vertices that are constructed from blowing up  $v \in V(R)$  are labelled  $v_a, v_b, v_c,$  and  $v_d$ . Under this labelling our strategy for  $R(4)$  is to have the cliques  $\{1_a, 2_a, 3_a\}, \{1_b, 4_a, 5_a\}, \{1_c, 6_a, 7_a\}, \{2_b, 3_b, 9_a\}, \{2_c, 3_c, 9_b\}, \{4_b, 5_b, 10_a\}, \{4_c, 5_c, 10_b\}, \{6_b, 7_b, 8_a\}$  and  $\{6_c, 7_c, 8_b\}$  play the complete graph strategy, and the remaining 13 vertices, which form a copy of  $R_c$ , to play the strategy for  $R_c$  as described in the proof of Theorem 6.3.  $\square$

Now that we have shown that Conjectures 4.3 and 5.2 are not true, we turn our attention to other open questions. Due to the limited tools and methods currently available, there are many seemingly trivial problems on guessing games which still remain unsolved. One such problem is the following.

**Problem 6.5.** *Does there exist an undirected graph whose asymptotic guessing number increases when a single directed edge is added?*

Adding a directed edge gives one of the players more information, which cannot lower the probability that the players win. However, surprisingly it seems extremely difficult to make use of the extra directed edge to increase the asymptotic guessing number. An exhaustive (but not completely rigorous) search on undirected graphs with 9 vertices or less did not yield any examples.

As such, we significantly weaken the requirements in Problem 6.5 by introducing the concept of a Superman vertex. We define a Superman vertex to be one that all other vertices can see. I.e., given a digraph  $G$ , we call vertex  $u \in V(G)$  a *Superman vertex* if  $uv \in E(G)$  for all  $v \in V(G) \setminus \{u\}$ . We can similarly define a Luthor vertex as one which sees all other vertices. To be precise  $u$  is a *Luthor vertex* if  $vu \in E(G)$  for all  $v \in V(G) \setminus \{u\}$ .

**Problem 6.6.** *Does there exist an undirected graph whose asymptotic guessing number increases when directed edges are added to change one of the vertices into a Superman vertex (or a Luthor vertex)?*

To change one of the vertices into a Superman or Luthor vertex will often involve adding multiple directed edges, meaning the players will have a lot more information at their disposal when making their guesses. We again searched all undirected graphs on 9 vertices or less and remarkably still could not find any examples.

With the discovery of the graph  $R$  and in particular the graph  $R_c$  we can show the answer is yes to Problem 6.5 and consequently Problem 6.6. We define the undirected graph  $R_c^-$  to be the same as the graph  $R_c$  but with the undirected edge between vertices 3 and 8 removed. We also define the directed graph  $R_c^+$

to be the same as  $R_c^-$  but with the addition of a single directed edge going from vertex 3 to vertex 8.

**Theorem 6.7.** *We have  $\text{gn}(R_c^-) = 53/6$  and  $\text{gn}(R_c^+) = 9$ .*

*Proof.* The Shannon bounds for  $R_c^-$  and  $R_c^+$  are  $53/6$  and  $9$  respectively (data files can be provided upon request).

We will prove  $\text{gn}(R_c^+) \geq 9$  by observing that the strategy for  $(R_c, 3)$  (see the proof of Theorem 6.3) is a valid strategy for  $(R_c^+, 3)$ . With the exception of player 3 all players in  $(R_c^+, 3)$  have access to the same information they did in  $(R_c, 3)$ . Player 3 however, now no longer has access to  $a_8$ . By studying the strategy player 3 uses in  $(R_c, 3)$  we will see that this is of no consequence. Summing conditions (5), (6), (8), and twice (7), gives

$$a_1 + 2a_2 + a_3 + 2a_4 + 2a_{8'} + 2a_9 \equiv 0 \pmod{3},$$

hence player 3 guesses  $-a_1 - 2a_2 - 2a_4 - 2a_{8'} - 2a_9 \pmod{3}$  in  $(R_c, 3)$ . Since player 3 makes no use of  $a_8$  this validates our claims.

We complete our proof by showing  $\text{gn}(R_c^-) \geq 53/6$ . We know  $\text{gn}(R_c^-) \geq \text{gn}(R_c^-, 3^6) = \text{gn}(R_c^-(6), 3)/6$  so it is enough to show  $\text{gn}(R_c^-(6), 3) \geq 53$ . Since  $R_c^-(6)$  had 78 vertices we can do this by finding a strategy on  $(R_c^-(6), 3)$  that wins with a probability of  $3^{-25}$ . To this end, let us label the vertices of  $R_c^-(6)$  such that the six vertices that are constructed from blowing up  $v \in V(R_c^-)$  are labelled  $v_a, v_b, v_c, v_d, v_e$ , and  $v_f$ . Under this labelling, our strategy for  $R_c^-(6)$  is to play the complete graph strategy on the cliques

$$\begin{array}{ccccc} \{1_a, 2_a, 3_a\}, & \{1_b, 2_b, 7_a\}, & \{1_c, 3_b, 4_a\}, & \{2_c, 3_c, 9'_a\}, & \{4_b, 5_a, 10'_a\}, \\ \{4_c, 5_b, 10'_b\}, & \{5_c, 6_a, 9'_b\}, & \{6_b, 7_b, 8_a\}, & \{6_c, 7_c, 8_b\}, & \{8_c, 9'_c, 10'_c\}, \\ \{8_d, 9'_d, 10'_d\}, & \{8_e, 9'_e, 10'_e\}, & \{8_f, 9'_f, 10'_f\}, & & \end{array}$$

and to play the  $R_c$  strategy on the vertices

$$\begin{array}{l} \{1_d, 2_d, 3_d, 4_d, 5_d, 6_d, 7_d, 8'_a, 8'_b, 9_a, 9_b, 10_a, 10_b\}, \\ \{1_e, 2_e, 3_e, 4_e, 5_e, 6_e, 7_e, 8'_c, 8'_d, 9_c, 9_d, 10_c, 10_d\}, \\ \{1_f, 2_f, 3_f, 4_f, 5_f, 6_f, 7_f, 8'_e, 8'_f, 9_e, 9_f, 10_e, 10_f\}. \end{array}$$

The probability of winning in each of these 13 cliques is  $3^{-1}$  while the probability of winning in each of the three copies of  $R_c$  is  $3^{-4}$ . So the overall probability of winning is indeed  $3^{-25}$ , therefore completing the proof.  $\square$

We finish this section by considering a problem motivated by the reversibility of networks in network coding. Given a digraph  $G$ , let  $\text{Reverse}(G)$  be the digraph formed from  $G$  by reversing all the edges, i.e.  $uv \in E(G)$  if and only if  $vu \in E(\text{Reverse}(G))$ .

**Problem 6.8.** *Does there exist a digraph  $G$ , such that  $\text{gn}(G) \neq \text{gn}(\text{Reverse}(G))$ .*

We were not able to solve this problem. We did however find a graph  $R^S$  for which the Shannon bound of  $R^S$  and the Shannon bound of  $\text{Reverse}(R^S)$  did not match.  $R^S$  is simply the digraph formed by making vertex 1 in  $R$  a Superman vertex. In other words, we add three directed edges to  $R$ : the edge going from 1 to 8, from 1 to 9, and from 1 to 10. Consequently  $\text{Reverse}(R^S)$  is the graph formed by making vertex 1 in  $R$  a Luthor vertex. As such, we will refer to it as  $R^L$ .

**Theorem 6.9.** *We have  $\text{Sh}(R^S) = 27/4 = 6.75$ . For  $R^L$  we have the following bounds:*

1.  $\text{Sh}(R^L) = 34/5 = 6.8$ .
2.  $\text{ZY}(R^L) = 61/9 = 6.777\dots$
3.  $\text{DFZ}(R^L) = 359/53 = 6.773\dots$
4.  $\text{Ingl}(R^L) = 27/4 = 6.75$ .

*Proof.* The proofs are given in data files which can be made available upon request. □

From the strategy on  $R$  we know that  $\text{gn}(R^S) \geq 27/4$  and  $\text{gn}(R^L) \geq 27/4$ . Hence we have  $\text{gn}(R^S) = 27/4$ . We do not however know the precise value of  $\text{gn}(R^L)$  so it is possible that the asymptotic guessing numbers of  $R^S$  and  $R^L$  do not match.

## 7 Speeding up the computer search

In this section we mention a few of the simple tricks we used in order to speed up the computer search which allowed us to search through all the 10 vertex graphs and find the graph  $R$ . We hope that this may be of use to others continuing this research.

The majority of time spent during the searches was spent determining the Shannon bound by solving a large linear program. By reducing the number of constraints that we add to the linear program we can speed up the optimisation. Given a graph on  $n$  vertices a naive formation of the linear program would result in considering all  $2^{3n}$  Shannon inequalities of the form

$$H(A, C) + H(B, C) - H(A, B, C) - H(C) \geq 0 \text{ for } A, B, C \subset X_G.$$

However most of these do not need to be added to the linear program. In fact it is sufficient to just include the inequalities given by the following lemma.

**Lemma 7.1.** *Given a set of discrete random variables  $X_G$ , the set of Shannon inequalities*

$$H(A, C) + H(B, C) - H(A, B, C) - H(C) \geq 0 \text{ for } A, B, C \subset X_G,$$

*is equivalent to the set of inequalities given by*

- (i)  $H(Y) \leq H(X_G)$  for  $Y \subset X_G$  with  $|Y| = |X_G| - 1$ .
- (ii)  $H(Y) + H(Z) - H(Y \cup Z) - H(Y \cap Z) \geq 0$  for  $Y, Z \subset X_G$  with  $|Y| = |Z| = |Y \cap Z| + 1$ .

Observe that for a graph on  $n$  vertices there are  $n$  inequalities of type (i) and  $n(n-1)2^{n-3}$  inequalities of type (ii). (Counting the inequalities of type (ii) is equivalent to counting the number of squares in the hypercube poset formed from looking at the subsets of  $X_G$ .) Overall, this is about the cube root of the initial number of inequalities.

*Proof of Lemma 7.1.* Setting  $A = X_G$ ,  $B = X_G$ , and  $C = Y$ , shows that the Shannon inequalities imply the set of inequalities described by (i). Setting  $A = Y$ ,  $B = Z$ , and  $C = Y \cap Z$ , shows that the Shannon inequalities imply (ii).

To show (i) and (ii) imply the Shannon inequalities we will first generalise (i) and (ii).

We will begin by showing that (ii) implies

$$H(Y) + H(Z) - H(Y \cup Z) - H(Y \cap Z) \geq 0$$

for any  $Y, Z \subset X_G$ . Let  $Y \setminus (Y \cap Z) = \{Y_1, Y_2, \dots, Y_n\}$  and  $Z \setminus (Y \cap Z) = \{Z_1, Z_2, \dots, Z_m\}$ , where  $Y_1, \dots, Y_n$  and  $Z_1, \dots, Z_m$  are single discrete random variables. Define  $Y'_i$  to be  $\{Y_1, \dots, Y_i\}$  for  $1 \leq i \leq n$  and  $Y'_0 = \emptyset$ . We define  $Z'_j$  similarly. Finally let  $X_{i,j} = (Y \cap Z) \cup Y'_i \cup Z'_j$ , and note that  $X_{0,0} = Y \cap Z$ ,  $X_{n,0} = Y$ ,  $X_{0,m} = Z$ , and  $X_{n,m} = Y \cup Z$ . By (ii) we have

$$0 \leq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} [H(X_{i+1,j}) + H(X_{i,j+1}) - H(X_{i+1,j+1}) - H(X_{i,j})].$$

Here, the right hand side is telescopic and simplifies to the desired expression

$$H(Y) + H(Z) - H(Y \cup Z) - H(Y \cap Z).$$

Next we will generalise (i) to show that for any  $Y \subset Z \subset X_G$  with  $|Y| = |Z| - 1$  we have  $H(Y) \leq H(Z)$ . Let us define  $\bar{Z}$  to be  $X_G \setminus Z$ . Then, by the generalised version of (ii) we know that

$$H(Z) + H(Y \cup \bar{Z}) - H(Z \cup (Y \cup \bar{Z})) - H(Z \cap (Y \cup \bar{Z})) \geq 0$$

which simplifies to

$$H(Z) + H(Y \cup \overline{Z}) - H(X_G) - H(Y) \geq 0. \quad (9)$$

Observe that  $|Y \cup \overline{Z}| = |X_G| - 1$ , so (i) tells us that  $H(X_G) - H(Y \cup \overline{Z}) \geq 0$  which when added to (9) gives the inequality  $H(Z) - H(Y) \geq 0$  as required.

We can now further generalise (i) to show that for any  $Y \subset Z \subset X_G$  we have that  $H(Y) \leq H(Z)$ . To do this, let  $Z \setminus Y = \{Z_1, Z_2, \dots, Z_n\}$ , where  $Z_1, \dots, Z_n$  are single discrete random variables. Then, by repeated applications of our generalisation of (i) we have

$$H(Y) \leq H(Y, Z_1) \leq H(Y, Z_1, Z_2) \leq \dots \leq H(Y, Z_1, \dots, Z_n) = H(Z).$$

It is now a trivial matter to show that (i) and (ii) imply Shannon's inequality. Simply set  $Y = A \cup C$  and  $Z = B \cup C$  in the generalised version of (ii) to get

$$H(A, C) + H(B, C) - H(A, B, C) - H(A \cap B, C) \geq 0$$

and since  $H(A \cap B, C) \geq H(C)$  by the improved version of (i), the result follows.  $\square$

It is also worth mentioning that  $H(\emptyset) = 0$  together with the Shannon inequalities imply  $H(Y, Z) \leq H(Y) + H(Z)$  for disjoint  $Y, Z$ . Hence, the constraints  $H(X) \leq |X|$  for all  $X$  in the Shannon bound linear program are not all necessary and can be reduced to  $H(X) \leq |X|$  for  $|X| = 0$ , or 1.

When determining each graph's asymptotic guessing number, the natural approach is to calculate the lower bound using the fractional clique cover number, then calculate the Shannon bound and check if they match. However the linear program that gives us the fractional clique cover number also gives us a regular fractional clique cover from which an explicit strategy can be constructed. It is easy to convert this strategy into a feasible point of the Shannon bound linear program. Hence we can save a significant amount of time by simply checking if this feasible point is optimal, rather than by calculating the Shannon bound from scratch. Note that we check for optimality by solving the same Shannon bound linear program with the modification that we remove those constraints for which equality is not achieved by the feasible point.

The modified Shannon bound linear program is still the most time consuming process in the search, so ideally we would like to avoid it when possible. Christofides and Markström [3] show that for an undirected graph  $G$

$$\text{gn}(G) \leq |V(G)| - \alpha(G),$$

where  $\alpha(G)$  is the number of vertices in the maximum independent set. This can be interpreted as a simple consequence of the fact that removing players increases the probability the remaining players will win. (If the probability of winning decreased, the players could just create fictitious replacement players before the game started.) As such we present a simple generalization of this result.

**Lemma 7.2.** *Given a digraph  $G$  and an induced subgraph  $G'$ ,*

$$\max_{\mathcal{F}} \mathbf{P}[\text{Win}(G, s, \mathcal{F})] \leq \max_{\mathcal{F}} \mathbf{P}[\text{Win}(G', s, \mathcal{F})]$$

*or equivalently  $\text{gn}(G, s) \leq |V(G)| - |V(G')| + \text{gn}(G', s)$ . Hence*

$$\text{gn}(G) \leq |V(G)| - |V(G')| + \text{gn}(G').$$

We do not provide a proof as it is trivial. Note that the result  $\text{gn}(G) \leq |V(G)| - \alpha(G)$  is a simple corollary of this result as an independent set has a guessing number of 0.

Given a graph  $G$ , if we can find a subgraph such that the upper bound given in Lemma 7.2 matches the fractional clique cover bound, then we have determined the asymptotic guessing number, and can avoid an expensive Shannon bound calculation. This approach is particularly fast when doing an exhaustive search as all the smaller graphs will have had their asymptotic guessing numbers already determined.

One issue with this method is that if we are looking for a counterexample to the Shannon bound being sharp, there is a possibility that we may miss them because we avoided calculating the Shannon bound for every graph. Consequently to alleviate our fear we need the following result.

**Lemma 7.3.** *Given a digraph  $G$  and an induced subgraph  $G'$ , we have*

$$\text{Sh}(G) \leq |V(G)| - |V(G')| + \text{Sh}(G').$$

*Proof.* It is sufficient to prove the result only for induced subgraphs  $G'$  for which  $|V(G)| - |V(G')| = 1$ , as the result then follows by induction on  $|V(G)| - |V(G')|$ . Let  $u \in V(G)$  be the vertex that is removed from  $G$  to produce  $G'$ .

The Shannon bound for  $G'$  comes from solving a linear program, and as such the solution to the dual program naturally gives us a proof that  $H(X_{G'}) \leq \text{Sh}(G')$ . In particular, this proof consists of summing appropriate linear combinations of the constraints. Suppose that in each such constraint we replace  $H(X)$  with  $H(X, X_u) - H(X_u)$  for every  $X \subset X_{G'}$ . This effectively would replace constraints from the linear program for  $G'$  with inequalities which are implied from the linear program for  $G$ . For example,  $H(X) \geq 0$  for  $G'$ , would become  $H(X, X_u) - H(X_u) \geq 0$  for  $G$  (which is true by Shannon's inequality). As another example,  $H(X) \leq |X|$  becomes  $H(X, X_u) - H(X_u) \leq |X|$  (which is true as  $H(X, X_u) \leq H(X) + H(X_u) \leq |X| + H(X_u)$ ). This shows that all constraints in Theorem 5.1 of types (1) and (2) can be replaced as claimed. The same happens for constraints of types (3) and (4). Consequently, under this transformation, the proof that  $H(X_{G'}) \leq \text{Sh}(G')$  becomes a proof that  $H(X_G) - H(X_u) \leq \text{Sh}(G')$ . Since  $H(X_u) \leq 1$  the result immediately follows.  $\square$

We have seen that by removing vertices from a graph  $G$  we make the game easier allowing us to upper bound  $\text{gn}(G)$ . Another way we can make the game

easier is by adding extra edges to  $G$ . Consequently we can avoid the Shannon bound calculation by also using the asymptotic guessing number of supergraphs of  $G$  which have the same number of vertices as  $G$ .

We end this section by considering the problem of how to calculate the non-Shannon bounds, i.e. the Zhang-Yeung bound, the Dougherty-Freiling-Zeger bound, and the Ingleton bound. They all involve inequalities on 4 variables and consequently a naive approach is to add at least  $2^{4n}$  inequalities to the linear program, where  $n$  is the order of the graph. Unfortunately such a linear program is far too large to be computationally feasible. Our approach is given by the following algorithm:

1. Let  $\mathcal{C}$  be the set of Shannon bound constraints.
2. Solve the linear program which consists only of constraints  $\mathcal{C}$ .
3. Check if the solution satisfies all required 4 variable information inequalities (e.g. the Zhang-Yeung inequalities if we are calculating the Zhang-Yeung bound).
  - (a) If all the inequalities are satisfied then terminate, returning the objective value.
  - (b) If one of the inequalities is not satisfied add this constraint to  $\mathcal{C}$  and go back to 2.

We note that due to the large number of inequalities, Step 3 can take a while. So it is advisable to add some extra constraints to the linear program to limit the search to a solution which is symmetric under the automorphisms of the graph (there always exists such a solution due to the linearity of the problem). This extra symmetry can be used to avoid checking a significant proportion of the inequalities in Step 3.

## 8 Open Problems

Problem 6.8 asks whether there exists an irreversible guessing game, i.e. a guessing game  $G$  such that  $\text{gn}(G) \neq \text{gn}(\text{Reverse}(G))$ . This can be answered in the affirmative if  $\text{gn}(R^L)$  can be shown to be strictly larger than  $\frac{27}{4} = 6.75$ . Unfortunately, this might be hard to prove as it would establish the existence of a non-linear guessing strategy that improves the lower bound we derived.

It would also be interesting to determine the exact value of  $\text{gn}(R^-)$  as  $R^-$  according to our calculations is the only undirected graph on at most 10 vertices whose guessing number remains undetermined. Any lower bound that implies  $\text{gn}(R^-) > \frac{20}{3}$  would show that there exists a non-linear guessing strategy that outperforms the fractional clique cover strategy for  $R^-$ .

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