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Supersymmetric Wilson Loops in Diverse Dimensions

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Abstract

We consider supersymmetric Wilson loops à la Zarembo in planar supersymmetric Yang-Mills theories in diverse dimensions. Using perturbation theory we show that these loops have trivial vacuum expectation values to second order in the 't Hooft coupling. We review the known superspace results which, for specific dimensions, extend this triviality to all orders in the 't Hooft coupling. Using the gauge/gravity correspondence, we construct the explicit dual fundamental string solutions corresponding to these Wilson loops for the case of circular geometry. We find that the regularized action of these string solutions vanishes. We also generalize the framework of calibrated surfaces to prove the vanishing of the regularized action for loops of general geometry. We propose a possible string-side manifestation of the gauge theory generalized Konishi anomaly in seven dimensions.

1 Introduction

By now there is mounting evidence in favor of both the usefulness as well as the validity of the gauge/gravity duality between $\mathcal{N}=4$ SYM in four dimensions and string theory on $AdS_5\times S^5$. It is thus natural to ask if this duality can be tested and utilized in the cases of gauge theories in dimensions other than four. Such investigations are naturally motivated by the need to understand how the gauge/gravity duality may be realized in non-conformal supersymmetric Yang-Mills theories. For the special cases of sixteen supercharge SYM theories in diverse dimensions, the gravity duals were proposed some time ago in [1]. While the feasibility of generic tests of gauge/gravity duality is not very clear for sixteen supercharge SYM theories in dimensions greater than four: the SYM theories are not renormalizable and the dual Dp-brane geometries suffer from the non-decoupling of the alpha-prime corrections, it is worthwhile to exploit the duality between SYM theories in $p+1$ dimensions and Dp-branes and test it in the case of protected operators whose vacuum expectation values are independent of the coupling g^2. A special class of Wilson loops, first proposed by Zarembo in the case of $\mathcal{N}=4$ SYM [2] are particularly well suited to this purpose. In this paper we generalize Zarembo's construction to perform a non-trivial test of the duality between Dp-brane theories and SYM in $p+1$ dimensions.

The Maldacena-Wilson loop [3, 4] has proven to be a very powerful probe of the AdS/CFT correspondence. In four dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory it is given by

$$W=\frac{1}{N}\,\mathrm{Tr}\,P\exp\oint d\tau\,\Big(i\dot{x}^\mu(\tau)A_\mu+|\dot{x}(\tau)|\Theta^I(\tau)\Phi^I\Big), \tag{1}$$

where N is the rank of the gauge group $SU(N)$, Φ^I are the six scalar fields of the theory, and $\Theta^I\Theta^I=1$. The path of the Wilson loop is defined by $x^\mu(\tau)$, but there is also the freedom to define a path on S^5 parametrized by $\Theta^I(\tau)$. The specific coupling to the scalar fields in (1) is chosen to ensure local supersymmetry; the amount of global supersymmetry respected by W is intimately connected with the correlation of the paths $x^\mu(\tau)$ and $\Theta^I(\tau)$. There is a "perfectly" correlated choice, found by Zarembo [2][1]

$$\Theta^I(\tau)=\frac{\dot{x}^\mu}{|\dot{x}|}\,M^I_\mu,\quad M^I_\mu M^I_\nu=\delta_{\mu\nu}, \tag{2}$$

where M^I_μ is a constant matrix, which assures that the vacuum expectation value of the Wilson loop is trivial

$$\langle W\rangle_{\text{Zarembo}}=1. \tag{3}$$

The amount of supersymmetry respected by the loop is found by requiring

$$\delta_\epsilon W\sim\dot{x}^\mu\Big(i\gamma_\mu+M^I_\mu\Gamma_I\Big)\epsilon=0. \tag{4}$$

[1]These Wilson loops are closely related to a class constructed later in [5, 6, 7] whose contours lie on a three-sphere.

This gives one halving of the supersymmetry[2] for each non-zero component of $\dot{x}^\mu$, so that, for example, a planar loop is 1/4 BPS. One can appreciate the result (3) from a few different perspectives. The first is that (2) ensures that the combined gauge and scalar field Feynman gauge propagator joining two points on the loop is zero

$$\begin{aligned}\left\langle \left(i\dot{x}^\mu(\tau) A_\mu + |\dot{x}(\tau)| \Theta_x^I(\tau) \Phi^I \right) \left(i\dot{y}^\nu(\sigma) A_\nu + |\dot{y}(\sigma)| \Theta_y^J(\sigma) \Phi^J \right) \right\rangle \\ = \frac{g^2}{4\pi^2} \frac{-\dot{x}\cdot\dot{y} + \Theta_x \cdot \Theta_y |\dot{x}||\dot{y}|}{(x-y)^2} = 0,\end{aligned} \tag{5}$$

which immediately precludes the contribution of ladder/rainbow diagrams. As shown by Zarembo [2], all interacting diagrams up to two loops can also, without an inordinate effort, be shown to vanish. A much stronger statement was made in [8], where superspace techniques were exploited to prove (3) for Wilson loops whose contours are contained in $\mathbb{R}^3$. The loops of Zarembo are also naturally described in terms of the twisting of $\mathcal{N}=4$ SYM to produce a topological theory; in this context the triviality of the vacuum expectation value for loops in the full $\mathbb{R}^4$ was proven in [9].

At strong coupling the vacuum expectation value of (1) is accessible via the dual string theory. It is given by the partition function of a fundamental string, the saddle points of which are minimal area embeddings in $AdS_5 \times S^5$ [3, 10]. In the following coordinates for $AdS_5 \times S^5$

$$ds^2 = U^2 dX^\mu dX^\mu + \frac{1}{U^2} dU^I dU^I, \tag{6}$$

one requires the following boundary conditions for the string embedding Σ at the boundary $U = \infty$

$$X^\mu|_{\partial\Sigma} = x^\mu, \qquad \left.\frac{U^I}{|U|}\right|_{\partial\Sigma} = \Theta^I. \tag{7}$$

The action of the string is then found to contain a generic divergence owing to the diverging area element of Anti-de Sitter space as the boundary is approached. This divergence is proportional to the circumference of the loop

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{\det \partial_a \mathbb{X}^M \partial_b \mathbb{X}^N G_{MN}} = \frac{\sqrt{\lambda}}{2\pi} \left(U_{\text{max.}} \oint d\tau |\dot{x}(\tau)| + A_{\text{reg.}} \right), \tag{8}$$

where $\mathbb{X}^M = (X^\mu, U^I)$, and may be removed via a Legendre transformation [10], leaving the regularized action $S_{\text{reg.}} = \sqrt{\lambda} A_{\text{reg.}}/(2\pi)$. The result for the vacuum expectation value of (1) is then

$$\langle W \rangle_{\lambda\to\infty} = \mathcal{V} \exp\left(-S_{\text{reg.}}\right), \tag{9}$$

where $\mathcal{V}$ is a prefactor stemming from integration over zero modes in the partition function. The disc partition function naturally involves three zero modes. If there is no extra parametric freedom in embedding the string, then $\mathcal{V} \sim \lambda^{-3/4}$, i.e. one

[2]The Poincaré and superconformal supersymmetries are halved independently of one another.

factor of $\lambda^{-1/4}$ for each zero mode. This is the case for the standard 1/2 BPS circle which sits at a point on S^5 [11]. The expectation therefore, for the string dual of the Zarembo loops, is that $\mathcal{V} = 1$, and $S_{\text{reg.}} = 0$. The first of these conditions has not been shown explicitly, and for other than planar loops remains a mystery. For the case of planar loops, it was argued in [2] that there are 3 compensating zero modes stemming from parametric freedom in embedding the string in an $S^2 \subset S^5$. For loops other than planar, it remains unclear how the contribution of the three basic zero modes is cancelled [12]. We discuss this issue further in section 3.3. The second condition, $S_{\text{reg.}} = 0$, was shown explicitly by Zarembo in [2] for the circular supersymmetric Wilson loop in $AdS_5 \times S^5$. There the string solution was found and the regularized action calculated. The analogous string-side embodiment of the results of [8] were realized[3] in [12], where it was proven that $S_{\text{reg.}} = 0$ for the string dual of a generic $\mathcal{N} = 4$ SYM Zarembo Wilson loop. This used the method of calibrated surfaces which we will review in section 3.2.

In the present work we will extend these results, to the degree it is possible, to maximally supersymmetric Yang-Mills theories in general spacetime dimensions. Indeed we may view (1) as arising from a toroidal compactification of the standard Wilson loop in $\mathcal{N} = 1$, $d = 10$ SYM, and in this sense we are free to compactify more or less directions than 6, namely $9 - p$ where p ranges from 0 to 9,

$$\begin{aligned} &\frac{1}{N} \operatorname{Tr} P \exp \oint d\tau \, i\dot{x}^M(\tau) A_M \\ &\qquad \to \frac{1}{N} \operatorname{Tr} P \exp \oint d\tau \left(i\dot{x}^\mu(\tau) A_\mu + |\dot{x}(\tau)| \Theta^I(\tau) \Phi^I \right), \end{aligned} \tag{10}$$

where $M = 1, \ldots, 10$, $\mu = 1, \ldots, p+1$, $I = 1, \ldots, 9-p$. We then require the same relations to hold relating the paths x^μ and Θ^I, i.e. (2). The supersymmetry relation (4) also continues to hold after this dimensional reduction. For the various spacetime dimensions d, we are restricted by (2) to curves $x^\mu(\tau)$ in various subspaces of $\mathbb{R}^d$, these are summarized in the table below.

d	1	2	3	4	5	6	7	8
Curves in	$\mathbb{R}^1$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^4$	$\mathbb{R}^3$	$\mathbb{R}^2$	$\mathbb{R}^1$

We will concentrate on the dimensions $2 \leq d \leq 7$, since the curves in $\mathbb{R}^1$ are the trivial 1/2 BPS straight lines.

On the gauge theory side, we perform our analyses using both perturbative and (non-perturbative) superspace techniques. From the perturbative point of view, we study the relevant gauge theories in a unified way, up to the next to leading order (NLO), or two loop approximation. This analysis allows us to perform a straightforward extension of the results presented in [2]. At this order in perturbation theory we find that the vacuum expectation value for the Zarembo loops in *all* dimensional reductions of the $d = 10$, $\mathcal{N} = 1$ SYM theories, down to $d = 1$, is identically '1'. Clearly, the NLO results beg the question if some or all of the gauge theories preserve the triviality of the Zarembo loops to higher or even all orders in perturbation theory.

[3] The issue of the zero mode prefactor $\mathcal{V}$ is still outstanding.

On a related note, one may also worry about the reasonability of perturbative methods in non-renormalizable gauge theories, which SYM in $d \geq 5$ are expected to be. Though we do not expect the perturbative results for generic gauge theory observables in these theories to be meaningful, we can use perturbation theory to gauge the validity of results believed to be protected by non-renormalization theorems. The non-renormalization theorems for the sixteen supercharge theories in question were derived in [8]. In that paper, the d dimensional SYM theories were reformulated in a $d-3$ superspace language. This reformulation, which is briefly reviewed in the next section, allows one to view the Wilson loops in question as elements of a chiral ring. Furthermore, the (superspace) equations of motion were shown to imply shape invariance of the loops embeddable in $\mathbb{R}^3$. These two results were used to formally establish the triviality of these Zarembo loops for all sixteen supercharge gauge theories in $7 > d \geq 3$. The appearance of a generalized Konishi anomaly in $d = 7$ [8] puts an upper bound (in terms of dimensions) on the gauge theories for which the perturbative results may be expected to hold to all loop orders. However, for gauge theories in $d < 3$, the superspace methods are simply limited by the construction/requirement of a $d-3$ dimensional superspace, with at least one dynamical supercoordinate. We can thus regard the perturbative results as a non-trivial verification of the predictions of [8] at the NLO, and a hint toward the potential for generalization of the triviality of the Zarembo loops to all loop orders for gauge theories in dimensions $3 > d \geq 1$.

On the gravity side, we use the string duals for the sixteen supercharge Yang-Mills theories proposed in [1]. These Dp-brane geometries (where $d = p + 1$) contain an S^{8-p}, the d boundary theory coordinates, and a U direction, so that the boundary is at $U = \infty$. We find the explicit fundamental string solutions corresponding to circular Zarembo-type Wilson loops in these backgrounds[4]. They wrap part of an $S^2 \subset S^{8-p}$ and extend in the U-direction from the boundary circle. We find that these solutions have the expected zero regularized area. This result is independent of the cut-off $U_{\text{max.}}$ where the boundary theory is defined; this is the string-side manifestation of the protection of these operators in the gauge theories, despite the issues of running couplings and non-renormalizability. In appendix A we analyze the supersymmetry respected by the solutions and find that they are indeed 1/4 BPS, as required. We also generalize the framework of calibrated surfaces given in [12] to the Dp-brane geometries, thereby proving that the regularized action vanishes for any Zarembo-type Wilson loop constructed in these theories, and as a check show that our circular string solutions also satisfy the appropriate equations. Finally, in section 3.3 we discuss the potential string-side manifestation of the gauge theory generalized Konishi anomaly for $d = 7$.

2 Gauge theory results

In this section we present the arguments in favor of the triviality of the vacuum expectation values of supersymmetric Wilson loops in 16 supercharge super Yang-

[4]String duals of generic Wilson loops have also been considered in the Dp-brane geometries, see [13].

Mills theories, from the perspective of the relevant gauge theories. To this end, we shall start with a perturbative point of view, and subsequently correlate the weak-coupling results with all-loop predictions based on superspace techniques obtained in [8].

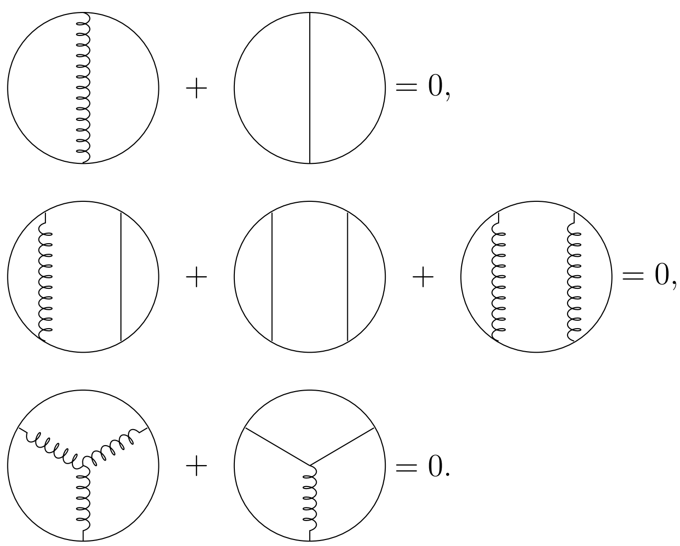
We start with a sixteen supercharge SYM action in $10 > 2\omega \geq 1$ dimensions given by

$$S = \frac{1}{g^2} \int d^{2\omega}x \; \mathrm{Tr} \left(\frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi^i)^2 - \frac{1}{2} [\Phi^i, \Phi^j]^2 + \bar{\Psi} \Gamma^\mu D_\mu \Psi + i \bar{\Psi} \Gamma^i [\Phi^i, \Psi] \right). \quad (11)$$

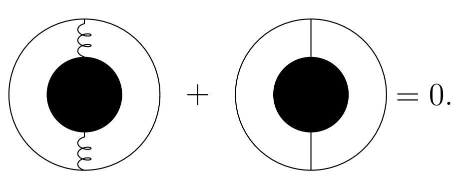
It is understood that the Lorentz indices $\mu, \nu = 1, \ldots, 2\omega$ while the number of scalars $i, j = 1, \ldots, (10 - 2\omega)$.

As was shown by Zarembo in [2], the triviality of the Wilson loop expectation value at the leading order in perturbation theory is simply a consequence of the equality of the gluon and scalar propagators in the Feynman gauge. Although the focus in [2] was on four dimensional gauge theory, this leading order result readily generalizes to all the dimensional reductions of the ten dimensional $\mathcal{N} = 1$ gauge theory.

At the next-to-leading order, the diagrams that do not involve loop corrections to propagators cancel due to the same reason as above. In other words, the following cancelations between Feynman diagrams occur for all dimensional reductions of $\mathcal{N} = 1$, $d = 10$ SYM theories, due to the same arguments put forward in the Feynman gauge for the four dimensional theory in [2]:



For the triviality of the Wilson loop expectation value to hold at the next-to-leading order, all that one needs to show is the equality between the one loop corrected gluon and scalar propagators in the Feynman gauge, such that the following cancelation takes place:



The one-loop gluon propagator in this gauge is given by

$$\Delta_{\mu\nu}^{ab} = g^2 \delta^{ab} \frac{1}{p^2} \left(\delta_{\mu\nu} - g^2 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} f_g(\omega) \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{4-2\omega}} \right), \tag{12}$$

where the function f_g encodes the contributions to the propagator from the various interaction vertices

$$f_g = 2(3\omega - 1) - N_s - N_f(\omega - 1). \tag{13}$$

The contribution of $2(3\omega - 1)$ in f_g is due to the combination of the gluon-gluon and ghost-gluon scattering in 2ω dimensions. The factor of N_s arises from the N_s real adjoint scalars running in loops, while the factor of N_f; the number of real fermionic degrees of freedom in the theory, is due to gluon-fermion scattering.

Using the same notation, we may write the one loop corrected scalar propagator as

$$\Delta_{mn}^{ab} = g^2 \delta^{ab} \frac{1}{p^2} \left(\delta_{mn} - g^2 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} f_s(\omega) \frac{\delta_{mn}}{p^{4-2\omega}} \right), \tag{14}$$

where

$$f_s(\omega) = 4(2\omega - 1) - \frac{N_f}{2}(2\omega - 1). \tag{15}$$

The contribution of $4(2\omega - 1)$ comes about due to the scalar-vector intermediate state, while the fermion loop contribution to the scalar propagator generates a factor of $\frac{N_f}{2}(2\omega - 1)$ with the opposite sign.

A necessary and sufficient condition for the supersymmetric Wilson loops to have unit vacuum expectation value at the one and two loop level is

$$f_g = f_s. \tag{16}$$

It is easy to check that this is indeed satisfied when the number of real scalars $N_s = 10 - 2\omega$ and $N_f = 16$.

We have thus established the triviality of the Wilson loop expectation value at the next to leading order for *all* dimensional reductions of the $\mathcal{N} = 1$ ten dimensional SYM.

The one loop corrected gluon and scalar propagators, as they have been expressed above, are also valid for the dimensional reduction of the six and four dimensional $\mathcal{N} = 1$ SYM theories as well. The equality of the loop corrected propagators continues to hold if we use either

$$N_f = 8, \quad N_s = 6 - 2\omega \quad \text{or} \tag{17}$$

$$N_f = 4, \quad N_s = 4 - 2\omega. \tag{18}$$

This fact proves the triviality of Wilson loop expectation value for eight (four) supercharge theories in dimensions less than or equal to five (three).

Thus, the following table summarizes the balance between the number of dimensions and the number of supersymmetries necessary for Wilson loops to have trivial expectation values at the next-to-leading order:

Number of Supercharges	Dimensions $\leq$
16	9
8	5
4	3

It is probably too optimistic to expect that all the gauge theories listed above retain the triviality of the Zarembo loops to all orders in perturbation theory. However, for the case of the sixteen supercharge theories, lower dimensional superspace techniques were successfully employed in [8] to probe the all-loop behavior of many of the gauge theories considered above. We shall briefly review these techniques and compare the superspace results with the perturbative computations reported above.

For the four dimensional theory, the starting point was a rewriting of the action in a $\mathcal{N}=2$, $d=1$ superspace, coordinatized by $t, \theta_\alpha, \bar{\theta}_\alpha$, where $\alpha = 1,2$ is an $SU(2)$ index. The action for the four dimensional gauge theory was shown to be [8][5]

$$S = \frac{1}{g^2}\int d^3x\, dt \Bigg[\mathrm{Tr}\left(\mathcal{W}_\alpha \mathcal{W}^\alpha \epsilon_{ijk}(\Phi_i \partial_j \Phi_k + \frac{2i}{3}\Phi_i\Phi_j\Phi_k) + cc \right)_{\theta\theta} + \mathrm{Tr}\left(\bar{\Omega}_i e^V \Omega_i e^{-V}\right)_{\theta\theta\bar{\theta}\bar{\theta}} \Bigg]. \tag{19}$$

In the quantum mechanical superspace, the three chiral superfields Φ_i contain the spatial components of the gauge potential A_i and three of the six real scalars[6] Φ^{i+3}. The bottom component of the chiral fields being given by $A_i + i\Phi^{i+3}$. The temporal component A_0 as well as $\Phi^{7,8,9}$ are contained in the vector superfield V. The superfields are also implicitly labeled by the coordinates x_i, which are treated simply as auxiliary indices from the quantum mechanical point of view. Ω is given by

$$\Omega_i = \Phi_i + e^{-V}(i\partial_i - \bar{\Phi}_i)e^V. \tag{20}$$

One of the main observations in the paper was that the Wilson loops of the type considered in this paper could be thought of as elements of a chiral ring from the lower dimensional superspace point of view. In particular the equation of motion for these loops took on the form

$$\left\langle \mathrm{Tr}\Big(W(C,x)\, \epsilon_{ijk}\mathcal{F}_{jk}(x) \Big) \right\rangle_{\theta=\bar{\theta}=0} = \mathcal{A}_i, \tag{21}$$

where,

$$\mathcal{F}_{jk} = \partial_j \Phi_k - \partial_k \Phi_j + i[\Phi_j, \Phi_k], \tag{22}$$

and where $W(C,x)$ is the untraced Wilson loop operator with a marked point x on the loop, and $\mathcal{A}_i$ is a possible anomaly term. In the absence of the anomaly term, the loop equation implied shape independence. In conjunction with the fact that the loop

[5] $\mathcal{W}_\alpha = \bar{D}\bar{D}e^V D_\alpha e^{-V}$, see [8] for further details.

[6] At the risk of abuse of notation, we denote both the chiral superspace fields as well as the real scalars by Φ. We hope that the difference will be clear from the context.

is an element of the chiral ring, the shape independence yielded a trivial expectation value of the loop. Note that this hinges upon the three-dimensional epsilon symbol and for this reason is limited to curves in $\mathbb{R}^3$.

Similar arguments were also applied to sixteen supercharge Yang-Mills theories in dimensions $3 \leq d \leq 7$. The key to the generalization was being able to write the action for the relevant gauge theories in a four supercharge $d-3$ dimensional superspace. It was further shown that only in the case of the seven dimensional gauge theory does one encounter a non-zero anomaly; this is the generalized Konishi anomaly.

Conjoining these superspace arguments with the evidence presented from the weak coupling perturbation theory, we conclude that sixteen supercharge SYM theories in dimensions $6 \geq d \geq 3$ possess supersymmetric Wilson loops with trivial vacuum expectation values.

It is also worth noting that lower dimensional superspace methods were also employed to analyze Wilson loops in SYM theories with 8 supercharges, and Wilson loops with trivial expectation values were found in $4 \geq d \geq 1$ dimensions in [14]. These results are consistent with the perturbative results reported earlier for the dimensional reductions of $\mathcal{N}=1$, $d=6$ SYM. The case of the five dimensional Yang-Mills theory, suffers from a non-vanishing anomaly, which was not seen in the perturbative calculations we presented above.

In summary, the next to leading order perturbation theory and the superspace arguments match up in the following cases:

Number of Supersymmetries	Dimensions
16	$3 \leq d \leq 6$
8	$1 \leq d \leq 4$

In the case of sixteen supercharge theories, we also have a dual gravity description available to us. In what follows, we reproduce and generalize the results for this case using the dual gravity picture. As the table above indicates, apart from the usefulness of the gravity computation as non-trivial test of the gauge gravity duality, can hope to shed some light on the non-perturbative behavior of the Zarembo loops for the gauge theories for which the lower dimensional superspace arguments do not exist, e.g. the case of $d=2$ SYM with sixteen supercharges.

3 String duals and strong coupling results

The string duals of the class of maximally supersymmetric Yang-Mills theories were presented in [1]. The holographic dual of the $d=p+1$ dimensional gauge theory is given by the string frame metric

$$\begin{aligned} ds^2 &= \alpha' \left(\frac{U^{(7-p)/2}}{C_p} dx_{\shortparallel}^2 + \frac{C_p}{U^{(7-p)/2}}\, dU^2 + C_p\, U^{(p-3)/2}\, d\Omega_{8-p}^2 \right), \\ e^{\phi} &= (2\pi)^{2-p} g^2 \left(\frac{C_p^2}{U^{7-p}} \right)^{(3-p)/4}, \quad C_p^2 = g^2 N\, 2^{7-2p} \pi^{(9-3p)/2}\, \Gamma\left(\frac{7-p}{2} \right), \end{aligned} \tag{23}$$

where g and N are the bare coupling and the number of colours of the dual Yang-Mills theory. There is also a p-form gauge potential which depends only on the U coordinate. These solutions are obtained from the field theory limit of Dp-brane solutions

$$g^2 = (2\pi)^{p-2}\, g_s\, \alpha'^{(p-3)/2} = \text{fixed}, \quad \alpha' \to 0, \tag{24}$$

where one can see that for $p > 3$, the string coupling $g_s \to \infty$ which indicates a breakdown of the limit, in the sense that α' corrections are not suppressed and the decoupling of bulk modes is not guaranteed. This is a reflection of the fact that the Yang-Mills theories with $d = p+1 > 4$ are nonrenormalizable. As discussed in the introduction, we are describing objects which are protected and therefore we can trust our solutions in spite of this breakdown. Indeed we will find that the regularized action of our string solutions vanishes independently of the choice of cut-off $U_{\text{max.}}$ - the coordinate dual to the boundary gauge theory energy scale.

3.1 Supersymmetric circular loops

We present here string solutions corresponding to circular supersymmetric Wilson loops in the background (23). We have a natural lower bound of $p = 1$, in order that the boundary has enough dimensions to accommodate the circle, namely two, and a natural upper bound of $p = 6$, since, as we will see below, we will require an S^2 to accommodate the coupling of the Wilson loop to the scalars of the dual gauge theory. We have analyzed the supersymmetry of these solutions in appendix A, where we show that they are 1/4 BPS.

We begin with the action of the fundamental string in Euclidean conformal gauge, in the background (23). We write

$$\begin{aligned} dx_{\shortparallel}^2 &= dr^2 + r^2 d\psi^2 + dx_{p-1}^2, \\ d\Omega_{8-p} &= d\theta^2 + \cos^2\theta\, d\phi^2 + \sin^2\theta\, d\Omega_{6-p}^2, \end{aligned} \tag{25}$$

where dx_{p-1}^2 is a $p-1$ dimensional metric on $\mathbb{R}^{1,p-2}$ or $\mathbb{R}^{p-1}$ (in the case $p = 1$ we are forced to take the Euclidean metric). Our solution ansatz is then

$$\psi = \phi = \tau,\ \tau \in [0, 2\pi], \quad r = r(U), \quad \theta = \theta(U), \quad dx_{p-1}^2 = d\Omega_{6-p}^2 = 0, \tag{26}$$

with which we can write the string action as

$$S = \frac{C_p}{4\pi} \int_0^{2\pi} d\tau \int d\sigma \left[\frac{U^{(7-p)/2}}{C_p^2} \left(r^2 + r'^2 \right) + \frac{U'^2}{U^{(7-p)/2}} + U^{(p-3)/2} \left(\theta'^2 + \cos^2\theta \right) \right], \tag{27}$$

where prime denotes differentiation w.r.t. σ. We must also satisfy the Virasoro constraint

$$\frac{U^{(7-p)/2}}{C_p^2}\, r'^2 + \frac{U'^2}{U^{(7-p)/2}} + U^{(p-3)/2}\theta'^2 = U^{(p-3)/2}\cos^2\theta + \frac{U^{(7-p)/2}}{C_p^2}\, r^2. \tag{28}$$

The solution we find is

$$
\begin{aligned}
R^2 - r(U)^2 &= \begin{cases} \frac{2\,C_p^2}{5-p}\,U^{p-5}, & p \neq 5 \\ -2\,C_5^2\,\log U, & p = 5 \end{cases}, \\
\sin\theta &= \frac{U_{\text{min.}}}{U}, \qquad r(U_{\text{min.}}) = 0,
\end{aligned} \tag{29}
$$

where R is the asymptotic radius of the circle at $U = \infty$. Note that for $p > 4$, $r(\infty) = \infty$, R becomes imaginary, and so the solution doesn't satisfy the usual boundary condition. We will cut the geometry off at $U_{\text{max.}}$ however, and so define the radius of the circle in the boundary theory as $r(U_{\text{max.}})$. The solution wraps one half of an $S^2 \subset S^{8-p}$; the string worldsheet's boundary lies along the equator[7]. In order to check that (29) is in fact a solution to the equations of motion we express r' and θ' in terms of U' and plug them into the Virasoro constraint (28) and solve for U' in terms of U. The result of this operation is

$$
U' = \begin{cases} \sqrt{\frac{2}{5-p}\left(U^{5-p}/U_{\text{min.}}^{5-p} - 1\right)(U^2 - U_{\text{min.}}^2)}, & p \neq 5 \\ \sqrt{2\log(U/U_{\text{min.}})\;(U^2 - U_{\text{min.}}^2)}, & p = 5. \end{cases} \tag{30}
$$

With this expression we can also express U'' in terms of U, and through (29), we can therefore also express r'' and θ'' in terms of U. The expression for U'' is

$$
\begin{aligned}
U'' &= \frac{U_{\text{min.}}^{p-5}}{5-p}\Big((5-p)\,U^{4-p}\,(U^2 - U_{\text{min.}}^2) + 2U(U^{5-p} - U_{\text{min.}}^{5-p})\Big), \quad p \neq 5, \\
U'' &= U^{-1}\,(U^2 - U_{\text{min.}}^2) + 2U\log(U/U_{\text{min.}}), \quad p = 5.
\end{aligned} \tag{31}
$$

It is then a straightforward, if somewhat tedious exercise to verify that the equations of motion for U, r, and θ are satisfied through the chain of substitutions.

We have plotted $U(r)$ in figure 3.1.

It remains to compute the action of the solutions. Using (28) we can express the action as twice the "prime" terms, i.e. those involving derivatives by σ. We express everything in terms of U and U', the latter we use to reexpress the integration over σ by integration over U.

$$
\begin{aligned}
S &= C_p \int_{U_{\text{min.}}}^{U_{\text{max.}}} \frac{dU}{U^{(7-p)/2}}\,\frac{\frac{2}{5-p}\,U^2\left(U^{5-p}/U_{\text{min.}}^{5-p} - 1\right) + U^2 - U_{\text{min.}}^2}{\sqrt{\frac{2}{5-p}\left(U^{5-p}/U_{\text{min.}}^{5-p} - 1\right)(U^2 - U_{\text{min.}}^2)}}, \quad p \neq 5, \\
S &= C_5 \int_{U_{\text{min.}}}^{U_{\text{max.}}} \frac{dU}{U}\,\frac{2U^2\log(U/U_{\text{min.}}) + U^2 - U_{\text{min.}}^2}{\sqrt{2\log(U/U_{\text{min.}})\;(U^2 - U_{\text{min.}}^2)}}, \quad p = 5.
\end{aligned} \tag{32}
$$

The integral is simple to evaluate. The result is

$$
S = U_{\text{max.}}\sqrt{1 - U_{\text{min.}}^2/U_{\text{max.}}^2}\;r(U_{\text{max.}}). \tag{33}
$$

[7]More precisely, when $U_{\text{max.}}$ is not strictly ∞, the boundary is shifted down towards the pole.

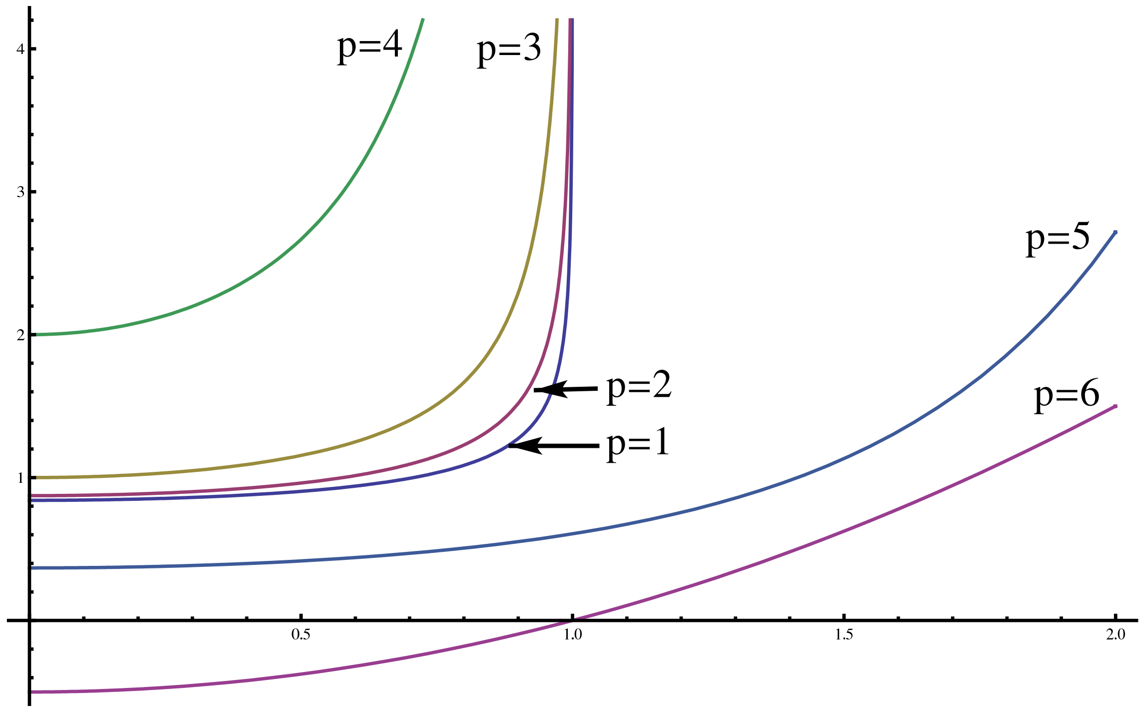


Figure 1: A plot of U vs. r for the solutions (29). We have set $R = C_p = 1$ ($R = i$ for $p > 4$). Note that for $p > 4$, r diverges toward $U = \infty$.

The prescription for removing the divergence from the action is to perform a Legendre transformation [10], as follows

$$\begin{aligned} S_{\text{reg.}} &= S - \int d\tau\, d\sigma\, \partial_\sigma \left(\mathcal{Y}^I \frac{\delta S}{\delta \partial_\sigma \mathcal{Y}^I} \right) \\ &= S - \int d\tau\, \mathcal{Y}^I \frac{\delta S}{\delta \partial_\sigma \mathcal{Y}^I} \bigg|_{\partial\Sigma}, \end{aligned} \tag{34}$$

where we are using the coordinates defined in (38). We then find, using (52) and (30),

$$\frac{\delta S}{\delta \partial_\sigma \mathcal{Y}^I} = \frac{C_p}{2\pi} U^{(p-7)/2} \left(U \hat{\theta}^I \right)' = \frac{1}{2\pi} r(U) \Big(-\sin\tau, \cos\tau, 0, \dots, 0 \Big), \tag{35}$$

and so

$$\int d\tau\, \mathcal{Y}^I \frac{\delta S}{\delta \partial_\sigma \mathcal{Y}^I} \bigg|_{\partial\Sigma} = U_{\text{max.}} \sqrt{1 - U_{\text{min.}}^2 / U_{\text{max.}}^2}\, r(U_{\text{max.}}). \tag{36}$$

We therefore have that

$$S_{\text{reg.}} = 0, \tag{37}$$

independent of $U_{\text{max.}}$ and consistent with our expectations.

3.2 Calibrated surfaces

In the paper [12] a method of calibrated surfaces was employed to prove that the string duals of the supersymmetric Wilson loops of general shape in the $p = 3$ case had the expected regularized action, namely zero. We now show that this machinery applies equally well to the case of general p. As a check on our work, we also show that it applies to the solutions (29).

In order to apply the technique we express the metric of S^{8-p} together with the dU^2 term from (23) as follows

$$\frac{dU^2}{U^2} + d\Omega_{8-p}^2 = \frac{d\mathcal{Y}^I d\mathcal{Y}^I}{\mathcal{Y}^2}, \quad \mathcal{Y}^I = U\hat{\theta}^I, \quad \hat{\theta}^I\hat{\theta}^I = 1, \quad I = 1, \ldots, 9-p. \tag{38}$$

For convenience we will rescale the $x_{\shortparallel}^\mu = C_p X^\mu$. We then have

$$ds^2 = \alpha' \, C_p \left(\mathcal{Y}^{(7-p)/2} dX^\mu dX^\mu + \mathcal{Y}^{(p-7)/2} d\mathcal{Y}^I d\mathcal{Y}^I \right). \tag{39}$$

Now we make a split in the $\mathcal{Y}^I$ coordinates

$$\mathcal{Y}^I = (Y^m, V^i), \quad m = 1, \ldots, p+1, \quad i = 1, \ldots, 8-2p, \tag{40}$$

so that Y^m and X^μ have the same number of components. It is clear that this can only be done for $p \leq 4$. For $p > 4$ we can choose instead to split the $X^\mu = (X^{\bar{I}}, V^i)$, so that $X^{\bar{I}}$ has the same number of components as $\mathcal{Y}^I$, and what follows is equally true (with the appropriate relabelling of indices). We use precisely the same definition for an almost complex structure proposed in [12]

$$J = J_{AB} \, d\mathbb{X}^A \wedge d\mathbb{X}^B = \delta_{\mu m} \, dX^\mu \wedge dY^m, \tag{41}$$

where $\mathbb{X}^A = (X^\mu, Y^m, V^i)$. We find that the following key relations used in [12] are equally true for the metric (39), namely

$$\begin{aligned} &J_A^B \, J_B^C = -\delta_A^\mu \, \delta_\mu^C - \delta_A^m \delta_m^C, \\ &G_{MN} J_\mu^M J_\nu^N = G_{\mu\nu}, \quad G_{MN} J_m^M J_n^N = G_{mn}, \quad G_{MN} J_i^M J_j^N = 0. \end{aligned} \tag{42}$$

That being the case everything follows as in [12]. We continue by reiterating the results of [12] in the interest of readability. One defines

$$\begin{aligned} \mathcal{P} &\equiv \frac{1}{4} \int d^2\sigma \sqrt{h} \, h^{ab} \, G_{MN} \, v_a^M v_b^N, \\ v_a^M &\equiv \partial_a \mathbb{X}^M - J_N^M \, \frac{h_{ac}\epsilon^{cb}}{\sqrt{h}} \, \partial_b \mathbb{X}^N, \end{aligned} \tag{43}$$

where h_{ab} is a positive definite metric on the worldsheet. Using (42) one can then show that

$$\mathcal{P} = \frac{1}{2} \int d^2\sigma \sqrt{h} \, h^{ab} \, G_{MN} \, \partial_a \mathbb{X}^M \partial_b \mathbb{X}^N - \int_\Sigma J - \frac{1}{4} \int d^2\sigma \sqrt{h} \, h^{ab} G_{ij} \, \partial_a V^i \partial_b V^j, \tag{44}$$

where Σ is the string worldsheet. Now suppose that $v_a^M = 0$. As can be easily checked in conformal gauge, this condition automatically implies that the string equations of motion and Virasoro constraints are satisfied. Further, this implies that $\mathcal{P} = 0$ and that $\partial_a V^i = 0$. We then have that

$$S = \frac{C_p}{2\pi} \int_\Sigma J, \tag{45}$$

that is, we have that the action of the string worldsheet is expressible as an integral of the closed 2-form J over the string worldsheet. This will integrate to a surface term

$$\int_\Sigma J = \int_\Sigma \delta_{\mu m}\, d(Y^m dX^\mu) = U_{\text{max.}} \int_{\partial\Sigma} \delta_{\mu m}\, \hat{\theta}^m\, dX^\mu. \tag{46}$$

Now examining the equation $v_a^M = 0$ in conformal gauge one finds

$$\dot{X}^\mu = U^{(p-7)/2}\, Y'^m = U^{(p-7)/2} \Big(U\, (\hat{\theta}^m)' + \hat{\theta}^m\, U' \Big). \tag{47}$$

If it is true that $U^{(p-5)/2}\, (\hat{\theta}^m)' \to 0$ as the boundary is approached, one then has that

$$\dot{X}^\mu(U_{\text{max.}}) \simeq U_{\text{max.}}^{(p-7)/2} (\hat{\theta}^m\, U')|_{U_{\text{max.}}}, \tag{48}$$

and so

$$\hat{\theta}^m|_{\partial\Sigma} = \frac{\dot{x}^\mu}{|\dot{x}|}, \tag{49}$$

where $x^\mu = X^\mu(U_{\text{max.}})$ is the Wilson loop contour. This is precisely the contour of the supersymmetric Wilson loop in the gauge theory (2), (7). Thus we have found a solution to the string equations of motion which also satisfies the necessary boundary conditions. Furthermore, in conformal gauge we have that

$$U^{(p-7)/2}\, Y'^m = \frac{2\pi}{C_p} \frac{\delta S}{\delta \partial_\sigma Y^m}, \tag{50}$$

and therefore (46) also gives

$$S = \oint d\tau\, Y^m \left. \frac{\delta S}{\delta \partial_\sigma Y^m} \right|_{\partial\Sigma}, \tag{51}$$

which is nothing but the divergence removed from the action by the Legendre transformation [10] to give $S_{\text{reg.}}$; thus we see that the regularized action vanishes for these solutions. Again this result is independent of the choice of cut-off in the coordinate U.

3.2.1 Checking the circular supersymmetric solutions

We can now verify that our solution (29) obeys the equations $v_a^M = 0$ and (48), thereby confirming our result (33). We begin by writing our solution (29) in the coordinates (39). We find (for example, for $p \neq 5$)[8]

$$
\begin{aligned}
X^1 &= r\cos\psi = \sqrt{\frac{2}{5-p}\left(U_{\text{min.}}^{p-5} - U^{p-5}\right)}\cos\tau, \\
X^2 &= r\sin\psi = \sqrt{\frac{2}{5-p}\left(U_{\text{min.}}^{p-5} - U^{p-5}\right)}\sin\tau, \\
Y^1 &= -U\cos\theta\sin\phi = -\sqrt{U^2 - U_{\text{min.}}^2}\sin\tau, \\
Y^2 &= U\cos\theta\cos\phi = \sqrt{U^2 - U_{\text{min.}}^2}\cos\tau, \\
Y^3 &= U\sin\theta = U_{\text{min.}}.
\end{aligned}
\tag{52}
$$

The equations $v_a^M = 0$ in conformal gauge then reduce to

$$
\begin{aligned}
X'^{\mu} + \left(Y^2 + V^2\right)^{(p-7)/4} \dot{Y}^{m=\mu} &= 0, \\
Y'^{m} - \left(Y^2 + V^2\right)^{(7-p)/4} \dot{X}^{\mu=m} &= 0, \\
\dot{X}^{\mu} - \left(Y^2 + V^2\right)^{(p-7)/4} Y'^{m=\mu} &= 0, \\
\dot{Y}^{m} + \left(Y^2 + V^2\right)^{(7-p)/4} X'^{\mu=m} &= 0,
\end{aligned}
\tag{53}
$$

which, through use of (30) may be shown to be satisfied. Finally we note that

$$
\begin{aligned}
U^{(p-5)/2}\,(\hat{\theta}^m)' &= \frac{U_{\text{min.}}^2}{U^2}\, r(U)\,\Big(-\sin\tau, \cos\tau, 0, \ldots, 0\Big), \\
U^{(p-7)/2}\hat{\theta}^m\, U' &= \left(1 - \frac{U_{\text{min.}}^2}{U^2}\right) r(U)\,\Big(-\sin\tau, \cos\tau, 0, \ldots, 0\Big),
\end{aligned}
\tag{54}
$$

and so (48) is also satisfied.

3.3 Zero modes and the generalized Konishi anomaly in $d = 7$

It seems a contradiction that in the gauge theory analysis discussed in section 2 there is an anomaly in the case $d = 7$ precluding $\langle W\rangle = 1$ for this theory, whereas the string solution seems to suffer no such issue. In fact, as discussed in the introduction, there is more to the vacuum expectation value of the Wilson loop than the regularized action; the prefactor $\mathcal{V}$ stemming from integration over zero modes in the partition function also plays a role. The issue, as regards supersymmetric Wilson loops, was first discussed in [2], for the case $p = 3$. Although there appears to be no parametric freedom for the minimal area embedding of a string in an AdS space with given boundary conditions, Zarembo argued that the supersymmetric circle may be embedded into the S^5 with a freedom given by a vector $\mathbf{n} \in S^3$ which chooses which S^2 the worldsheet occupies. This gives a natural reason for the cancellation of the prefactor in $\langle W\rangle$, as these three zero modes could cancel the effect of the basic three coming from the AdS embedding. This reasoning is limited to the case of planar curves, and

[8] The $p = 5$ case follows similarly.

[12] noted the lack of resolution of this problem for general curves. Specifically, in order for the R-symmetry of the Wilson loop defined in the gauge theory to match the string solution, these zero modes must be integrated over.

In our case we note the fact that uniquely in the case of $d = 7$ (i.e. $p = 6$) do we have that the spherical product space S^{8-p} is an S^2. In this case we are restricted to curves in $\mathbb{R}^2$, and we will concentrate on our explicit solution for the supersymmetric circle, although we expect the following comments to be true for general closed, planar curves. The fact that the spherical product space is an S^2 precludes the existence of zero modes on the spherical side of the geometry, and thus, assuming the absence of any parametric freedom in the embedding on the analogue of the AdS side of the geometry, precludes the possible cancellation of the three basic zero modes of the string worldsheet. We would thus expect a non-zero prefactor $\mathcal{V} \sim \lambda^{-3/4}$ and therefore our prediction for the vacuum expectation value of the Wilson loop at strong coupling is

$$\langle W \rangle_{d=7} \sim \left(\frac{\lambda}{R^3} \right)^{-3/4} , \tag{55}$$

where R is a scale setting the size of the Wilson loop. This seems to be the string-side manifestation of the generalized Konishi anomaly in $d = 7$ discussed in section 2. It would be very interesting to try to recover this result from gauge theory.

The situation is extremely reminiscent of the circular Wilson loop for $\mathcal{N} = d = 4$ SYM obtained by a "large" conformal transformation of the straight line. In that case, the Wilson loop expectation value is a non-trivial function of the 't Hooft coupling. However, the vacuum expectation value for the loop is entirely determined by an anomaly; namely, the conformal anomaly [11]. For the seven dimensional gauge theory, the generalized Konishi anomaly seems to play a similar role. It is tempting to speculate that it might similarly be possible to recover the strong coupling result mentioned above from the gauge theory end, by reducing the problem to a matrix model computation[9].

4 Summary and outlook

In this paper we have generalized the construction of supersymmetric Wilson loops in $\mathcal{N} = 4$, $d = 4$ SYM at weak and strong coupling to the general case of SYM theories with 16 supercharges in d dimensions (and in the case of $d \leq 4$ ($d \leq 3$) at weak coupling, with 8 (4) supercharges). We have given two-loop perturbative evidence and reviewed the applicability of evidence from superspace techniques, that these loops have trivial vacuum expectation values. Using the gauge/strings duality we have also described the 16 supercharge theory supersymmetric Wilson loops at strong coupling and also found strong evidence of trivial expectation values; the dual string solutions have zero regularized action. We have found the explicit fundamental string solutions for the case of circular supersymmetric loops in general d. In the

[9] To this end, it might be interesting to investigate if the methods of [15] can be adapted to the analysis of the gauge theory in $d = 7$.

case of $d = 7$ where superspace techniques indicate a non-zero expectation value on the gauge theory side, we have found a strong candidate dual manifestation of this phenomena at strong coupling, namely the disappearance of string worldsheet zero modes. Based on this we have given a prediction for the strong coupling behavior of the vacuum expectation value of supersymmetric Wilson loops in the $d = 7$ theory.

Looking beyond the issues addressed in this paper, it would doubtless be interesting to try and extend the present results to more general Wilson loops and to other instances of gauge/gravity dualities. For example, for the theories we considered with $d > 4$, the various UV completions were discussed in [1] (see also [13]) and involve lifting to M-theory (in the case of odd d), or the application of S-duality in the IIB case. We expect these theories to retain the trivial Wilson loop operators we have constructed here as a natural consequence of coupling independence. On a different note, it was pointed out earlier in the paper, using both perturbative as well as superspace methods, that eight supercharge SYM theories in $1 \leq d \leq 4$ admit Zarembo loops. Clearly, this fact can be used to carry out non-trivial tests for any candidate gravity dual for these theories.

In the special case of three spacetime dimensions, the recent developments due to Bagger, Lambert and Gustavsson (BLG) [16, 17, 18, 19] and Aharony, Bergman, Jafferis and Maldacena (ABJM) [20] relate the sixteen supercharge SYM theory to superconformal Chern-Simons (SCS) theories. The $\mathcal{N} = 8$ SCS theory proposed by BLG, which can also be recovered as a special case of the ABJM model, is believed to be related to the IR limit of the sixteen supercharge SYM theory. It is interesting to note that Wilson loops that preserve global supersymmetries have also been constructed for the ABJM model in [21, 22, 23, 24]. In the present paper, we have shown that Zarembo loops exist in the SYM theory both at weak and at strong coupling. It thus seems plausible that one can uncover a precise relationship between the Zarembo loops of the SYM theory and the corresponding operators in the BLG model. Perhaps, the formal relationship between the Lagrangians of the two theories, elucidated in [25], can prove to be fruitful to uncover this aspect of the M2/D2 duality.

On a related note, it would be extremely interesting to explore connections between Wilson loops and scattering amplitudes. The relations between these two classes of gauge theory observables, first studied in the context of $\mathcal{N} = 4$ SYM in $d = 4$ [26, 27, 28, 29, 30] can potentially exist for three dimensional Yang-Mills theories as well. The matrix structure of all $2 \leftrightarrow 2$ scattering amplitudes for $\mathcal{N} \geq 4$ SCS theories was recently explored in [31]. This study includes the BLG model, which is expected to be the strong coupling dual of the sixteen supercharge SYM theory. A further study of Wilson loops in the three dimensional gauge theory is obviously needed to fill the missing connections between Wilson loops and scattering amplitudes both in the SYM theory as well as its dual strong-coupling description as a SCS theory.

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A Supersymmetry of string solutions

The supersymmetry analysis of the case $p = 3$ is given in [32], and is rather special due to the constancy of the dilaton. Therefore here we will present the analysis for the cases $p \neq 3$.

The Killing spinor equations for the geometries (23) are obtained by demanding that the variation of the dilatino λ and gravitino ψ_M vanish on the supergravity solution. We use the "democratic formalism"[10] developed in [33] (see for example [34], appendix B therein),

$$\begin{aligned} \delta\psi_M &= D_M\,\epsilon + \frac{e^{\phi}}{16}\,\widetilde{F}_{(p+2)}\,\gamma_M\,\mathcal{P}_p\,\epsilon = 0, \\ \delta\lambda &= \widetilde{\partial}\phi\,\epsilon + \frac{e^{\phi}}{8}\,(-1)^p\,\widetilde{F}_{(p+2)}\,\mathcal{P}_p\,\epsilon = 0, \end{aligned} \tag{56}$$

where we have used the fact that the p-brane solutions have only a dilaton ϕ and a $(p+2)$-form field strength $F_{(p+2)}$ turned on. The Killing spinor is denoted by ϵ while γ_M are the real 10-d curved space gamma matrices in Lorentzian mostly positive signature. The covariant derivative $D_M = \partial_M + \frac{1}{4}\omega_M^{ab}\Gamma_{ab}$ where Γ_a denote tangent space gamma matrices. The constant matrices $\mathcal{P}_p$ are given in [34] but won't concern us here. Finally we have adopted the notation

$$\widetilde{F}_{(p+2)} \equiv F_{M_1\ldots M_{p+2}}\gamma^{M_1\ldots M_{p+2}}, \quad \widetilde{\partial}\phi \equiv \gamma^M\partial_M\phi. \tag{57}$$

By acting with γ_M from the left on the second equation in (56) we may eliminate the field strength term in the first equation and obtain

$$D_M\,\epsilon - \frac{1}{2}\frac{s_M}{(3-p)}\,\widetilde{\partial}\phi\,\gamma_M\,\epsilon = 0, \quad s_M = \begin{cases} 1 & \text{if} \quad M = 0,\ldots,p \\ -1 & \text{otherwise} \end{cases}, \tag{58}$$

where we have used the fact that on the solution (23) the dilaton ϕ depends only on the coordinate U, while $F_{(p+2)} = F_{0\ldots pU}$ where $0,\ldots,p$ denote the $p+1$ coordinates $x_{\shortparallel}$.

For convenience we scale the C_p and α' dependence out of the metric, which is equivalent to replacing $\alpha', C_p \to 1$. We also specialize to those coordinates relevant

[10] In the democratic formalism the number of Ramond-Ramond potentials $C_{(n)}$ is doubled so that $n = 0, 2, \ldots, 10$ for IIB and $n = 1, 3, \ldots, 9$ for IIA. The extra potentials, in the absence of fermionic and NS-NS fields, are simply given by the action of Hodge duality upon the field strengths.

to the string solution (26). We then employ the following basis of one-forms

$$\begin{aligned} e^{\bar{U}} = U^{(p-7)/4}\, dU, \quad e^{\bar{r}} = U^{(7-p)/4}\, dr, \quad e^{\bar{\psi}} = rU^{(7-p)/4}\, d\psi, \\ e^{\bar{\theta}} = U^{(p-3)/4}\, d\theta, \quad e^{\bar{\phi}} = U^{(p-3)/4} \cos\theta\, d\phi, \end{aligned} \tag{59}$$

using which the relevant components of the spin-connection are

$$\begin{aligned} \omega_r^{\bar{U}\bar{r}} = \frac{p-7}{4} U^{(5-p)/2}, \quad \omega_\psi^{\bar{U}\bar{\psi}} = \frac{p-7}{4}\, r\, U^{(5-p)/2}, \quad \omega_\psi^{\bar{r}\bar{\psi}} = -1, \\ \omega_\theta^{\bar{U}\bar{\theta}} = \frac{3-p}{4}, \quad \omega_\phi^{\bar{U}\bar{\phi}} = \frac{3-p}{4} \cos\theta, \quad \omega_\phi^{\bar{\theta}\bar{\phi}} = \sin\theta. \end{aligned} \tag{60}$$

The Killing spinor equations (58) are then given by

$$\begin{aligned} &\partial_U\, \epsilon + \frac{p-7}{8U}\, \epsilon = 0, \\ &\partial_r\, \epsilon = 0, \\ &\partial_\psi\, \epsilon - \frac{1}{2} \Gamma_{\bar{r}\bar{\psi}}\, \epsilon = 0, \\ &\partial_\theta\, \epsilon - \frac{1}{2} \Gamma_{\bar{U}\bar{\theta}}\, \epsilon = 0, \\ &\partial_\phi\, \epsilon + \frac{1}{2} \sin\theta\, \Gamma_{\bar{\theta}\bar{\phi}} - \frac{1}{2} \cos\theta\, \Gamma_{\bar{U}\bar{\theta}}\, \epsilon = 0, \end{aligned} \tag{61}$$

and solved by

$$\epsilon = U^{(7-p)/8}\, e^{\frac{\theta}{2}\Gamma_{\bar{U}\bar{\theta}}}\, e^{\frac{\psi}{2}\Gamma_{\bar{r}\bar{\psi}}}\, e^{-\frac{\phi}{2}\Gamma_{\bar{\phi}\bar{U}}}\, \epsilon_0. \tag{62}$$

The supersymmetry projector for the fundamental string is given by

$$\partial_\tau \mathbb{X}^M \partial_\sigma \mathbb{X}^N\, \gamma_{MN}\, \epsilon = \sqrt{-\det \partial_a \mathbb{X}^M \partial_b \mathbb{X}^N G_{MN}}\, \mathcal{P}\, \epsilon = \mathcal{L}\mathcal{P}\, \epsilon\,, \tag{63}$$

where

$$\mathcal{P} = \begin{cases} \Gamma_0 \ldots \Gamma_9, & \text{IIA, i.e. } p \text{ even} \\ KI, & \text{IIB, i.e. } p \text{ odd} \end{cases}, \tag{64}$$

where $KI = -IK$, K acts by complex conjugation upon spinors while I acts as $-i$, see [35, 36, 37]. On our solution (29) we find

$$\begin{aligned} \partial_\tau \mathbb{X}^M \partial_\sigma \mathbb{X}^N\, \gamma_{MN} = U'\, r\, \Gamma_{\bar{\psi}\bar{U}} + U'\, U^{(p-5)/2}\, \cos\theta\, \Gamma_{\bar{\phi}\bar{U}} \\ + r'\, r\, U^{(7-p)/2}\, \Gamma_{\bar{\psi}\bar{r}} + r'\, U\, \cos\theta\, \Gamma_{\bar{\phi}\bar{r}} \\ + \theta'\, U\, r\, \Gamma_{\bar{\psi}\bar{\theta}} + \theta' U^{(p-3)/2} \cos\theta\, \Gamma_{\bar{\phi}\bar{\theta}}. \end{aligned} \tag{65}$$

The Killing spinor also simplifies to

$$\epsilon = U^{(7-p)/8}\, e^{\frac{\theta}{2}\Gamma_{\bar{U}\bar{\theta}}}\, e^{\frac{\tau}{2}\left(\Gamma_{\bar{r}\bar{\psi}} - \Gamma_{\bar{\phi}\bar{U}}\right)}\, \epsilon_0. \tag{66}$$

In order to find solutions to the projector equation, we find that we must remove the τ dependence from the Killing spinor by requiring

$$\Gamma_{\bar{r}\bar{\psi}}\, \epsilon_0 = \Gamma_{\bar{\phi}\bar{U}}\, \epsilon_0. \tag{67}$$

The projector equation is then

$$e^{-\frac{\theta}{2}\Gamma_{\bar{U}\bar{\theta}}}\Bigg[U'\,r\,\Gamma_{\bar{\psi}\bar{U}}+U'\,U^{(p-5)/2}\,\cos\theta\,\Gamma_{\bar{\phi}\bar{U}}+r'\,r\,U^{(7-p)/2}\,\Gamma_{\bar{\psi}\bar{r}}+r'\,U\,\cos\theta\,\Gamma_{\bar{\phi}\bar{r}}$$
$$+\theta'\,U\,r\,\Gamma_{\bar{\psi}\bar{\theta}}+\theta' U^{(p-3)/2}\cos\theta\,\Gamma_{\bar{\phi}\bar{\theta}}\Bigg]\,e^{\frac{\theta}{2}\Gamma_{\bar{U}\bar{\theta}}}\,\epsilon_0=\mathcal{L}\mathcal{P}\epsilon_0. \tag{68}$$

Expanding out the LHS of this expression and using (67) one finds

$$\begin{aligned}
&-\Big(\sin\theta\,r\,U'+\cos\theta\,\theta'\,r\,U\Big)\,\Gamma_{\bar{\theta}\bar{\psi}}\,\epsilon_0-\Big(\sin\theta\cos\theta\,U'\,U^{(p-5)/2}+\theta'\,U^{(p-3)/2}\,\cos^2\theta\Big)\,\Gamma_{\bar{\theta}\bar{\phi}}\,\epsilon_0\\
&+\Big(U'\,U^{(p-5)/2}\,\cos^2\theta-r'\,r\,U^{(7-p)/2}-\theta'\,\sin\theta\,\cos\theta\,U^{(p-3)/2}\Big)\,\Gamma_{\bar{\phi}\bar{U}}\,\epsilon_0\\
&+\Big(U'\,r\,\cos\theta+r'\,U\,\cos\theta-\theta'\,r\,U\,\sin\theta\Big)\,\Gamma_{\bar{\psi}\bar{U}}\,\epsilon_0.
\end{aligned} \tag{69}$$

One then finds that the first three bracketed expressions are zero on the solution (29), while the last bracketed expression is equal to $\sqrt{\det\partial_a\mathbb{X}^M\partial_b\mathbb{X}^N G_{MN}}$, which by the Virasoro constraint (28) is the square-root of a perfect square. In addition to (67) we therefore also have

$$\Gamma_{\bar{\psi}\bar{U}}\,\epsilon_0=i\mathcal{P}\,\epsilon_0. \tag{70}$$

The two conditions (67) and (70) each reduce the supersymmetry by half, thus the solutions respect a quarter of the original 16 supersymmetries, i.e. they are 1/4 BPS.

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