Scalar soliton quantization with generic moduli

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ABSTRACT: We canonically quantize multi-component scalar field theories in the presence of solitons. This extends results of Tomboulis [1] to general soliton moduli spaces. We derive the quantum Hamiltonian, discuss reparameterization invariance and explicitly show how, in the semiclassical approximation, the dynamics of the full theory reduce to quantum mechanics on the soliton moduli space. We emphasize the difference between the semiclassical approximation and a truncation of the dynamical variables to moduli. Both procedures produce quantum mechanics on moduli space, but the two Hamiltonians are generically different.

Contents

1 Introduction and summary

The topic of quantization in soliton sectors is a rich one with a long list of applications. Foundational work on this subject was carried out in the mid 70's and includes [1–9]. For thoroughly pedagogical reviews we refer the reader to [10, 11].

In [1] Tomboulis quantized a simple two-dimensional scalar theory in the one-soliton sector by introducing a canonical transformation from the original fields to a dynamical modulus — i.e. a collective coordinate — plus fluctuations around the classical soliton solution, while imposing a set of constraints which preserves the total number of degrees of freedom. A key assumption in that work was that the soliton solution has a single modulus associated with translations in the spatial direction, as is the case e.g. for a kink in ϕ^4 theory. This greatly simplifies some conceptual and calculational aspects of the analysis. Similarly, other contemporary approaches to soliton quantization around static classical solutions, including canonical transformations with unconstrained variables as well as path integral techniques, primarily dealt with systems in which all collective coordinate degrees of freedom correspond to translational modes.[1]

Several years later, a more geometrical framework for understanding collective coordinates emerged from studies of the Bogomolny equation, describing 't Hooft-Polyakov monopoles [12, 13] in four-dimensional Yang-Mills-Higgs theory in the BPS limit of vanishing potential [14, 15]. In this theory, the minimal-energy solution set of the static field

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[1]Some aspects of the analysis of [7], in particular the derivation of the soliton sector Hamiltonian, are more general and do not require the linear motion assumption for the collective coordinates made elsewhere in the paper — an assumption which is based on an identification of the collective coordinates with translational degrees of freedom.

equations with fixed boundary conditions, corresponding to a particular topological charge sector, is a finite-dimensional Riemannian manifold, $(\mathcal{M}, G)$. The metric is the natural one induced from the (flat) metric on field configuration space. The framework suggested by Manton [16] is that, for slowly varying field configurations, the dynamics of the full system is well approximated by promoting the moduli to time-dependent variables — the collective coordinates — in which case the field theory equations of motion reduce to the geodesic equation on $\mathcal{M}$. The usefulness of this framework was beautifully demonstrated by Atiyah and Hitchin's analysis of two-to-two monopole scattering [17].

Generically, $\mathcal{M}$ has curvature and not all moduli correspond to broken symmetries such as translations. Nevertheless in asymptotic regions of $\mathcal{M}$, corresponding to field configurations with well-separated and localized lumps of energy, one can associate the parameters with locations and internal phases of constituent solitons. This suggests that Manton's paradigm of motion on moduli space is applicable in any theory admitting static multi-soliton solutions.

It should be emphasized that Manton's prescription is for constructing approximate time-dependent solutions to *classical* field equations. However, as demonstrated earlier by Gervais, Jevicki, and Sakita [8], it is also natural to assume small velocities for the collective coordinates in the semiclassical analysis of a soliton sector of a quantum theory. The approximate classical solution provides an approximate saddle point for the semiclassical expansion of the path integral. One would like the corrections coming from performing the saddle-point approximation to be comparable to those due to expanding around an approximate solution; the latter are controlled by the collective coordinate velocities.[2] Hence the geometry $(\mathcal{M}, G)$ provides a natural starting point for the quantum analysis of a soliton sector, and the Manton approximation is incorporated as part of the semiclassical expansion.

This point of view was first considered in [19] and has since been used to great effect, e.g. in the context of $\mathcal{N} = 2$ supersymmetric four-dimensional gauge theory [20–22] where semiclassical results can be compared against the quantum-exact ones of Seiberg and Witten [23, 24]. In these analyses one typically *truncates* the classical degrees of freedom to the collective coordinates and then quantizes the resulting finite-dimensional system, yielding a (supersymmetric) quantum mechanical sigma model with target $\mathcal{M}$. This is sufficient for answering basic questions about the original quantum field theory, such as the existence of soliton states and what charges these carry. The first corrections to masses and charges, obtained from one-loop determinants, have also been considered [25, 26]. However, to our knowledge, the exact quantum Hamiltonian describing the full dynamics of a quantum field theory around a (multi-) soliton sector has not been studied within the general geometrical framework.

In this work we extend the canonical transformation of [1] to multi-component scalar field theories with general multi-soliton moduli spaces. This naturally requires using geo-

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metric quantities on the moduli space of classical solutions. Our primary goal is to extract the quantum Hamiltonian for this system, which may be useful in various contexts, such as the study of scattering processes involving both solitons and perturbative particles [27].

Our secondary goal is to establish a formalism that facilitates extending this inquiry to (supersymmetric) theories with gauge fields and fermions, e.g. involving monopoles in four dimensions or instanton-solitons in five dimensions. While we intend to return to this in the near future, the restriction to scalar fields helps highlight the main qualitative results against the added technical details required for those applications.

In that vein, we emphasize several conceptual points as they arise in the explicit analysis. We demonstrate how one recovers a reparameterization-invariant theory for the dynamical moduli when the fluctuations are switched off. This sector has knowledge of both the intrinsic and extrinsic geometry of the system. Furthermore, we show how the full quantum Hamiltonian of the field theory can be expanded in the perturbative coupling, when one additionally requires the solitons to be slowly moving. We organize and present this semiclassical expansion to the first few orders and briefly discuss how Lorentz invariance can be recovered in perturbation theory. Finally, we exhibit how, when restricting to incoming and outgoing states which do not involve perturbative excitations, the leading-order dynamics reduce to quantum mechanics on the soliton moduli space.

We would like to point out that a detailed reduction to the quantum mechanics on the soliton moduli space has also been performed from the point of view of embedded submanifolds [28, 29]. This approach is based on the formalism developed for studying quantum mechanical systems constrained to move on a sumbanifold of their full configuration space [30, 31]. It can be extended to quantum field theory in the presence of a soliton, by viewing the moduli space as embedded inside the infinite-dimensional space of field configurations. The quantum mechanics on moduli space then emerges by solving the functional Schrödinger equation in the Born-Oppenheimer (or 'thin-layer') approximation. This effectively integrates out the massive modes and produces geometric corrections due to both the intrinsic and extrinsic curvature, which are missed by a naive truncation. Our quantum mechanics, obtained via conventional second quantization, exactly agrees with that of [28, 29].

The rest of this paper is organized as follows: In section 2 we set up the background, introduce the change of variables and canonically quantize the theory. In section 3 we obtain the quantum Hamiltonian. Section 4 deals with the reparameterization invariance of the collective coordinate sector, while in section 5 we present the semiclassical expansion. Finally, section 6 discusses the reduction to quantum mechanics on the moduli space.

2 The change of variables

We begin with a general class of real scalar field theories with classical Lagrangian

$$L = \int d\mathbf{x} \left\{ \frac{1}{2}\dot{\Phi} \cdot \dot{\Phi} - \frac{1}{2}\partial_{\mathbf{x}}\Phi \cdot \partial_{\mathbf{x}}\Phi - V(\Phi) \right\} . \tag{2.1}$$

We work in flat D-dimensional Minkowski space, with $\mathbf{x}$ a $(D-1)$-dimensional position vector and $d\mathbf{x}$ shorthand for $d^{D-1}x$. Φ is an n-tuple and $\cdot$ denotes the Euclidean inner

product on $\mathbb{R}^n$. When necessary we will use indices $a, b, \ldots$ to label components of n-tuples. Let $M_{\text{vac}} = \{\Phi \mid V(\Phi) = 0\} \subset \mathbb{R}^n$ denote the space of vacua where the potential energy function vanishes. A finite-energy field configuration must approach some point in M_{vac} as $\mathbf{x} \to \infty$ in any direction. Thus the space of static, finite-energy field configurations decomposes into topological sectors labeled by $\pi_{D-2}(M_{\text{vac}})$, the set of homotopy equivalence classes of maps from the $(D-2)$-sphere at spatial infinity into the vacuum manifold. A (multi-) soliton solution[3] will be a field configuration of minimal energy in a nontrivial topological sector. In particular, M_{vac} should have multiple components in order for solitons to exist when $D = 2$.

The Hamiltonian, $H[\Phi, \Pi]$ associated with the Lagrangian $L[\Phi, \dot{\Phi}]$ is

$$H = \int \mathrm{d}\mathbf{x} \left[\frac{1}{2}\Pi \cdot \Pi + \frac{1}{2}\partial_{\mathbf{x}}\Phi \cdot \partial_{\mathbf{x}}\Phi + V(\Phi)\right] . \tag{2.2}$$

We assume that Φ, Π at fixed time t are Darboux coordinates on phase space

$$\begin{aligned}
\{\Phi^a(t, \mathbf{x}), \Phi^b(t, \mathbf{y})\} = \{\Pi^a(t, \mathbf{x}), \Pi^b(t, \mathbf{y})\} &= 0 \\
\{\Phi^a(t, \mathbf{x}), \Pi^b(t, \mathbf{y})\} &= \delta^{ab}\delta(\mathbf{x} - \mathbf{y}),
\end{aligned} \tag{2.3}$$

where $\delta(\mathbf{x} - \mathbf{y})$ is a $(D-1)$-dimensional Dirac delta function and the Poisson bracket is given by

$$\{F[\Phi, \Pi], \tilde{F}[\Phi, \Pi]\} := \int \mathrm{d}\mathbf{z} \left\{\frac{\delta F}{\delta\Phi(\mathbf{z})} \cdot \frac{\delta\tilde{F}}{\delta\Pi(\mathbf{z})} - \frac{\delta F}{\delta\Pi(\mathbf{z})} \cdot \frac{\delta\tilde{F}}{\delta\Phi(\mathbf{z})}\right\} . \tag{2.4}$$

In the quantum theory, Φ, Π are promoted to operators[4] $\hat{\Phi}, \hat{\Pi}$ and the Poisson bracket to a commutator

$$\{\,,\,\} \to [\,,\,] = i\{\,,\,\}, \tag{2.5}$$

such that

$$[\hat{\Phi}^a(t, \mathbf{x}), \hat{\Pi}^b(t, \mathbf{y})] = i\delta^{ab}\delta^{(D-1)}(\mathbf{x} - \mathbf{y}) . \tag{2.6}$$

We consider a fixed topological sector and assume there exists a finite-dimensional smooth family of classical static soliton solutions, parameterized by moduli U^M,

$$\Phi(x) = \phi(\mathbf{x}; U^M), \tag{2.7}$$

where M runs over the dimension of the moduli space $\dim_{\mathbb{R}}\mathcal{M}$, such that

$$-\partial_{\mathbf{x}}^2\phi + \left.\frac{\delta V}{\delta\Phi}\right|_{\Phi=\phi} = 0, \qquad -\partial_{\mathbf{x}}^2 + \left.\frac{\delta^2 V}{\delta\Phi\delta\Phi}\right|_{\Phi=\phi} =: \Delta(U) \geq 0 . \tag{2.8}$$

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[3]Although Derrick's theorem [32, 33] precludes the existence of soliton solutions for $D > 2$, it is no more difficult to leave D arbitrary. Doing so will facilitate the extension to theories with gauge interactions where one can have $D > 2$.

[4]For added clarity in this section we use hats to distinguish quantum operators from their classical counterparts. In later sections we will be working exclusively at the quantum level and will drop this convention in favor of brevity.

The inequality $\Delta(U) \geq 0$ is meant to signify that $\Delta(U)$ is a positive operator, such that all of its eigenvalues are non-negative, and the notation is to emphasize that this operator depends on where we are in moduli space.

In order to study the behavior of the theory around the soliton configuration (2.7), one makes a change of variables from the original field $\Phi(x)$ to collective coordinates $U^M = U^M(t)$ and fluctuations $\chi(x; U^M(t))$ about the solution:

$$\Phi(x) = \phi(\mathbf{x}; U^M(t)) + \chi(x; U^M(t)) . \tag{2.9}$$

To preserve the number of degrees of freedom, there should be as many constraints on these new variables as there are coordinates U^M. We note that $\partial_M \phi$ will be a zero-mode of the linear differential operator Δ. One would like to exclude such zero-frequency modes from the mode expansion of χ. This can be done by imposing the constraints

$$\psi_M^{(1)} = \int \mathrm{d}\mathbf{x}\, \chi \cdot \partial_M \phi = 0 . \tag{2.10}$$

We introduce momentum variables $(p_M, \pi(x; U^M))$ conjugate to (U^M, χ) and extend this transformation to phase space. We treat these as Darboux coordinates in an extended phase space

$$\{U^M(t), p_N(t)\}' = \delta^M{}_N , \qquad \{\chi^a(t, \mathbf{x}; U(t)), \pi^b(t, \mathbf{y}; U(t))\}' = \delta^{ab}\delta(\mathbf{x} - \mathbf{y}) , \tag{2.11}$$

with Poisson structure $\{\,,\,\}'$, defined by

$$\{F[U, \chi; p, \pi], \tilde{F}[U, \chi; p, \pi]\}' := \frac{\partial F}{\partial U^M} \cdot \frac{\partial \tilde{F}}{\partial p_M} - \frac{\partial F}{\partial p_M} \cdot \frac{\partial \tilde{F}}{\partial U^M} +$$
$$+ \int \mathrm{d}\mathbf{z} \left\{ \frac{\delta F}{\delta \chi(\mathbf{z})} \cdot \frac{\delta \tilde{F}}{\delta \pi(\mathbf{z})} - \frac{\delta F}{\delta \pi(\mathbf{z})} \cdot \frac{\delta \tilde{F}}{\delta \chi(\mathbf{z})} \right\} . \tag{2.12}$$

We also extend the coordinate transformation (2.9) to a phase space transformation with the ansatz

$$\Pi(x) = \Pi_0^M[U, \chi; p_M, \pi]\, \partial_M \phi(\mathbf{x}; U(t)) + \pi(x, U(t)) , \tag{2.13}$$

where the functionals Π_0^M will be determined below. In analogy with (2.10) we impose

$$\psi_M^{(2)} = \int \mathrm{d}\mathbf{x}\, \pi \cdot \partial_M \phi = 0 . \tag{2.14}$$

The constraints are second-class as the Poisson brackets are non-vanishing:

$$\{\psi_M^{(1)}, \psi_N^{(1)}\}' = \{\psi_M^{(2)}, \psi_N^{(2)}\}' = 0 ,$$
$$\{\psi_M^{(1)}, \psi_N^{(2)}\}' = \int \mathrm{d}\mathbf{z}\, \partial_M \phi \cdot \partial_N \phi =: G_{MN}(U) . \tag{2.15}$$

Here $G_{MN}(U)$ is the metric on the moduli space of soliton solutions. Restriction of the dynamics to the constraint surface is achieved through the introduction of Dirac brackets,

$$\{F, \tilde{F}\}'_{\mathrm{D}} := \{F, \tilde{F}\}' + \{F, \psi_M^{(1)}\}' G^{MN} \{\psi_N^{(2)}, \tilde{F}\}' - \{F, \psi_M^{(2)}\}' G^{MN} \{\psi_N^{(1)}, \tilde{F}\}' . \tag{2.16}$$

Geometrically, the Dirac bracket is the pullback of the Poisson bracket to the constraint surface and satisfies all the properties of the ordinary Poisson bracket. The appearance of the moduli space metric G_{MN} in the Dirac bracket is quite natural and can be viewed as a motivation for choosing the momentum constraint as in (2.14).

One can straightforwardly work out the Dirac brackets of our Darboux coordinates. The nonzero brackets with the constraints are

$$
\begin{aligned}
\{\psi_N^{(1,2)}, p_M\}' &= \partial_M \psi_N^{(1,2)} \\
\{\chi, \psi_M^{(2)}\}' &= \partial_M \phi \\
\{\psi_M^{(1)}, \pi\}' &= \partial_M \phi \,,
\end{aligned}
\tag{2.17}
$$

so that we have

$$
\begin{aligned}
\{U^M, p_N\}'_{\mathrm{D}} &= \delta^M{}_N \,, \\
\{p_M, p_N\}'_{\mathrm{D}} &= -(\partial_M \psi_P^{(1)}) G^{PQ}(\partial_N \psi_Q^{(2)}) + (\partial_M \psi_P^{(2)}) G^{PQ}(\partial_N \psi_Q^{(1)}) \,, \\
\{p_M, \chi(\mathbf{x})\}'_{\mathrm{D}} &= -\partial_M \chi(\mathbf{x}) + (\partial_M \psi_P^{(1)}) G^{PQ} \partial_Q \phi(\mathbf{x}) \,, \\
\{p_M, \pi(\mathbf{x})\}'_{\mathrm{D}} &= -\partial_M \pi(\mathbf{x}) + (\partial_M \psi_P^{(2)}) G^{PQ} \partial_Q \phi(\mathbf{x}) \,, \\
\{\chi^a(\mathbf{x}), \pi^b(\mathbf{y})\}'_{\mathrm{D}} &= \delta^{ab}\delta(\mathbf{x}-\mathbf{y}) - \partial_M \phi^a(\mathbf{x}) G^{MN} \partial_N \phi^b(\mathbf{y}) \,,
\end{aligned}
\tag{2.18}
$$

with the rest vanishing. Here we suppressed all non-essential arguments of the fields. These brackets appear complicated at first, but we have not yet specified the functional dependence of χ, π on U^M. One can freely do this, since the degrees of freedom contained in χ should comprise a basis for $L^2[\mathbb{R}^{D-1}]$ and not $L^2[\mathbb{R}^{D-1} \times \mathcal{M}]$. Indeed, it is always possible to choose the U-dependence of χ, π such that

$$
\partial_M \psi_N^{(1,2)} \approx 0 \,,
\tag{2.19}
$$

where $\approx$ denotes 'upon restriction to the constraint surface'; see appendix A for details. Having done so, the non-vanishing Dirac brackets become

$$
\begin{aligned}
\{U^M, p_N\}'_{\mathrm{D}} &= \delta^M{}_N \\
\{p_M, \chi(\mathbf{x})\}'_{\mathrm{D}} &\approx -\partial_M \chi(\mathbf{x}) \\
\{p_M, \pi(\mathbf{x})\}'_{\mathrm{D}} &\approx -\partial_M \pi(\mathbf{x}) \\
\{\chi^a(\mathbf{x}), \pi^b(\mathbf{y})\}'_{\mathrm{D}} &= \delta^{ab}\delta(\mathbf{x}-\mathbf{y}) - \partial_M \phi^a(\mathbf{x}) G^{MN} \partial_N \phi^b(\mathbf{y}) \,.
\end{aligned}
\tag{2.20}
$$

We remind that for systems with second-class constraints it is the Dirac bracket that is promoted to the commutator in the quantum theory

$$
\{\, , \,\}'_{\mathrm{D}} \to [\, , \,]' = i\{\, , \,\}'_{\mathrm{D}} \,.
\tag{2.21}
$$

In order for the quantum theory in the old and new variables to be equivalent, we must require that the transformation $(\Phi; \Pi) \to (U^M, \chi; p_M, \pi)$ defined by (2.9) and (2.13) be canonical. Then $\{\, , \,\} = \{\, , \,\}'_{\mathrm{D}}$ and hence $[\, , \,] = [\, , \,]'$.[5] This latter condition can

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[5] In particular, the restriction of the new extended phase space to the constraint surface should give back the original phase space.

be used to fix the functionals Π_0^M in (2.13). In order to implement this requirement we compute $\{\Phi^a(x), \Phi^b(y)\}'_{\mathrm{D}}$, $\{\Phi^a(x), \Pi^b(y)\}'_{\mathrm{D}}$, and $\{\Pi^a(x), \Pi^b(y)\}'_{\mathrm{D}}$ by inserting the change of variables (2.9), (2.13) and using the brackets (2.20). In the process, we find Π_0^M such that the results are consistent with (2.3). The full computations are tedious but straightforward; some intermediate results are recorded in appendix B for the reader interested in the derivation.

We summarize these results as follows. The $[\hat{\Phi}, \hat{\Phi}]$ commutator is trivial since $\hat{U}, \hat{\chi}$ are commuting operators. Thus

$$[\hat{\Phi}^a(\mathbf{x}), \hat{\Phi}^b(\mathbf{y})]' = [\phi^a(\mathbf{x}; \hat{U}) + \hat{\chi}^a(\mathbf{x}; \hat{U}), \phi^b(\mathbf{y}; \hat{U}) + \hat{\chi}^b(\mathbf{y}; \hat{U})]' = 0 \ . \tag{2.22}$$

The calculation of $[\hat{\Phi}, \hat{\Pi}]'$ fixes the form of Π_0^M. At the classical level one finds

$$\Pi_0^N \approx \left(p_M - \int d\mathbf{z}\, \pi(\mathbf{z}; U) \cdot \partial_M \chi(\mathbf{z}; U) \right) [(G - \Xi)^{-1}]^{MN}, \tag{2.23}$$

where

$$\Xi_{MN}(U) := \int d\mathbf{z}\, \chi(\mathbf{z}; U) \cdot \partial_M \partial_N \phi(\mathbf{z}; U) \ . \tag{2.24}$$

At the quantum level one must be careful about operator ordering in eq. (2.13). The symmetrized ansatz

$$\hat{\Pi}(x) = \frac{1}{2} \left(\hat{a}^M \partial_M \phi(\mathbf{x}; \hat{U}(t)) + (\partial_M \phi(\mathbf{x}; \hat{U}(t)))\hat{\bar{a}}^M \right) + \hat{\pi}(x; \hat{U}(t)) , \tag{2.25}$$

where

$$\begin{aligned} \hat{a}^M &:= (\hat{p}_N - \int \hat{\pi} \cdot \partial_N \hat{\chi}) [(\hat{G} - \hat{\Xi})^{-1}]^{NM} \\ \hat{\bar{a}}^M &:= [(\hat{G} - \hat{\Xi})^{-1}]^{MN} (\hat{p}_N - \int \partial_N \hat{\chi} \cdot \hat{\pi}) , \end{aligned} \tag{2.26}$$

provides a natural generalization of the ansatz in [1]. Here we have begun using the shorthand $\int d\mathbf{z}\, \pi(\mathbf{z}; U) \cdot \partial_N \chi(\mathbf{z}; U) = \int \pi \cdot \partial_N \chi$. It is also useful to introduce the combination

$$\hat{C}^{MN} := [(\hat{G} - \hat{\Xi})^{-1}]^{MN} \ . \tag{2.27}$$

Note that $\hat{C}^{MN} = \hat{C}^{(MN)}$ and that (2.25) reduces to (2.23) when the operators become commuting fields. With the form of the change of momentum variables fixed, it is now a nontrivial task to check whether $[\hat{\Pi}, \hat{\Pi}]' = 0$. Explicit evaluation leads to the expected result.[6]

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[6]In place of (2.9) one could have also used an alternative change of variables as in [7] — see also [28] — where the fluctuation field χ can be directly expanded in terms of only non-zero-modes. Then one does not need to impose constraints and the p plus π-modes are canonically conjugate to the U plus χ-modes. It can be explicitly seen that this approach also leads to the relation (2.23) and hence the same soliton sector Hamiltonian.

3 The soliton sector Hamiltonian

We are now in a position to implement the change of variables (2.9) and (2.25) in the Hamiltonian (2.2). Squaring (2.25) leads to[7]

$$
\int \mathrm{d}\mathbf{x}\, \Pi \cdot \Pi = A^M G_{MN} A^N + \int \pi \cdot \pi - \frac{1}{4} C^{MP} C^{NQ} \int \partial_M \partial_P \phi \cdot \partial_N \partial_Q \phi
$$
$$
+ \frac{1}{2} C^{MP} C^{NQ} \Gamma_{MNR} C^{RS} \Big(\Gamma_{PQS} + 2\Gamma_{QSP} - \int \chi \cdot \partial_P \partial_Q \partial_S \phi \Big)
$$
$$
- \frac{1}{2} C^{MP} C^{NQ} \partial_P \Gamma_{QMN} \,, \tag{3.1}
$$

where

$$
A^M := \frac{1}{2}(a^M + \bar{a}^M) \,. \tag{3.2}
$$

All terms beyond the first two result from the evaluation of two commutators and should be thought of as $\mathcal{O}(\hbar^2)$. We have also introduced

$$
\Gamma_{PMN} := \frac{1}{2} \left(\partial_M G_{PN} + \partial_N G_{PM} - \partial_P G_{MN} \right) = \int \partial_P \phi \cdot \partial_M \partial_N \phi \tag{3.3}
$$

to define Christoffel symbols on the moduli space.

At this point we note that, in the special case where the moduli space consists of a single modulus associated with translations, $\mathcal{M} = \mathbb{R}$, all terms in the second and third lines vanish. The terms in the first line then reproduce the analogous result in [1], including the 'quantum correction' term $\int (\partial^2 \phi)^2$.

The full Hamiltonian follows trivially from (3.1) by adding the potential, which can be expanded around the solution $\Phi = \phi$:

$$
H = v(U) + \frac{1}{2} A^M G_{MN} A^N - \frac{1}{8} C^{MP} C^{NQ} \Big(\int \partial_M \partial_P \phi \cdot \partial_N \partial_Q \phi \Big) +
$$
$$
+ \frac{1}{4} C^{MP} C^{NQ} \Big[-\partial_P \Gamma_{QMN} + \Gamma_{MNR} C^{RS} \Big(\Gamma_{PQS} + 2\Gamma_{QSP} - \int \chi \cdot \partial_P \partial_Q \partial_S \phi \Big) \Big]
$$
$$
+ \int \Big[\frac{1}{2} \pi \cdot \pi + s(\mathbf{x}; U) \cdot \chi + \frac{1}{2} \chi \cdot \Delta(\mathbf{x}; U) \chi + V_I(\chi) \Big] \,. \tag{3.4}
$$

In the above

$$
v(U) := \int \Big(\frac{1}{2} \partial_{\mathbf{x}} \phi \cdot \partial_{\mathbf{x}} \phi + V(\phi) \Big), \qquad s(\mathbf{x}; U) := -\partial_{\mathbf{x}}^2 \phi + \frac{\partial V}{\partial \Phi} \Big|_{\Phi = \phi} \tag{3.5}
$$

and $V_I(\chi)$ denotes cubic and higher-order interaction terms in the fluctuations χ coming from the original potential. If $\phi(\mathbf{x}, U)$ parameterizes a family of *exact* static solutions then: a) $v(U)$ will be a constant, by definition the classical soliton mass and b) the source term $s(\mathbf{x}; U)$ will vanish. However, we will see shortly that it is natural to also allow for a small deviation from an exact solution. In that case $\mathcal{M}$ is not really a true moduli space, as evidenced by the appearance of the potential $v(U)$.

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[7]From now on we will drop hats as well as the $\approx$ notation, since all expressions are understood as restricted to the constraint surface.

Eq. (3.4) is the final, exact result for the quantum Hamiltonian[8] of the theory. It is valid for all values of soliton moduli U^M and conjugate momenta p_N.

4 Covariance

Given the form of (3.4), it appears that the Hamiltonian is not invariant under arbitrary reparameterizations of the moduli U. This is not the case and the manifestly invariant form of the Hamiltonian can be recovered once we properly order the kinetic term operators.

The canonical change of variables in configuration space (2.9) from Φ to U, χ — plus constraints — effectively maps a Cartesian coordinate system to a curvilinear one. This map describes how the curved moduli space is embedded in the total infinite-dimensional space of modes. The orthogonal directions to $\mathcal{M}$, within the constraint surface, are parameterized by the massive oscillator modes of χ. It is possible to use the theory of embedded surfaces in order to construct the exact metric for the infinite-dimensional Cartesian space in the new curvilinear coordinate system. This was explicitly done by Fujii et al. in [28], who found that the Hamiltonian can be expressed in terms of the Laplace-Beltrami operator in the curvilinear coordinate frame. Hence the full theory, in the new set of variables, is reparameterization invariant as expected.

It is interesting to see how covariance becomes manifest in the subsector of the theory with all fluctuations switched off. We have, from (3.4),

$$
\begin{aligned}
H|_{\chi,\pi=0} = {} & \frac{1}{2}p_M G^{MN} p_N + v(U) + \frac{1}{8}(\partial_P G^{PM}) G_{MN}(\partial_Q G^{QN}) - \frac{1}{4}\partial_M \partial_N G^{MN} + \\
& - \frac{1}{4}G^{MP} G^{NQ}\Big[\frac{1}{2}\int \partial_M \partial_P \phi \partial_N \partial_Q \phi + \partial_P \Gamma_{QMN}\Big] + \\
& + \frac{1}{4}\Gamma^{PQS}\Big[\Gamma_{PQS} + 2\Gamma_{QSP}\Big] ,
\end{aligned}
\tag{4.1}
$$

where the last two terms in the first line are obtained from commutators upon appropriately ordering the momentum operators in the kinetic term. This expression can be manipulated as follows. First note that

$$
-\frac{1}{8}G^{MP} G^{NQ}\int \partial_M \partial_P \phi \partial_N \partial_Q \phi = -\frac{1}{8}\int (\nabla^2 \phi)^2 - \frac{1}{8}\Gamma_{RM}{}^M \Gamma^R{}_N{}^N .
\tag{4.2}
$$

Second, we have

$$
\partial_M \partial_N G^{MN} = 4Y - R + \Gamma_{RMS}\Gamma^{SMR} ,
\tag{4.3}
$$

where R is the scalar curvature on moduli space

$$
\begin{aligned}
R &= G^{MP} G^{NQ} R_{MNPQ} \\
&= G^{MP} G^{NQ}\Big[\partial_N \Gamma_{QMP} - \partial_M \Gamma_{QNP} + \Gamma^R{}_{NP}\Gamma_{RQM} - \Gamma^R{}_{MP}\Gamma_{RQN}\Big]
\end{aligned}
\tag{4.4}
$$

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[8]We have suppressed the appearance of counter-terms in our discussion, as they are model dependent. A renormalizable theory will require a finite number of local counter-terms to be added to the action. These counter-terms can be determined from the UV divergences in the perturbative sector of the theory. The resulting counter-term Hamiltonian should also be transformed to the soliton sector and included in (3.4). It is a nontrivial test of renormalizability that the resulting counter-term Hamiltonian is sufficient to cancel all UV divergences for processes computed in the soliton sector.

and [28]

$$Y := -\frac{1}{2}\partial_M(G^{MN}\Gamma^S{}_{NS}) - \frac{1}{4}\Gamma^{SN}{}_S\Gamma^R{}_{NR} \ . \tag{4.5}$$

Using this definition we can re-express (4.3) as

$$\begin{aligned}-\frac{1}{4}\partial_M\partial_N G^{MN} = \ &-\frac{1}{2}Y + \frac{1}{4}R - \frac{1}{4}\Gamma_{RMS}\Gamma^{SMR}+ \\ &+\frac{1}{4}\partial_M(G^{MN}G^{SR}\Gamma_{RNS}) + \frac{1}{8}\Gamma^{SN}{}_M\Gamma^R{}_{NR} \ .\end{aligned} \tag{4.6}$$

Substituting (4.2) and (4.6) into (4.1), one finds that all bilinears in the Γ's as well as the terms involving derivatives of Γ's mutually cancel to leave

$$\begin{aligned}H|_{\chi,\pi=0} = \ &\frac{1}{2}p_M G^{MN}p_N + v(U) - \frac{1}{2}Y + \frac{1}{4}R - \frac{1}{8}\int(\nabla^2\phi)^2 \\ = \ &\frac{1}{2}G^{-1/4}p_M G^{1/2}G^{MN}p_N G^{-1/4} + v(U) + \frac{1}{4}R - \frac{1}{8}\int(\nabla^2\phi)^2 \ . \end{aligned} \tag{4.7}$$

In the last step we noted that

$$G^{-1/4}p_M G^{1/2}G^{MN}p_N G^{-1/4} = p_M G^{MN}p_N - Y \ . \tag{4.8}$$

The l.h.s. of (4.8) is in fact a covariant quantity when understood as a Hamiltonian acting on a wavefunction Ψ with canonical normalization $\int_{\mathcal{M}} \mathrm{d}^d U \Psi^*\Psi = 1$. The time-independent Schrödinger equation takes the form

$$\frac{1}{2}G^{-1/4}\partial_M(G^{1/2}G^{MN}\partial_N(G^{-1/4}\Psi)) = E\Psi \ , \tag{4.9}$$

Redefining $\Psi = G^{1/4}\tilde{\Psi}$ leads to the correct curved space normalization $\int_{\mathcal{M}} \mathrm{d}^d U \sqrt{G}\tilde{\Psi}^*\tilde{\Psi} = 1$ and modifies (4.9) to

$$\frac{1}{2}G^{-1/2}\partial_M(G^{1/2}G^{MN}\partial_N(\tilde{\Psi})) = E\tilde{\Psi} \ , \tag{4.10}$$

where the l.h.s. is now the Laplace-Beltrami operator on the soliton moduli space.

Hence we have arrived at the explicitly covariant expression

$$H|_{\chi,\pi=0} = \frac{1}{2}G^{-1/4}p_M G^{1/2}G^{MN}p_N G^{-1/4} + v(U) + \frac{1}{4}R - \frac{1}{8}\int(\nabla^2\phi)^2 \ . \tag{4.11}$$

In fact, the quantity $\int(\nabla^2\phi)^2$ has a nice geometric interpretation. Using the results of [28] one finds that[9]

$$\int(\nabla^2\phi)^2 = d^2\mathcal{H}^2 \ , \tag{4.12}$$

where $\mathcal{H}$ is the extrinsic mean curvature and $d = \dim_{\mathbb{R}}\mathcal{M}$. This curvature invariant encodes information about how the moduli space $\mathcal{M}$ is embedded as a submanifold into the infinite-dimensional flat configuration space.

---

[9]The notation of [28] is rather different from the one used here, so it is useful to describe the precise map: We have $\partial_P\phi^a(\mathbf{x}) \to B_a^{Ax}$. Then $\nabla^2\phi^a(\mathbf{x}) \to g^{ab}\nabla_a B_b^{Ax} = n\overline{H}^{Ax}$ and the integration over $\mathbf{x}$ is performed by contracting the $\overline{H}$'s with $\eta_{Ax,By}$.

Our covariant result is then simply

$$H|_{\chi,\pi=0} = \frac{1}{2}G^{-1/4}p_M G^{1/2}G^{MN}p_N G^{-1/4} + v(U) + \frac{1}{4}R - \frac{d^2}{8}\mathcal{H}^2 \ . \qquad (4.13)$$

This Hamiltonian defines a quantum mechanics with target $\mathcal{M}$; we will refer to (4.13) as the 'truncated Hamiltonian' in the following. The curvature terms are $\mathcal{O}(\hbar^2)$ effects and may be viewed as intrinsic and extrinsic 'quantum potentials.' The appearance of the Ricci scalar is well documented in background-independent approaches to quantum mechanics on curved spaces. In that context, it has been observed that the coefficient of the Ricci scalar term is ambiguous, depending on the operator ordering prescription [34]. Here there is no ambiguity because the correct ordering prescription is inherited from the parent theory, which is defined on a flat configuration space. It is also interesting to observe that even in the limit where the fluctuations have been completely decoupled, the Hamiltonian for the collective coordinates encodes information about the extrinsic geometry [28].

5 Semiclassical analysis

Up to this point we have not explicitly kept track of powers of coupling constants. As is typical in the soliton literature [4, 10, 11], we will assume that there is effectively a single coupling g such that, in terms of the canonically normalized field $\tilde{\Phi}$, the potential $\tilde{V}(\tilde{\Phi};g)$ has the scaling property

$$\tilde{V}(\tilde{\Phi};g) = \frac{1}{g^2}\tilde{V}(g\tilde{\Phi};1) =: \frac{1}{g^2}V(g\tilde{\Phi}) \ . \qquad (5.1)$$

Thus, if we define the rescaled field $\Phi = g\tilde{\Phi}$, then then the entire coupling dependence of the Lagrangian (2.1) becomes

$$L(\tilde{\Phi},\dot{\tilde{\Phi}};g) = \frac{1}{g^2}L(\Phi,\dot{\Phi};1) \ . \qquad (5.2)$$

We will assume that we have been working with the rescaled field Φ all along and that we previously set the coefficient of g^{-2} in front of (2.1) to one. Note that if ϕ is the rescaled classical solution, it will be independent of g and hence the canonically normalized classical solution $\tilde{\phi}$ will go as g^{-1}, which is the usual behavior we expect from a soliton configuration.

Under the assumption (5.2), it is clear from the path integral point of view that g^2 plays the role of $\hbar$ and the semiclassical expansion is a g expansion. Once the factor of g^{-2} is restored in front of the Lagrangian, the Hamiltonian, (2.2), becomes

$$H = \int d\mathbf{x}\left[\frac{g^2}{2}\Pi \cdot \Pi + \frac{1}{g^2}\left(\frac{1}{2}\partial_{\mathbf{x}}\Phi \cdot \partial_{\mathbf{x}}\Phi + V(\Phi)\right)\right] \ . \qquad (5.3)$$

Meanwhile, the definitions of the metric and potential on moduli space read

$$G_{MN} := \frac{1}{g^2}\int \partial_M\phi \cdot \partial_N\phi \ , \qquad \Xi_{MN} := \frac{1}{g^2}\int \chi \cdot \partial_M\partial_N\phi \qquad (5.4)$$

and

$$v(U) := \frac{1}{g^2} \int \left(\frac{1}{2} \partial_{\mathbf{x}} \phi \cdot \partial_{\mathbf{x}} \phi + V(\phi) \right), \qquad s(\mathbf{x}; U) := \frac{1}{g^2} \left(-\partial_{\mathbf{x}}^2 \phi + \frac{\partial V}{\partial \Phi} \Big|_{\Phi = \phi} \right) . \qquad (5.5)$$

The canonical transformations are given by

$$\begin{aligned} \Phi &= \phi + g\,\chi \\ \Pi &= \frac{1}{2} \left(a^M \partial_M \phi + \partial_M \phi\, \bar{a}^M \right) + \frac{1}{g} \pi , \end{aligned} \qquad (5.6)$$

where

$$a^M = \frac{1}{g^2} (p_N - \int \pi \cdot \partial_N \chi) C^{MN} , \qquad \bar{a}^M = \frac{1}{g^2} C^{MN} (p_N - \int \partial_N \chi \cdot \pi) , \qquad (5.7)$$

with $C^{MN} = [(G - g\Xi)^{-1}]^{MN}$. In the above we have rescaled the fluctuations χ, π so that they are canonically normalized fields, while the power of g^{-2} in (5.7) originates from the definitions (5.4).

Setting $A^M = \frac{1}{2}(a^M + \bar{a}^M)$ as before, our full quantum Hamiltonian (3.4) can now be re-written as

$$\begin{aligned} H = {}& \frac{g^4}{2} A^M G_{MN} A^N + v(U) - \frac{1}{8g^2} C^{MP} C^{NQ} \int \partial_M \partial_P \phi \cdot \partial_N \partial_Q \phi \\ &+ \frac{1}{4} C^{MP} C^{NQ} \Big[-\partial_P \Gamma_{QNM} + \Gamma_{MNR} C^{RS} \Big(\Gamma_{PQS} + 2\Gamma_{QSP} - \frac{1}{g} \int \chi \cdot \partial_P \partial_Q \partial_S \phi \Big) \Big] \\ &+ \int \left[\frac{1}{2} \pi \cdot \pi + g\, s \cdot \chi + \frac{1}{2} \chi \cdot \Delta \chi + V_I(\chi) \right] . \end{aligned} \qquad (5.8)$$

We can then expand the $A^M G_{MN} A^N$ term in powers of the coupling as follows:

$$\begin{aligned} g^4 A^M G_{MN} A^N = {}& p_M \Big(G^{MN} + 2g(G^{-1} \Xi G^{-1})^{MN} + 3g^2 (G^{-1} \Xi G^{-1} \Xi G^{-1})^{MN} + \mathcal{O}(g^5) \Big) p_N \\ &+ \left[\frac{1}{4} (\partial_P G^{PM}) G_{MN} (\partial_Q G^{QN}) - \frac{1}{2} \partial_M \partial_N G^{MN} + \mathcal{O}(g^3) \right] \\ &- \frac{1}{2} \Big[p_M \Big(G^{MN} + 2g(G^{-1} \Xi G^{-1})^{MN} + \mathcal{O}(g^4) \Big) \int [\partial_N \chi^a, \pi^a]_+ \\ &\quad + \int [\pi^a, \partial_M \chi^a]_+ \Big(G^{MN} + 2g(G^{-1} \Xi G^{-1})^{MN} + \mathcal{O}(g^3) \Big) p_N \Big] \\ &+ \frac{1}{4} \int [\pi^a, \partial_M \chi^a]_+ \Big(G^{MN} + \mathcal{O}(g^3) \Big) \int [\pi^b, \partial_N \chi^b]_+ , \end{aligned} \qquad (5.9)$$

where $[A, B]_+ := AB + BA$. Note that, as in (4.1), the terms in the second line come from commutators when expanding $A^M G_{MN} A^N$ and moving p_M to the far left and p_N to the far right of the expression.

Notice also that the first line contains a term linear in the fluctuations χ through Ξ. The presence of this tadpole is due to the fact that $\phi(\mathbf{x}; U(t))$ is not an exact solution to the time-dependent equations of motion, irrespective of whether or not $\phi(\mathbf{x}; U)$ is an exact solution to the time-independent ones. This is what motivates the *small velocity assumption*: As it stands, (5.9) is valid for all values of soliton momenta but makes little

sense in perturbation theory, since the scalar propagator would be higher order in the coupling compared to the tadpole. However, if one considers appropriately slowly-moving solitons, $p^2\chi$ can be viewed as a legitimate interaction term.

In a similar vein, since we do not solve the time-dependent equations of motion exactly, there is no need to insist on an exact solution to the time-independent equations. We merely require an approximate solution so that the tadpole term, $s(\mathbf{x}; U) \cdot \chi$, coming from the potential may also be viewed as an interaction term.

Thus we will continue by making the assumptions

$$\dot{U}^M \sim \mathcal{O}(g) \quad \Rightarrow \quad p_M \sim \mathcal{O}(1/g)\,, \qquad s(\mathbf{x}; U) \sim \mathcal{O}(1)\,, \tag{5.10}$$

so that we are expanding around an approximate solution to the time-dependent equations of motion. Note that the latter condition implies that

$$v(U) = M_{\mathrm{cl}} + \delta v(U)\,, \qquad \text{where} \quad M_{\mathrm{cl}} \sim \mathcal{O}(1/g^2)\,, \quad \delta v(U) \sim \mathcal{O}(1)\,. \tag{5.11}$$

In other words, the integral of the potential evaluated on the classical solution is constant up to $\mathcal{O}(g^2)$-suppressed corrections, which may be moduli dependent. The constant M_{cl} is interpreted as the classical — or leading order — contribution to the soliton mass, while the corrections give a U-dependent potential on the moduli space.

In this small-velocity and small-potential approximation, the semiclassical expansion of the full Hamiltonian becomes

$$H = H^{(-2)} + H^{(0)} + H^{(1)} + H^{(2)} + \mathcal{O}(g^3)\,, \tag{5.12}$$

where

$$
\begin{aligned}
H^{(-2)} &= M_{\mathrm{cl}}\,, \\
H^{(0)} &= \frac{1}{2} p_M G^{MN} p_N + \delta v(U) + \frac{1}{2} \int \left(\pi \cdot \pi + \chi \cdot \Delta \chi \right)\,, \\
H^{(1)} &= \int \left\{ \frac{1}{g}\, p_M G^{MP} \left(\chi \cdot \partial_P \partial_Q \phi \right) G^{PN} p_N + g\, s \cdot \chi + \frac{g}{3!} V^{(3)}_{abc}(\phi) \chi^a \chi^b \chi^c \right. \\
&\qquad \left. - \frac{1}{4} \left([\pi^a, \partial_M \chi^a]_+ G^{MN} p_N + p_M G^{MN} [\pi^a, \partial_N \chi^a]_+ \right) \right\}\,, \\
H^{(2)} &= \frac{3g^2}{2} p_M \left(G^{-1} \Xi G^{-1} \Xi G^{-1} \right)^{MN} p_N + \frac{g^2}{4!} \int V^{(4)}_{abcd}(\phi) \chi^a \chi^b \chi^c \chi^d \\
&\quad - \frac{g}{2} \left([\pi^a, \partial_M \chi^a]_+ \left(G^{-1} \Xi G^{-1} \right)^{MN} p_N + p_M \left(G^{-1} \Xi G^{-1} \right)^{MN} [\pi^a, \partial_N \chi^a]_+ \right) \\
&\quad + \frac{1}{8} \left(\int [\pi^a, \partial_M \chi^a]_+ \right) G^{MN} \left(\int [\pi^b, \partial_N \chi^b]_+ \right) \\
&\quad + \frac{1}{4} R - \frac{1}{2} Y - \frac{1}{8g^2} \int (\nabla^2 \phi)^2\,. \tag{5.13}
\end{aligned}
$$

Here, $H^{(n)}$ is $\mathcal{O}(g^n)$ provided that (5.10) and (5.11) hold, and we recall that $G^{MN} \sim \mathcal{O}(g^2)$. $V^{(3,4)}(\phi)$ denote the third and fourth derivatives of the potential, evaluated on the soliton solution ϕ. Finally, we have used the results of section 4 to simplify the terms in $H^{(2)}$ that are zeroth order in fluctuations.

Let us briefly discuss the issue of Lorentz invariance. Eq. (5.12) is in principle a double expansion: a quantum expansion in the coupling, as well as an expansion in small soliton velocities. A subset of the collective coordinates, $\{U^i\}_{i=1}^{D-1} \subset \{U^M\}$, correspond to the center-of-mass position of the soliton solution ϕ. The conjugate variables, p_i, correspond to the center-of-mass momentum. One expects that any observable computed exactly in the quantum theory should be covariant under Lorentz transformations. On the one hand, expanding around slowly-moving solitons — in particular $p_i \sim \mathcal{O}(1/g)$ — naturally breaks the Lorentz symmetry of the original theory. On the other, the scaling (5.10) suggests that relativistic corrections should appear as quantum effects associated with the $p_i^2 \chi$ tadpoles. In fact, it can be explicitly seen for the case of kink solitons in two-dimensional ϕ^4 theory that re-summing all the tree-level diagrams obtained by gluing together the $p_i^2 \chi$ tadpole interactions restores Lorentz invariance for the soliton energy [8, 35]. This computation should be extendable to the class of theories we are studying, but we will not explicitly consider it here.

6 Reduction to QM on the soliton moduli space

It is straightforward to use our results for the semiclassical expansion of the Hamiltonian to determine the behavior of the leading-order dynamics. Keeping terms in H through $\mathcal{O}(1)$, we have

$$H = M_{\rm cl} + \frac{1}{2}p_M G^{MN} p_N + \delta v(U) + \frac{1}{2}\int (\pi \cdot \pi + \chi \cdot \Delta\chi) + \mathcal{O}(g) . \tag{6.1}$$

Let us focus on the fluctuation terms. We make a mode expansion

$$\chi(x;U) = \int \frac{{\rm d}\mathbf{k}}{(2\pi)^{D-1}} \frac{1}{\sqrt{2\omega_\mathbf{k}}}\Big[a_\mathbf{k}(t) + a^\dagger_{-\mathbf{k}}(t)\Big]\zeta_\mathbf{k}(\mathbf{x};U)$$

$$\pi(x;U) = \int \frac{{\rm d}\mathbf{k}}{(2\pi)^{D-1}}(-i)\sqrt{\frac{\omega_\mathbf{k}}{2}}\Big[a_\mathbf{k}(t) - a^\dagger_{-\mathbf{k}}(t)\Big]\zeta_\mathbf{k}(\mathbf{x};U) , \tag{6.2}$$

where the ζ's are eigenfunctions of the operator $\Delta(U)$ with *strictly positive* eigenvalues $\omega_\mathbf{k}^2$: $\Delta(U)\zeta_\mathbf{k} = \omega_\mathbf{k}^2(U)\zeta_\mathbf{k}$. They are orthonormal

$$\int {\rm d}\mathbf{x}\, \zeta_\mathbf{k}(\mathbf{x};U)\zeta_{\mathbf{k}'}(\mathbf{x};U) = (2\pi)^{D-1}\delta(\mathbf{k} - \mathbf{k}') \tag{6.3}$$

and satisfy the completeness relation

$$\int \frac{{\rm d}\mathbf{k}}{(2\pi)^{D-1}}\zeta_\mathbf{k}(\mathbf{x};U)\zeta_\mathbf{k}(\mathbf{y};U) = \delta(\mathbf{x} - \mathbf{y}) - \frac{1}{g^2}\partial_M\phi(\mathbf{x};U) \cdot G^{MN}\partial_N\phi(\mathbf{y};U) . \tag{6.4}$$

The modified completeness relation is due to the fact that we have excluded the zero-eigenvalue modes from the expansion. The $\zeta_\mathbf{k}(\mathbf{x}, U)$ form a basis for the subspace of configuration space orthogonal to the tangent space $T_U\mathcal{M}$. Using (6.4), one can show that the commutator $[\chi, \pi]$, (2.20), is equivalent to the standard creation and annihilation commutators

$$[a_\mathbf{k}, a_{\mathbf{k}'}] = [a^\dagger_\mathbf{k}, a^\dagger_{\mathbf{k}'}] = 0 , \qquad [a_\mathbf{k}, a^\dagger_{\mathbf{k}'}] = (2\pi)^{D-1}\delta(\mathbf{k} - \mathbf{k}') . \tag{6.5}$$

We have written the mode expansions (6.2) as though the non-zero spectrum of Δ is purely continuous. While the spectrum of Δ is guaranteed to have a continuous component,[10] there could additionally be a discrete component beyond the zero-modes. Strictly positive discrete eigenvalues correspond to breather-like modes, and the mode expansion should include a sum over them. We will understand '$\int d\mathbf{k}$' in the above and following expressions as representing the integral over the continuous spectrum plus the sum over the breather-like modes, if present.

Using (6.2), (6.3), and (6.5), it is then easy to see that

$$\frac{1}{2}\int d\mathbf{x}\,(\pi\cdot\pi + \chi\cdot\Delta\chi) = \int \frac{d\mathbf{k}}{(2\pi)^{D-1}}\omega_{\mathbf{k}}\Big(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + \frac{1}{2}[a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}]\Big) \tag{6.6}$$

so that the full Hamiltonian is

$$H \simeq M_{\mathrm{cl}} + \frac{1}{2}p_M G^{MN}p_N + \delta v(U) + \int \frac{d\mathbf{k}}{(2\pi)^{D-1}}\omega_{\mathbf{k}}\Big(\frac{1}{2}[a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}] + a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}\Big)\,. \tag{6.7}$$

In particular, when acting on a state which does not involve massive fluctuations, the last term above vanishes and one is left with the zero-point energy of the fluctuation fields. In a renormalizable theory, the divergent part of this quantity can be removed, after vacuum energy subtraction, by mass renormalization; see footnote 8. The finite piece then generates a one-loop correction to the potential $M_{\mathrm{cl}} + \delta v(U) \to M_{\text{1-loop}} + \delta v(U)_{\text{1-loop}}$ [1–3]. Hence, the final result for the leading contribution in the semiclassical approximation *and when restricting to incoming and outgoing states that do not contain perturbative excitations* is

$$H_{\mathrm{s.c.}} = M_{\text{1-loop}} + \frac{1}{2}p_M G^{MN}p_N + \delta v(U)_{\text{1-loop}}\,, \tag{6.8}$$

which is a quantum mechanics on the soliton moduli space. We will refer to (6.8) as the 'semiclassical Hamiltonian'.

This quantum mechanics, as written, is not covariant with respect to general coordinate transformations on $\mathcal{M}$. However, following the discussion around (4.9), it can be trivially made covariant by replacing $p_M G^{MN}p_N \to G^{-1/4}p_M G^{1/2}G^{MN}p_N G^{-1/4}$. These two quantities differ by Y, which is higher order in the g-expansion and hence can be neglected in (6.8).

It is interesting to note that, even after this replacement, the two quantum mechanical systems on $\mathcal{M}$ defined by the truncated Hamiltonian (4.13) and the semiclassical Hamiltonian (6.8) are different. Although the intrinsic and extrinsic quantum potentials of the truncated Hamiltonian are present in the semiclassical expansion (5.12), it would be inconsistent to include them in the semiclassical Hamiltonian (6.8), without first accounting

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[10] This statement can be justified as follows: Since classical solitons are localized objects, we expect the difference between the operator $\Delta(U)$ and the operator $\Delta^0 := -\delta_{ab}\partial_{\mathbf{x}}^2 + V_{ab}^{(2),\infty}(\hat{\mathbf{x}})$, to be a compact operator. Here $V_{ab}^{(2),\infty}(\hat{\mathbf{x}}) \geq 0$ is the asymptotic form of the second derivative of the potential evaluated on the soliton solution as $\mathbf{x} \to \infty$, and $\hat{\mathbf{x}}$ parameterizes the $(D-2)$-sphere at infinity. Weyl's theorem then implies that the continuous part of the spectra of $\Delta(U)$ and Δ^0 must agree. If $\min_{\hat{\mathbf{x}}} V_{ab}^{(2),\infty}(\hat{\mathbf{x}}) > 0$, then the continuous spectrum of Δ^0 will have a mass gap, while if $\min_{\hat{\mathbf{x}}} V_{ab}^{(2),\infty}(\hat{\mathbf{x}}) = 0$ it will extend down to zero. In either case there will be a continuous spectrum that we can label by $\mathbf{k}$.

for all $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$ corrections from integrating out the fluctuations. Furthermore, the semiclassical approximation demands that the $\mathcal{O}(1)$, 'one-loop' corrections from χ, π be accounted for in the semiclassical Hamiltonian: They are of the same order as the kinetic term and moduli-dependent classical potential $\delta v(U)$, due to the necessity of imposing (5.10) and (5.11).

Acknowledgments

A Mode expansions for χ, π

For a fixed value of the moduli, χ and π are simply n-tuples of scalar fields on $\mathbb{R}^{D-1}$; they can be expanded in any complete basis for the Hilbert space $L^2[\mathbb{R}^{D-1}, \mathbb{R}^n]$. A particular basis that is naturally adapted to the problem is the basis of eigenfunctions of the Hermitian operator $\Delta(U)$, defined in (2.8). Since the form of this operator depends on the moduli, so will its eigenfunctions; we denote the complete set of eigenfunctions by $\{\zeta_{\mathcal{I}}(\mathbf{x}; U)\}$, where $\mathcal{I}$ runs over an indexing set. This set will include both the continuous part of the spectrum, as well as the discrete part of the spectrum, which includes the zero-modes and may additionally contain other massive breather-like modes. We write schematically

$$\chi(x; U) = \sum_{\mathcal{I}} \chi^{\mathcal{I}}(t)\zeta_{\mathcal{I}}(\mathbf{x}, U), \qquad \pi(x; U) = \sum_{\mathcal{I}} \pi^{\mathcal{I}}(t)\zeta_{\mathcal{I}}(\mathbf{x}, U), \qquad (A.1)$$

where $\chi^{\mathcal{I}}(t), \pi^{\mathcal{I}}(t)$ comprise the complete set of degrees of freedom in $\chi(x; U), \pi(x; U)$. The $\zeta_{\mathcal{I}}$ satisfy

$$\int d\mathbf{x}\zeta_{\mathcal{I}}(\mathbf{x}; U) \cdot \zeta_{\mathcal{J}}(\mathbf{x}, U) = \delta_{\mathcal{I}\mathcal{J}}, \qquad \delta^{ab}\delta(\mathbf{x} - \mathbf{y}) = \sum_{\mathcal{I}} \zeta_{\mathcal{I}}^a(\mathbf{x}; U)\zeta_{\mathcal{I}}^b(\mathbf{y}; U), \qquad (A.2)$$

where by '$\delta_{\mathcal{I}\mathcal{J}}$' and '$\sum_{\mathcal{I}}$' we mean $(2\pi)^{D-1}\delta(\mathbf{k} - \mathbf{k}')$ and $\int \frac{d\mathbf{k}}{(2\pi)^{D-1}}$ in the case of the continuous spectrum.

Let $e^A = e^A{}_M \, dU^M$ be a vielbein for the moduli space such that $\delta_{AB}e^A{}_M e^A{}_N = G_{MN}$, where δ_{AB} is the flat Euclidean metric on the tangent space, and let $e_A{}^M$ denote the inverse vielbein satisfying $\delta^{AB}e_A{}^M e_B{}^N = G^{MN}$. Then we know that the orthonormal eigenfunctions for the zero-modes are

$$\zeta_{\mathcal{I}=A}(\mathbf{x}; U) = e_A{}^M(U)\partial_M\phi(\mathbf{x}; U) . \qquad (A.3)$$

Then, substituting (A.1) into the constraints and using (A.2), we have

$$\psi_N^{(1)} = \chi^A e_A{}^M(U) \int \partial_M \phi(\mathbf{x}; U) \cdot \partial_N \phi(\mathbf{x}; U) = \chi^A e_A{}^M(U) G_{MN}(U) = \chi_A e^A{}_N(U), \quad \text{(A.4)}$$

and similarly

$$\psi_N^{(2)} = \pi_A e^A{}_N(U) . \quad \text{(A.5)}$$

From here we can explicitly see that the constraint surface corresponds to $\chi_A = \pi_A = 0$. Meanwhile,

$$\partial_M \psi_N^{(1)} = \chi_A \partial_M e^A{}_N(U) \approx 0, \qquad \partial_M \psi_N^{(2)} = \pi_A \partial_M e^A{}_N(U) \approx 0 . \quad \text{(A.6)}$$

Another identity that follows trivially from (A.1) and (A.2) and will be useful below is

$$\partial_M \left(\int \mathrm{d}\mathbf{x} \chi(\mathbf{x}; U) \cdot \pi(\mathbf{x}; U) \right) = \partial_M \left(\sum_I \chi_I \pi^I \right) = 0, \quad \text{(A.7)}$$

or $\int \partial_M \chi \cdot \pi = - \int \chi \cdot \partial_M \pi$. Finally, if I, J index the non-zero modes, which include the continuous spectrum and any possible breather-like modes, then the commutator of the fields χ, π is equivalent to

$$[\chi^I, \chi^J] = [\pi^I, \pi^J] = 0, \qquad [\chi^I, \pi^J] = i\delta^{IJ} . \quad \text{(A.8)}$$

These can be used to show, for example, that

$$\begin{aligned} [\chi^a(\mathbf{x}; U), \partial_M \chi^b(\mathbf{y}; U)] &\approx 0 \\ [\pi^a(\mathbf{x}; U), \partial_M \pi^b(\mathbf{y}; U)] &\approx 0 . \end{aligned} \quad \text{(A.9)}$$

More generally, the commutator of any U-derivative of χ with another U-derivative of χ is zero, and similarly for π.

B Some details on the canonical transformation

First let us consider $\{\Phi(\mathbf{x}), \Pi(\mathbf{y})\}'_{\mathrm{D}}$ in order to derive the classical form of Π_0^M as given in (2.23). Substituting in (2.9), (2.13) for Φ, Π and using (2.20), we can write the result as

$$\begin{aligned} \{\Phi^a(\mathbf{x}), \Pi^b(\mathbf{y})\}'_{\mathrm{D}} &\approx \delta^{ab}\delta(\mathbf{x} - \mathbf{y}) + \partial_M \phi^a(\mathbf{x}) \left(\frac{\partial \Pi_0^N}{\partial p_M} - G^{MN} \right) \partial_N \phi^b(\mathbf{y}) \\ &\quad - \partial_Q \chi^a(\mathbf{x}) \frac{\partial \Pi_0^N}{\partial p_M} \partial_N \phi^b(\mathbf{y}) \\ &\quad + \int \mathrm{d}z \left(\delta^{ac}\delta(\mathbf{x} - \mathbf{z}) - \partial_M \phi^a(\mathbf{x}) G^{MQ} \partial_Q \phi^c(\mathbf{z}) \right) \frac{\delta \Pi_0^N}{\delta \pi^c(\mathbf{z})} \partial_N \phi^b(\mathbf{y}) . \quad \text{(B.1)} \end{aligned}$$

The first term is what we want; thus we must choose the functional Π_0^M so that the remaining terms vanish. Consider the $\mathbf{x}$ dependence of these remaining terms. The term in the first line is tangential to $T_U\mathcal{M} \subset L^2[\mathbb{R}^{D-1}_{(\mathbf{x})}]$ since it is proportional to the zero-mode $\partial_M \phi(\mathbf{x})$, while the term in the last line is in the orthogonal complement $(T_U\mathcal{M})^\perp$ since it

involves the projection operator $\delta(\mathbf{x}-\mathbf{z}) - \partial_M\phi(\mathbf{x})G^{MQ}\partial_Q\phi(\mathbf{z})$. The term involving $\partial_Q\chi(\mathbf{x})$ can be decomposed into a piece along $T_U\mathcal{M}$ and a piece orthogonal to it. Substituting this into (B.1), we find that $\{\Phi(\mathbf{x}),\Pi(\mathbf{y})\}'_{\mathrm{D}} = \delta(\mathbf{x}-\mathbf{y})$ if and only if both of the following hold:

$$
\begin{aligned}
0 &\approx \frac{\partial\Pi_0^N}{\partial p_M} - G^{MN} - \frac{\partial\Pi_0^N}{\partial p_M}G^{MP}\int \partial_P\phi\cdot\partial_Q\chi\,, \\
0 &\approx \partial_Q\chi^c(\mathbf{z})\frac{\partial\Pi_0^N}{\partial p_M} + \frac{\delta\Pi_0^N}{\delta\pi^c(\mathbf{z})}\;.
\end{aligned}
\tag{B.2}
$$

Note that we can write

$$
\int \partial_P\phi\cdot\partial_Q\chi = \partial_Q\psi_P^{(1)} - \int \chi\cdot\partial_P\partial_Q\phi \approx -\int \chi\cdot\partial_P\partial_Q\phi = -\Xi_{PQ}\,,
\tag{B.3}
$$

where Ξ_{PQ} was defined in (2.24). The first of (B.2) implies

$$
\frac{\partial\Pi_0^N}{\partial p_M} \approx \left[(G-\Xi)^{-1}\right]^{MN} =: C^{MN}
\tag{B.4}
$$

whence the second equation implies that

$$
\Pi_0^N \approx \left(p_M - \int \pi\cdot\partial_M\chi\right)C^{MN}\;.
\tag{B.5}
$$

Here we have omitted the possible addition of a term depending only on the coordinates (U,χ). Consideration of $\{\Pi(\mathbf{x}),\Pi(\mathbf{y})\}'_{\mathrm{D}}$ shows that it is consistent to set this term to zero.

We observe that if we set $\chi,\pi = 0$, then the momentum transformation (2.13) with (B.5) reduces to $\Pi(\mathbf{x}) = p^M\partial_M\phi(\mathbf{x};U)$. This is exactly what one would expect for the classical momentum density of the moving soliton.

At the quantum level we take the change of momentum variables to be (2.25). The basic commutators are the right-hand sides of (2.20), multiplied by a factor of i. Using these we have

$$
\begin{aligned}
[f(U),a^M]' &= [f(U),\bar{a}^M]' = iC^{MN}\partial_N f(U)\,, \\
[\chi(\mathbf{x};U),a^M]' &= [\chi(\mathbf{x};U),\bar{a}^M]' \approx i\left(G^{MN} - C^{MN}\right)\partial_N\phi(\mathbf{x};U)\,,
\end{aligned}
\tag{B.6}
$$

where f is any function of U. Then one easily obtains the desired relation,

$$
[\Phi^a(\mathbf{x}),\Pi^b(\mathbf{y})]' \approx i\delta^{ab}\delta(\mathbf{x}-\mathbf{y})\;.
\tag{B.7}
$$

For $[\Pi(\mathbf{x}),\Pi(\mathbf{y})]'$ we first note that

$$
[\pi(\mathbf{x}),C^{MN}] = -iC^{MP}\left(\nabla_P\partial_Q\phi(\mathbf{x})\right)C^{QN}\,,
\tag{B.8}
$$

from which it follows that

$$
\begin{aligned}
[\pi(\mathbf{x}),a^M]' &\approx -i\Theta_{PN}C^{NM}G^{PQ}\partial_Q\phi(\mathbf{x}) - ia^N C^{MP}\nabla_N\partial_P\phi(\mathbf{x})\,, \\
[\pi(\mathbf{x}),\bar{a}^M]' &\approx -i\left(\nabla_P\partial_N\phi(\mathbf{x})\right)C^{MP}\bar{a}^N - i\left(\partial_Q\phi(\mathbf{x})\right)G^{QP}C^{MN}\Theta_{NP}
\end{aligned}
\tag{B.9}
$$

and where we have defined

$$\Theta_{MN} := \int \pi \cdot \partial_M \partial_N \phi \, . \tag{B.10}$$

Note that $\Theta_{MN} = \Theta_{(MN)}$. Using this, one can express $[\Pi, \Pi]$ in the form

$$
\begin{aligned}
{[\Pi^a(\mathbf{x}), \Pi^b(\mathbf{y})]'} \approx {}& - i a^P C^{Q[M} \Gamma^{N]}{}_{PQ} \partial_M \phi^a(\mathbf{x}) \partial_N \phi^b(\mathbf{y}) + i \partial_M \phi^a(\mathbf{x}) \partial_N \phi^b(\mathbf{y}) \Gamma^{[M}{}_{PQ} C^{N]P} \bar{a}^Q \\
& + i \left(C\Theta G^{-1} - G^{-1}\Theta C \right)^{[MN]} \partial_M \phi^a(\mathbf{x}) \partial_N \phi^b(\mathbf{y}) \\
& + \frac{1}{4} [a^M, a^N] \partial_M \phi^a(\mathbf{x}) \partial_N \phi^b(\mathbf{y}) + \frac{1}{4} \partial_M \phi^a(\mathbf{x}) \partial_N \phi^b(\mathbf{y}) [\bar{a}^M, \bar{a}^N] \\
& + \frac{1}{4} \partial_N \phi^b(\mathbf{y}) [a^M, \bar{a}^N] \partial_M \phi^a(\mathbf{x}) - \frac{1}{4} \partial_N \phi^a(\mathbf{x}) [a^M, \bar{a}^N] \partial_M \phi^b(\mathbf{y}) \, . \tag{B.11}
\end{aligned}
$$

What one needs is then the commutators of the a's and $\bar{a}$'s. Equations (B.6) and (B.9) together with

$$[p_Q, C^{MN}] = - C^{MR}[p_Q, G_{RS} - \Xi_{RS}]C^{SN} = iC^{MR} \left(\partial_Q G_{RS} - \partial_Q \Xi_{RS} \right) C^{SN} \tag{B.12}$$

can be used to show

$$
\begin{aligned}
\left[C^{PM}, p_Q - \int \pi \cdot \partial_Q \chi \right] = \left[C^{PM}, p_Q - \int \partial_Q \chi \cdot \pi \right] \approx {}& \\
\approx {}& -iC^{PR} C^{MS} S_{QRS} + iC^{PR} C^{MS} \Gamma^T{}_{RS} (C^{-1})_{TQ} \, , \tag{B.13}
\end{aligned}
$$

where we have defined

$$S_{QRS} := \partial_Q G_{RS} + \Gamma_{QRS} - \int \chi \cdot \partial_Q \partial_R \partial_S \phi \, , \tag{B.14}$$

which is totally symmetric, $S_{QRS} = S_{(QRS)}$. Making note of the comment below (A.9) and using (A.7), one also finds that

$$
\begin{aligned}
{[(p_P - \int \pi \cdot \partial_P \chi), (p_Q - \int \pi \cdot \partial_Q \chi)]} \approx {}& - 2i \Theta_{[P|R} G^{RS} (C^{-1})_{S|Q]} \, , \\
{[(p_P - \int \partial_P \chi \cdot \pi), (p_Q - \int \partial_Q \chi \cdot \pi)]} \approx {}& 2i (C^{-1})_{[P|R} G^{RS} \Theta_{S|Q]} \, , \\
{[(p_P - \int \pi \cdot \partial_P \chi), (p_Q - \int \partial_Q \chi \cdot \pi)]} \approx {}& i \Theta_{QR} G^{RS} (C^{-1})_{SP} - i (C^{-1})_{QR} G^{RS} \Theta_{SP} \\
& + i \int [\partial_Q \partial_P \chi^a, \pi^a] \\
& - \int d\mathbf{z}\, d\mathbf{w} [\partial_Q \chi^b(\mathbf{w}), \pi^a(\mathbf{z})][\partial_P \chi^a(\mathbf{z}), \pi^b(\mathbf{w})] \, . \tag{B.15}
\end{aligned}
$$

These imply

$$
\begin{aligned}
{[a^M, a^N]} &\approx 2i \left(a^P C^{Q[M} \Gamma^{N]}{}_{PQ} + (G^{-1}\Theta C)^{[MN]} \right) \\
{[\bar{a}^M, \bar{a}^N]} &\approx -2i \left(\Gamma^{[M}{}_{PQ} C^{N]P} \bar{a}^Q + (C\Theta G^{-1})^{[MN]} \right) \tag{B.16}
\end{aligned}
$$

and

$$[a^M, \bar{a}^N] \approx ia^P C^{QM} \Gamma^N{}_{PQ} - i\Gamma^M{}_{PQ} C^{NP} \bar{a}^Q + i \left(C\Theta G^{-1} - G^{-1}\Theta C \right)^{NM}$$
$$- ia^P C^{MQ} C^{NR} S_{PQR} + iS_{PQR} C^{MP} C^{NQ} \bar{a}^R$$
$$- C^{NQ} \Bigg\{ \Gamma^R{}_{SP} \Gamma^S{}_{RQ} + C^{RS} C^{TV} S_{RTP} S_{SVQ} - 2\Gamma^R{}_{(P|S} C^{ST} S_{TR|Q)}$$
$$- \int [\partial_Q \partial_P \chi^a, \pi^a] + \int d\mathbf{z}\, d\mathbf{w} [\partial_Q \chi^b(\mathbf{w}), \pi^a(\mathbf{z})][\partial_P \chi^a(\mathbf{z}), \pi^b(\mathbf{w})] \Bigg\} C^{MP} \,. \tag{B.17}$$

When substituting (B.17) into (B.11), the last two lines of (B.17) do not contribute because they commute with $\partial_M \phi$ and are symmetric in M, N. Furthermore, after commuting all a's to the far left and all $\bar{a}$'s to the far right, and using the symmetry properties of C^{MN}, S_{MNP}, there are additional cancellations and one is left with

$$\partial_N \phi^b(\mathbf{y})[a^M, \bar{a}^N]\partial_M \phi^a(\mathbf{x}) - \partial_N \phi^a(\mathbf{x})[a^M, \bar{a}^N]\partial_M \phi^b(\mathbf{y}) \approx$$
$$\approx 2ia^P C^{Q[M} \Gamma^{N]}{}_{PQ} \partial_M \phi^a(\mathbf{x})\partial_N \phi^b(\mathbf{y}) - 2i\partial_M \phi^a(\mathbf{x})\partial_N \phi^b(\mathbf{y}) \Gamma^{[M}{}_{PQ} C^{N]P} \bar{a}^Q$$
$$+ 2i \left(G^{-1}\Theta C - C\Theta G^{-1} \right)^{[MN]} \partial_M \phi^a(\mathbf{x})\partial_N \phi^b(\mathbf{y}) \,. \tag{B.18}$$

Using (B.16) and (B.18) in the calculation of (B.11) leads to complete cancellation on the constraint surface:

$$[\Pi^a(\mathbf{x}), \Pi^b(\mathbf{y})]' \approx 0 \,. \tag{B.19}$$

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References

[1] E. Tomboulis, *Canonical Quantization of Nonlinear Waves*, *Phys. Rev.* **D 12** (1975) 1678 [INSPIRE].

[2] R.F. Dashen, B. Hasslacher and A. Neveu, *Nonperturbative Methods and Extended Hadron Models in Field Theory. 1. Semiclassical Functional Methods*, *Phys. Rev.* **D 10** (1974) 4114 [INSPIRE].

[3] R.F. Dashen, B. Hasslacher and A. Neveu, *Nonperturbative Methods and Extended Hadron Models in Field Theory. 2. Two-Dimensional Models and Extended Hadrons*, *Phys. Rev.* **D 10** (1974) 4130 [INSPIRE].

[4] J. Goldstone and R. Jackiw, *Quantization of Nonlinear Waves*, *Phys. Rev.* **D 11** (1975) 1486 [INSPIRE].

[5] J.-L. Gervais and B. Sakita, *Extended Particles in Quantum Field Theories*, *Phys. Rev.* **D 11** (1975) 2943 [INSPIRE].

[6] C.G. Callan Jr. and D.J. Gross, *Quantum Perturbation Theory of Solitons*, *Nucl. Phys.* **B 93** (1975) 29 [INSPIRE].

[7] N.H. Christ and T.D. Lee, *Quantum Expansion of Soliton Solutions*, *Phys. Rev.* **D 12** (1975) 1606 [inSPIRE].

[8] J.-L. Gervais, A. Jevicki and B. Sakita, *Perturbation Expansion Around Extended Particle States in Quantum Field Theory. 1.*, *Phys. Rev.* **D 12** (1975) 1038 [inSPIRE].

[9] E. Tomboulis and G. Woo, *Soliton Quantization in Gauge Theories*, *Nucl. Phys.* **B 107** (1976) 221 [inSPIRE].

[10] R. Jackiw, *Quantum Meaning of Classical Field Theory*, *Rev. Mod. Phys.* **49** (1977) 681 [inSPIRE].

[11] R. Rajaraman, *Solitons and Instantons: An introduction to solitons and instantons in Quantum Field Theory*, North Holland (1982).

[12] G. 't Hooft, *Magnetic Monopoles in Unified Gauge Theories*, *Nucl. Phys.* **B 79** (1974) 276 [inSPIRE].

[13] A.M. Polyakov, *Particle Spectrum in the Quantum Field Theory*, *JETP Lett.* **20** (1974) 194 [inSPIRE].

[14] E.B. Bogomolny, *Stability of Classical Solutions*, *Sov. J. Nucl. Phys.* **24** (1976) 449 [inSPIRE].

[15] M.K. Prasad and C.M. Sommerfield, *An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon*, *Phys. Rev. Lett.* **35** (1975) 760 [inSPIRE].

[16] N.S. Manton, *A Remark on the Scattering of BPS Monopoles*, *Phys. Lett.* **B 110** (1982) 54 [inSPIRE].

[17] M. Atiyah and N.J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, *M.B. Porter Lectures* (1988).

[18] R.F. Dashen, B. Hasslacher and A. Neveu, *The Particle Spectrum in Model Field Theories from Semiclassical Functional Integral Techniques*, *Phys. Rev.* **D 11** (1975) 3424 [inSPIRE].

[19] G.W. Gibbons and N.S. Manton, *Classical and Quantum Dynamics of BPS Monopoles*, *Nucl. Phys.* **B 274** (1986) 183 [inSPIRE].

[20] S. Sethi, M. Stern and E. Zaslow, *Monopole and Dyon bound states in $N = 2$ supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 457** (1995) 484 [hep-th/9508117] [inSPIRE].

[21] M. Cederwall, G. Ferretti, B.E.W. Nilsson and P. Salomonson, *Low-energy dynamics of monopoles in $N = 2$ SYM with matter*, *Mod. Phys. Lett.* **A 11** (1996) 367 [hep-th/9508124] [inSPIRE].

[22] J.P. Gauntlett and J.A. Harvey, *S duality and the dyon spectrum in $N = 2$ super Yang-Mills theory*, *Nucl. Phys.* **B 463** (1996) 287 [hep-th/9508156] [inSPIRE].

[23] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [*Erratum ibid.* **B 430** (1994) 485-486] [hep-th/9407087] [inSPIRE].

[24] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099] [inSPIRE].

[25] R.K. Kaul, *Monopole Mass in Supersymmetric Gauge Theories*, *Phys. Lett.* **B 143** (1984) 427 [inSPIRE].

[26] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, *Quantum mass and central charge of*

supersymmetric monopoles: Anomalies, current renormalization and surface terms,
JHEP **06** (2006) 056 [`hep-th/0601029`] [ɪɴSPIRE].

[27] C. Papageorgakis and A.B. Royston, *Revisiting Soliton Contributions to Perturbative Amplitudes*, `arXiv:1404.0016` [ɪɴSPIRE].

[28] K. Fujii, N. Ogawa, S. Uchiyama and N.M. Chepilko, *Geometrically induced gauge structure on manifolds embedded in a higher dimensional space*, *Int. J. Mod. Phys.* **A 12** (1997) 5235 [`hep-th/9702191`] [ɪɴSPIRE].

[29] I.G. Moss and N. Shiiki, *Quantum mechanics on moduli spaces*, *Nucl. Phys.* **B 565** (2000) 345 [`hep-th/9904023`] [ɪɴSPIRE].

[30] H. Jensen and H. Koppe, *Quantum mechanics with constraints*, *Annals Phys.* **63** (1971) 586 [ɪɴSPIRE].

[31] P. Maraner, *Monopole gauge fields and quantum potentials induced by the geometry in simple dynamical systems*, *Annals Phys.* **246** (1996) 325 [`hep-th/9406004`] [ɪɴSPIRE].

[32] G.H. Derrick, *Comments on nonlinear wave equations as models for elementary particles*, *J. Math. Phys.* **5** (1964) 1252 [ɪɴSPIRE].

[33] N. Manton and P. Sutcliffe, *Topological solitons*, Cambridge University Press, Cambridge U.K. (2004).

[34] B.S. DeWitt, *Dynamical theory in curved spaces. 1. A Review of the classical and quantum action principles*, *Rev. Mod. Phys.* **29** (1957) 377 [ɪɴSPIRE].

[35] J.-L. Gervais, A. Jevicki and B. Sakita, *Collective Coordinate Method for Quantization of Extended Systems*, *Phys. Rept.* **23** (1976) 281 [ɪɴSPIRE].