

Deconstructing Supersymmetric S -matrices in $D \leq 2 + 1$

Abhishek Agarwal^a and Donovan Young^b

^a*Physical Review Letters, American Physical Society, 1 Research Road, Ridge, NY
11961, USA*

and

Physics Department, City College of CUNY, New York, NY 10031 USA

^b*Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100
Copenhagen, Denmark*

^aabhishek@ridge.aps.org, ^bdyoung@nbi.dk

Abstract

Global supersymmetries of the S -matrices of $\mathcal{N} = 2, 4, 8$ supersymmetric Yang-Mills theories in three spacetime dimensions (without matter hypermultiplets) are shown to be $SU(1|1)$, $SU(2|2)$ and $SU(2|2) \otimes SU(2|2)$ respectively. These symmetries are not manifest in the off-shell Lagrangian formulations of these theories. A direct map between these symmetries and their representations in terms of the Yang-Mills degrees of freedom and the corresponding quantities in Chern-Simons-Matter theories with $\mathcal{N} \geq 4$ supersymmetry is also obtained. Dimensional reduction of the on-shell observables of the Yang-Mills theories to two spacetime dimensions is also discussed.

1 Introduction and Summary

In this paper we continue with investigations of hidden symmetries of S -matrices of three dimensional supersymmetric Yang-Mills (SYM) theories and their relation to the corresponding quantities for supersymmetric Chern-Simons matter (SCS) theories. In a previous publication [1], we showed that the S -matrices of SYM theories with $\mathcal{N} \geq 2$ supersymmetry (without additional matter hypermultiplets) have additional bosonic symmetries that are not manifest in their off-shell Lagrangian formulations. In particular, the bosonic symmetries of the $\mathcal{N} \geq 2$ S -matrices was shown to be $SO(\mathcal{N})$, while only a global $SO(\mathcal{N} - 1)$ R -symmetry is explicitly realized in the Lagrangians. In this note we uncover the supersymmetric completion of the bosonic symmetries of the S -matrices and find them to be $SU(1|1)$, $SU(2|2)$, and $SU(2|2) \otimes SU(2|2)$ for $\mathcal{N} = 2, 4$ and 8 SYM theories respectively.

A related class of gauge theories of much recent interest are SCS models with $\mathcal{N} \geq 4$ supersymmetry. In particular the $\mathcal{N} = 6$ ABJM model [6] and the $\mathcal{N} = 8$ BLG theories [7] have been investigated in great detail in the recent literature focusing on M2-branes. For instance, the S -matrix of the superconformal ABJM model has been shown to have numerous fascinating hidden structures, including a potential infinite dimensional Yangian symmetry [2, 8]. Since the SYM and SCS theories are expected to be related by renormalization group flows (at least in the case of maximal supersymmetry through the flow of the D2-brane theory to M2) one might expect some aspects of the symmetries of the SCS S -matrices to be evident in on-shell properties of the SYM theories as well. A puzzling aspect of the D2 to M2 flow is the lack of a direct off-shell connection between the symmetries and degrees of freedom of the respective worldvolume theories¹. For instance, the ABJM model has four complex scalars and a $SU(2|2) \times g_2$ supersymmetry invariance, while the $\mathcal{N} = 8$ SYM theory on the other hand has a $SO(7)$ R -symmetry relating the seven real scalars of the theory. Part of what we do in this paper is show that the on-shell supersymmetry algebras of the SYM and SCS theories can be mapped to one another. We also provide a dictionary connecting the on-shell physical gauge invariant degrees of freedom of these two classes of theories.

The organization of the paper and a summary of our results are as follows. Starting with a particularly generic off-shell formulation of the $\mathcal{N} = 2, 4, 8$ SYM theories in three dimensions we review the arguments from [1] showing that the S -matrices for these theories possess a $SO(\mathcal{N})$ symmetry. We also briefly comment upon the three dimensional analog of the spinor-“helicity” formalism developed in [1] that allows this symmetry enhancement to be manifest.

In the next sub-section we briefly review some salient aspects of the on-shell supersymmetry symmetry algebra of SCS theories with $\mathcal{N} = 4, 6$, and 8. In particular we focus on the $SU(2|2)$ structures that are naturally present as the symmetries of the S -matrices of these theories. Much of our discussion on SCS theories is based on the formalism introduced in [2].

¹The M2-brane theory can be shown to be related to the D2-brane theory through a Higgs mechanism [9].

The following section contains the central results of this note where we present a clear connection between the the degrees of freedom and the underlying supersymmetry algebras of the SYM and SCS theories. In particular we show how to construct the $SO(\mathcal{N})$ covariant “ ρ ” tensors – that fix both the off-shell Lagrangians as well as the on-shell SUSY algebra of the SYM theories – from the single particle representation of the on-shell SUSY algebra of the SCS models. As a result we are able to extend the results of [1] and uncover the full global supersymmetry algebras for the SYM theories which are shown to be $SU(1|1)$, $SU(2|2)$, and $SU(2|2) \otimes SU(2|2)$ for $\mathcal{N} = 2, 4$, and 8 SYM theories respectively². This is to be contrasted with the global off-shell bosonic symmetries of these theories which are $SO(\mathcal{N} - 1)$. As noted in [1], this symmetry enhancement is due to $\mathcal{N} - 1$ extra $U(1)$ generators that couple the scalar degree of freedom arising from the on-shell gluon to the scalars transforming under the R -symmetry generators. The $\mathcal{N} - 1$ extra $U(1)$ generators enhance the bosonic symmetries to $SO(\mathcal{N})$ and the $SU(1|1)$, $SU(2|2)$, and $SU(2|2) \otimes SU(2|2)$ superalgebras are the supersymmetric completion of the enhanced bosonic symmetries of the S -matrices. As a consequence of this construction we are also able to relate the degrees of freedom of the SYM and SCS theories; namely identify the degrees of freedom which furnish a representation of the part of the superalgebra that is common to both these theories. In the case of $\mathcal{N} = 4$ supersymmetry we find the complex combination of the real degrees of freedom of the SYM theory that carry a representation of the $SU(2|2)$ superalgebra carried by the matter hypermultiplets of the SCS theory. For $\mathcal{N} = 8$ supersymmetry, the same construction is doubled, in a precise sense outlined later. Furthermore, we also find the precise truncation, in terms of the SYM supercharges, of the $\mathcal{N} = 8$ superalgebra to $\mathcal{N} = 6$ – the superalgebra of the ABJM theory.

In the final section of the paper we show that the on-shell supersymmetry algebras of the SYM theories considered in this paper survive a dimensional reduction to $1 + 1$ dimensions. In particular we show how the dimensional reduction of the spinor-“helicity” formalism allows one to eliminate the two dimensional gluon via gauge transformations while turning the three-dimensional gluon into an on-shell pseudo-scalar in two spacetime dimensions. We hope that this note will be useful in the further analysis of the on-shell symmetries and potential integrable structures for D2 and D1-brane theories.

2 Invariant on and off shell formulations of $D = 2+1$ SYM and SCS theories

We begin with a unified off-shell presentation of the $\mathcal{N} = 2, 4, 8$ SYM theories obtained in [1]. The action for these theories can be written in the following compact notation

²For other recent applications of $SU(2|2)$ -type symmetries to studies of lower dimensional gauge theories see [10].

(a is a gauge-group index)

$$\begin{aligned}
S = & -\frac{1}{4} \int F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} \int (D_\mu \phi^i)^a (D^\mu \phi^i)^a + \frac{i}{2} \int \bar{\lambda}_I^a \gamma^\mu D_\mu \lambda_I^a \\
& + \frac{i}{2} \int f^{abc} \rho_{AB}^i \bar{\lambda}_A^a (\phi^i)^b \lambda_B^c - \frac{1}{4} \int f^{abc} f^{amn} (\phi^i)^b (\phi^j)^c (\phi^i)^m (\phi^j)^n.
\end{aligned} \tag{1}$$

The capital indices A, B, I run from $1 \cdots \mathcal{N}$ while the number of adjoint scalars (or the range of the lower case indices, i, j) is $\mathcal{N} - 1$. The global R symmetries of these theories are obviously $SO(\mathcal{N} - 1)$.

While most of the terms in the action are the standard ones for supersymmetric Yang-Mills theories in any number of dimensions, the ρ tensors appearing as the Yukawa coupling are specific to three dimensions. Their explicit form depends on how copies of three dimensional gamma matrices are embedded in the gamma matrices of the higher dimensional minimally supersymmetric theories, of which the $D = 3$ theories can be thought of as dimensional reductions.

As shown in [1], the ρ tensors are key to understanding how the on-shell S -matrices of these theories have an enhanced symmetry, namely $SO(\mathcal{N})$. The hidden $SO(\mathcal{N})$ symmetry can readily be glimpsed by the following observation. Combining the ρ tensors that dictate the Yukawa couplings with the obvious $SO(\mathcal{N})$ invariant, namely, the delta function δ_{AB} , we get a tensor that has natural transformation properties under $SO(\mathcal{N})$

$$\rho_{AB}^C = \{\rho_{AB}^1 = \delta_{AB}, \rho_{AB}^i\} \rightarrow \rho_{AC}^D \rho_{BC}^E + \rho_{AC}^E \rho_{BC}^D = 2\delta^{DE} \delta_{AB}. \tag{2}$$

For example, for $\mathcal{N} = 2$ we have

$$\rho_{AB}^C = \{\delta_{AB}, \epsilon_{AB}\}, \tag{3}$$

i.e. the two $SO(2)$ invariants. For $\mathcal{N} = 8$, ρ_{BC}^A are the well known $\mathbf{8}_{s,c,v}$ symbols relating the three eight dimensional representations of $SO(8)$. As we shall see later on, the ρ tensors also dictate the bosonic part of the on-shell symmetries of these theories. A main result reported later in this paper is the full supersymmetry algebra whose bosonic part – namely $SO(\mathcal{N})$ – is captured by the ρ tensors.

Before moving on to other issues we note that (1) is invariant under the following off-shell supersymmetry transformations

$$\begin{aligned}
\delta A_\mu^a &= -2i \bar{\lambda}_A^a \gamma_\mu \epsilon_A, \\
\delta (\phi^i)^a &= -2i \rho_{AB}^i \bar{\lambda}_A^a \epsilon_B, \\
\delta \lambda_A^a &= F_{\mu\nu}^a \gamma^{\mu\nu} \epsilon_A + 2(D_\mu \phi^i)^a \rho_{AB}^i \gamma^\mu \epsilon_B - f^{abc} \rho_{AB}^i \rho_{BC}^j (\phi^i)^b (\phi^j)^c \epsilon_C.
\end{aligned} \tag{4}$$

Supersymmetry invariance requires that the ρ tensors satisfy the following identities³

$$\begin{aligned}
\rho_{AB}^i &= -\rho_{BA}^i, \\
\rho_{AB}^i \rho_{CB}^j + \rho_{AB}^j \rho_{CB}^i &= 2\delta^{ij} \delta_{AC}, \\
\rho_{AB}^i \rho_{CD}^i &= \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} - \epsilon_{ABCD},
\end{aligned} \tag{5}$$

³The signs in front of the epsilon tensors are sensitive to the ordering of the ρ^A 's. We have used the conventions of section 3 below.

for the case of $\mathcal{N} = 2, 4$. In the case of maximal supersymmetry the last of the three identities needs to be modified to

$$\rho_{AB}^i \rho_{CD}^i = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} - \epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} + \epsilon_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} - \epsilon_{\hat{A}\tilde{B}\tilde{C}\hat{D}}, \quad (6)$$

where the hatted indices run from $1, \dots, 4$ and the tilded indices from $5, \dots, 8$. The last term indicates ϵ_{3456} , and is totally antisymmetric under permutations. Further details and additional properties of the ρ tensors relevant to proving off-shell SUSY invariance are provided in appendix A.

Focusing now on on-shell quantities⁴, it is convenient to introduce a three dimensional polarization vector

$$\epsilon_\mu(p, k) = \frac{\langle p | \gamma_\mu | k \rangle}{\langle kp \rangle}, \quad p_\mu \epsilon^\mu(p, k) = k_\mu \epsilon^\mu(p, k) = 0, \quad (7)$$

(in the notation of [1]) for the oscillator expansion of the gauge potential. p is the physical momentum of the gluon while k is an auxiliary momentum whose choice is tantamount to gauge fixing. After carrying out the oscillator expansions, the supersymmetry algebra (4) translates into the following $SO(\mathcal{N})$ covariant transformations for the on-shell fields

$$Q_A |a^B\rangle = \frac{u}{2} \rho_{AC}^B |\lambda_C\rangle, \quad Q_A |\lambda_B\rangle = -\frac{u}{2} \rho_{AB}^C |a^C\rangle. \quad (8)$$

$|a^1\rangle$ is the on-shell scalar obtained from the gluon, while $|a^2\rangle, \dots, |a^{\mathcal{N}}\rangle$ are the on-shell versions of the real scalars present in the Lagrangian⁵. u is a real three dimensional Majorana spinor, whose form in the conventions of [1] is

$$u(p) = \frac{1}{\sqrt{p_0 - p_1}} \begin{pmatrix} p_2 \\ p_1 - p_0 \end{pmatrix}. \quad (9)$$

Since the supersymmetry algebra above is manifestly $SO(\mathcal{N})$ covariant, the S -matrix for these theories must necessarily be $SO(\mathcal{N})$ invariant for it to commute with the on-shell supercharges. This statement was also illustrated explicitly at the level of the four particle amplitudes in [1].

As is evident, the hidden enhanced bosonic symmetry of the SYM S -matrices are encoded in the the ρ tensors, which also fix the Yukawa couplings of the corresponding off-shell Lagrangians. In section 3 we shall construct these tensors from representations of the on-shell superalgebras relevant to SCS theories which will allow us to both find the supersymmetric completion of the bosonic $SO(\mathcal{N})$ symmetries and relate the superalgebras underlying the S -matrices of the SYM and SCS theories.

⁴For recent reviews of on-shell methods see [11].

⁵This symmetry can also be uncovered upon a linearization of the recently constructed gauge invariant formalism for SYM theories. For a discussion on this matter, we refer to [12]; especially the last section of this reference.

2.1 Supersymmetric Chern-Simons-Matter theories

Before relating the symmetries of the S -matrices of SYM and SCS theories in the next section, let us briefly review some details of Chern-Simons matter theories with $\mathcal{N} \geq 4$ supersymmetry. In the case of $\mathcal{N} = 4$ supersymmetry, one has two complex scalars ϕ_a and two compensating fermionic degrees of freedom ψ_a transforming under two different $SU(2)$ groups (denoted by the dotted and undotted indices) [3]. Since we shall be concerned only with color ordered amplitudes, we will not delve into the possible gauge groups and the representations compatible with $\mathcal{N} \geq 4$ supersymmetry, except to refer to [2–5]. One can add twisted matter hypermultiplets $\tilde{\phi}_a, \tilde{\psi}_a$ which, in general, can carry a different representation of the gauge group without losing $\mathcal{N} = 4$ supersymmetry [4, 5]. However, when the twisted and untwisted hypermultiplets carry the same representation, one has $\mathcal{N} = 5$ supersymmetry. The special case of the hypermultiplets being in the bifundamental representation of $SU(N)$ corresponds to the ABJM model with $\mathcal{N} = 6$ superconformal invariance [6], while the particular case of $N = 2$ produces the maximally supersymmetric $\mathcal{N} = 8$ BLG theory [7]. In the absence of twisted hypermultiplets, one has four supercharges⁶ $\mathcal{Q}_{\alpha\dot{b}\dot{c}}$ which act linearly on the on-shell fields as [2]

$$\mathcal{Q}_{\alpha\dot{b}\dot{c}}|\phi_d\rangle = \epsilon_{bd}u_\alpha|\psi_{\dot{c}}\rangle, \quad \mathcal{Q}_{\alpha\dot{b}\dot{c}}|\psi_{\dot{d}}\rangle = \epsilon_{\dot{c}\dot{d}}u_\alpha|\phi_b\rangle, \quad (10)$$

where u_α are solutions of the massless Dirac equation in three dimensions (9). The on-shell supersymmetry algebra for the conformal $\mathcal{N} = 4$ SCS theories is simply $SU(2|2)$

$$\{\mathcal{Q}_{\alpha\dot{b}\dot{c}}, \mathcal{Q}_{\beta\dot{e}\dot{f}}\} = \epsilon_{be}\epsilon_{\dot{c}\dot{f}}P_{\alpha\beta}. \quad (11)$$

In the case of $\mathcal{N} = 6$ supersymmetry one has two additional supercharges $\tilde{\mathcal{Q}}_\alpha^\pm$ forming a g_2 algebra which relate the twisted and untwisted matter fields. In the case of maximal supersymmetry one has two complete copies of $SU(2|2)$ furnishing the eight supercharges needed for the BLG theory. For a detailed exposition of the on-shell symmetry algebra for a more general class of theories (which include potential mass-deformations) we refer to [2], where a Lagrangian formulation of these gauge theories can also be found.

In the case of the SCS theories, the on-shell symmetries are also reflected in the off-shell Lagrangians, which is in contrast to the case of SYM theories [1]. In the next section, we show that the underlying on-shell supersymmetry algebras of the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ SYM and SCS theories are the same. Furthermore, we obtain the appropriate truncation of the $\mathcal{N} = 8$ superalgebra of the SYM theories that reproduces the $\mathcal{N} = 6$ on-shell algebra of the ABJM models. In the process, we also obtain a precise map between the on-shell gauge invariant degrees of freedom of these two different classes of gauge theories.

⁶ α is the three dimensional Lorentz index.

3 $SU(2|2)$ structure for S -matrices

In this section we will show that the on-shell superalgebra (8) respected by the S -matrices of the $\mathcal{N} = 4$ ($\mathcal{N} = 8$) SYM theories may be re-cast into the $SU(2|2)$ ($SU(2|2) \otimes SU(2|2)$) supersymmetry algebras obeyed by the $\mathcal{N} = 4$ ($\mathcal{N} = 8$) SCS theories considered in [2]. We will also show that the intermediate case, $\mathcal{N} = 6$, corresponds to removing two of the 8 SUSY generators of the $\mathcal{N} = 8$ SYM theory.

The on-shell degrees of freedom of the $\mathcal{N} = 2, 4, 8$ SYM theory consist of \mathcal{N} real bosons $|a^A\rangle$ and an equal number of Majorana fermions $|\lambda_C\rangle$. The superalgebra is then given by the action of the supercharges $Q_{B,\alpha}$ upon these on-shell states [1]

$$Q_{B,\alpha}|a^A\rangle = \frac{1}{2}u_\alpha \rho_{BC}^A |\lambda_C\rangle, \quad Q_{B,\alpha}|\lambda_C\rangle = -\frac{1}{2}u_\alpha \rho_{BC}^A |a^A\rangle, \quad (12)$$

where the indices $A, B, C = 1, \dots, \mathcal{N}$ and u_α is a 2-spinor ($\alpha = 1, 2$), while $\rho^1 = \mathbb{1}$. The ρ^I obey

$$\rho^I (\rho^J)^T + \rho^J (\rho^I)^T = 2\delta^{IJ} \mathbb{1}. \quad (13)$$

As discussed in the previous section, the SCS theories considered in [2] have, for the $\mathcal{N} = 4$ theory, on-shell complex bosons $|\phi_b\rangle$ and complex fermions $|\psi_i\rangle$, obeying the algebra (10). In the case of $\mathcal{N} = 6$ and $\mathcal{N} = 8$ there are also twisted versions of these states, denoted with a tilde. In these cases the states also carry an additional label corresponding to representations of the g_2 and g_4 algebras for the $\mathcal{N} = 6$ and $\mathcal{N} = 8$ cases respectively [2]. In [2] this additional label was denoted with a \pm for the $\mathcal{N} = 6$ case,

$$|\phi_{b\pm}\rangle, |\tilde{\phi}_{b\pm}\rangle, |\psi_{i\pm}\rangle, |\tilde{\psi}_{i\pm}\rangle, \quad (14)$$

and by the addition of an extra $SU(2)$ index (hatted for the untwisted states and tilded for the twisted ones) for the $\mathcal{N} = 8$ case

$$|\phi_{b\hat{c}}\rangle, |\tilde{\phi}_{b\hat{c}}\rangle, |\psi_{i\hat{c}}\rangle, |\tilde{\psi}_{i\hat{c}}\rangle. \quad (15)$$

It should be noted that the field content of the theory under consideration is obtained by fixing a value for these extra indices. For example, in the $\mathcal{N} = 6$ case choosing $+$ or $-$ amounts to a choice of sign for the central charge in the algebra [2]. The $\mathcal{N} = 4$ fields have no such extra indices, but it is convenient for us to decorate them as in the $\mathcal{N} = 8$ case, with a (in this case superfluous) extra index, so that (10) becomes

$$\mathcal{Q}_{abc}|\phi_{d\hat{e}}\rangle = u_\alpha \epsilon_{bd} |\psi_{c\hat{e}}\rangle, \quad \mathcal{Q}_{abc}|\psi_{i\hat{e}}\rangle = u_\alpha \epsilon_{\hat{c}\hat{d}} |\phi_{b\hat{e}}\rangle. \quad (16)$$

The $\mathcal{N} = 8$ case essentially amounts to a doubling of this algebra, and the addition of new supercharges $\tilde{\mathcal{Q}}_{\alpha\tilde{b}\tilde{c}}$ which relate twisted and untwisted states. This gives the $SU(2|2) \otimes SU(2|2)$ algebra

$$\begin{aligned} \mathcal{Q}_{abc}|\phi_{d\hat{e}}\rangle &= u_\alpha \epsilon_{bd} |\psi_{c\hat{e}}\rangle, & \mathcal{Q}_{abc}|\tilde{\phi}_{i\hat{e}}\rangle &= u_\alpha \epsilon_{\hat{c}\hat{d}} |\tilde{\psi}_{b\hat{e}}\rangle, \\ \mathcal{Q}_{abc}|\psi_{i\hat{e}}\rangle &= u_\alpha \epsilon_{\hat{c}\hat{d}} |\phi_{b\hat{e}}\rangle, & \mathcal{Q}_{abc}|\tilde{\psi}_{d\hat{e}}\rangle &= u_\alpha \epsilon_{bd} |\tilde{\phi}_{c\hat{e}}\rangle, \\ \tilde{\mathcal{Q}}_{\alpha\tilde{b}\tilde{c}}|\phi_{d\hat{e}}\rangle &= u_\alpha \epsilon_{\hat{c}\hat{e}} |\tilde{\psi}_{d\tilde{b}}\rangle, & \tilde{\mathcal{Q}}_{\alpha\tilde{b}\tilde{c}}|\psi_{i\hat{e}}\rangle &= -u_\alpha \epsilon_{\hat{c}\hat{e}} |\tilde{\phi}_{d\tilde{b}}\rangle, \\ \tilde{\mathcal{Q}}_{\alpha\tilde{b}\tilde{c}}|\tilde{\psi}_{d\hat{e}}\rangle &= u_\alpha \epsilon_{\tilde{b}\tilde{e}} |\phi_{d\hat{c}}\rangle, & \tilde{\mathcal{Q}}_{\alpha\tilde{b}\tilde{c}}|\tilde{\phi}_{i\hat{e}}\rangle &= -u_\alpha \epsilon_{\tilde{b}\tilde{e}} |\psi_{i\hat{c}}\rangle. \end{aligned} \quad (17)$$

We now give an explicit map between the on-shell degrees of freedom of the $\mathcal{N} = 8$ SCS and SYM theories which translates the SUSY algebras (12) and (17) into one another; the cases with less supersymmetry then follow in a straightforward way. We begin by breaking-up the $SO(8)$ indices A, B , and C in (12) into two indices, so that $A = (\hat{A}, \tilde{A}) = (1, \dots, 4, 5, \dots, 8)$, etc.. We then take two copies of the the Pauli matrices σ^i along with the unit matrix

$$\sigma_{\hat{A}} = \sigma_{\tilde{A}} = (\mathbb{1}, i\sigma^1, i\sigma^2, i\sigma^3). \quad (18)$$

Using the map⁷

$$\begin{aligned} \mathcal{Q} &= Q_{\hat{A}} \sigma_{\hat{A}}, & \tilde{\mathcal{Q}} &= Q_{\tilde{A}} \sigma_{\tilde{A}}, & |\phi\rangle &= |a^{\hat{A}}\rangle \sigma_{\hat{A}}, & |\tilde{\phi}\rangle &= |a^{\tilde{A}}\rangle \sigma_{\tilde{A}}, \\ |\psi\rangle &= |\lambda_{\hat{A}}\rangle \sigma_{\hat{A}}^T \epsilon, & |\tilde{\psi}\rangle &= -|\lambda_{\tilde{A}}\rangle \epsilon \sigma_{\tilde{A}}^T, \end{aligned} \quad (19)$$

one finds that

$$\begin{aligned} \rho_{\hat{B}\hat{C}}^{\hat{A}} &= -\frac{1}{2} \text{Tr} \left(\epsilon [\sigma_{\hat{C}}^T]^{-1} \sigma_{\hat{B}}^T \epsilon \sigma_{\hat{A}} \right), \\ \rho_{\tilde{B}\tilde{C}}^{\hat{A}} &= -\frac{1}{2} \text{Tr} \left(\epsilon \sigma_{\tilde{B}}^T [\sigma_{\tilde{C}}^T]^{-1} \epsilon \sigma_{\hat{A}} \right), \\ \rho_{\hat{B}\hat{C}}^{\tilde{A}} &= \frac{1}{2} \text{Tr} \left([\sigma_{\hat{C}}^T]^{-1} \epsilon \sigma_{\hat{B}} \epsilon \sigma_{\tilde{A}} \right), \\ \rho_{\tilde{B}\tilde{C}}^{\tilde{A}} &= -\frac{1}{2} \text{Tr} \left(\sigma_{\tilde{A}} \epsilon \sigma_{\tilde{B}} \epsilon [\sigma_{\tilde{C}}^T]^{-1} \right). \end{aligned} \quad (20)$$

It is then straightforward to verify that $\rho^1 = \mathbb{1}$ and that (5), (6), and (12) are obeyed. The map is given more explicitly by

$$\begin{aligned} |a_1\rangle &= \frac{1}{2}(|\phi_{11}\rangle + |\phi_{22}\rangle), & |a_2\rangle &= -\frac{i}{2}(|\phi_{12}\rangle + |\phi_{21}\rangle), \\ |a_3\rangle &= \frac{1}{2}(|\phi_{12}\rangle - |\phi_{21}\rangle), & |a_4\rangle &= -\frac{i}{2}(|\phi_{11}\rangle - |\phi_{22}\rangle), \\ |a_5\rangle &= \frac{1}{2}(|\tilde{\phi}_{11}\rangle + |\tilde{\phi}_{22}\rangle), & |a_6\rangle &= -\frac{i}{2}(|\tilde{\phi}_{12}\rangle + |\tilde{\phi}_{21}\rangle), \\ |a_7\rangle &= \frac{1}{2}(|\tilde{\phi}_{12}\rangle - |\tilde{\phi}_{21}\rangle), & |a_8\rangle &= -\frac{i}{2}(|\tilde{\phi}_{11}\rangle - |\tilde{\phi}_{22}\rangle), \end{aligned} \quad (21)$$

$$\begin{aligned} Q_1 &= \frac{1}{2}(\mathcal{Q}_{11} + \mathcal{Q}_{22}), & Q_2 &= -\frac{i}{2}(\mathcal{Q}_{12} + \mathcal{Q}_{21}), \\ Q_3 &= \frac{1}{2}(\mathcal{Q}_{12} - \mathcal{Q}_{21}), & Q_4 &= -\frac{i}{2}(\mathcal{Q}_{11} - \mathcal{Q}_{22}), \\ Q_5 &= \frac{1}{2}(\tilde{\mathcal{Q}}_{11} + \tilde{\mathcal{Q}}_{22}), & Q_6 &= -\frac{i}{2}(\tilde{\mathcal{Q}}_{12} + \tilde{\mathcal{Q}}_{21}), \\ Q_7 &= \frac{1}{2}(\tilde{\mathcal{Q}}_{12} - \tilde{\mathcal{Q}}_{21}), & Q_8 &= -\frac{i}{2}(\tilde{\mathcal{Q}}_{11} - \tilde{\mathcal{Q}}_{22}), \end{aligned} \quad (22)$$

⁷We suppress the spinor index α on the supercharges while the $SU(2)$ indices of the SCS fields are understood to be carried by the Pauli matrices, the first index corresponding to the row. Furthermore, $\epsilon \equiv i\sigma^2$ is understood to act on the Pauli matrices by usual matrix multiplication.

$$\begin{aligned}
|\lambda_1\rangle &= \frac{1}{2}(|\psi_{12}\rangle - |\psi_{21}\rangle), & |\lambda_2\rangle &= \frac{i}{2}(|\psi_{11}\rangle - |\psi_{22}\rangle), \\
|\lambda_3\rangle &= \frac{1}{2}(|\psi_{11}\rangle + |\psi_{22}\rangle), & |\lambda_4\rangle &= -\frac{i}{2}(|\psi_{12}\rangle + |\psi_{21}\rangle), \\
|\lambda_5\rangle &= -\frac{1}{2}(|\tilde{\psi}_{12}\rangle - |\tilde{\psi}_{21}\rangle), & |\lambda_6\rangle &= \frac{i}{2}(|\tilde{\psi}_{11}\rangle - |\tilde{\psi}_{22}\rangle), \\
|\lambda_7\rangle &= -\frac{1}{2}(|\tilde{\psi}_{11}\rangle + |\tilde{\psi}_{22}\rangle), & |\lambda_8\rangle &= -\frac{i}{2}(|\tilde{\psi}_{12}\rangle + |\tilde{\psi}_{21}\rangle).
\end{aligned} \tag{23}$$

The $\mathcal{N} = 4$ case then follows simply by deleting the $\tilde{\phi}$, $\tilde{\psi}$, and \tilde{Q} , or equivalently, by ignoring the $\tilde{A}, \tilde{B}, \tilde{C}$ indices. The $\mathcal{N} = 2$ case is a further truncation of this, where in addition, the range of the $\tilde{A}, \tilde{B}, \tilde{C}$ indices is taken to be from 1 to 2, or equivalently, where we take

$$\begin{aligned}
\mathcal{Q}_{11} &\rightarrow \mathcal{Q}_{22}, & \mathcal{Q}_{12} &\rightarrow \mathcal{Q}_{21}, & |\phi_{11}\rangle &\rightarrow |\phi_{22}\rangle, & |\phi_{12}\rangle &\rightarrow |\phi_{21}\rangle, \\
|\psi_{22}\rangle &\rightarrow -|\psi_{11}\rangle, & |\psi_{21}\rangle &\rightarrow -|\psi_{12}\rangle.
\end{aligned} \tag{24}$$

Then the relations (16) reduce to a $SU(1|1)$ algebra.

The $\mathcal{N} = 6$ SCS theory has a superalgebra given by [2]

$$\begin{aligned}
\mathcal{Q}_{abc}|\phi_{d\pm}\rangle &= u_\alpha \epsilon_{bd} |\psi_{c\pm}\rangle, & \mathcal{Q}_{abc}|\tilde{\phi}_{d\pm}\rangle &= u_\alpha \epsilon_{cd} |\tilde{\psi}_{b\pm}\rangle, \\
\mathcal{Q}_{abc}|\psi_{d\pm}\rangle &= u_\alpha \epsilon_{cd} |\phi_{b\pm}\rangle, & \mathcal{Q}_{abc}|\tilde{\psi}_{d\pm}\rangle &= u_\alpha \epsilon_{bd} |\tilde{\phi}_{c\pm}\rangle, \\
\tilde{Q}_\alpha^\pm |\phi_{b\mp}\rangle &= u_\alpha |\tilde{\psi}_{b\mp}\rangle, & \tilde{Q}_\alpha^\pm |\psi_{b\mp}\rangle &= -u_\alpha |\tilde{\phi}_{b\mp}\rangle, \\
\tilde{Q}_\alpha^\mp |\tilde{\psi}_{b\mp}\rangle &= -u_\alpha |\phi_{b\mp}\rangle, & \tilde{Q}_\alpha^\mp |\tilde{\phi}_{b\mp}\rangle &= u_\alpha |\psi_{b\mp}\rangle,
\end{aligned} \tag{25}$$

with all other actions of the supercharges upon the states producing zero. We make the following map between the \pm index on the scalars and fermions and the tilded and hatted indices of (17)

$$+ \rightarrow 1, \quad - \rightarrow 2, \tag{26}$$

and then (25) are equivalent to (17) with

$$\tilde{Q}^+ \rightarrow \tilde{Q}_{21}, \quad \tilde{Q}^- \rightarrow -\tilde{Q}_{12}, \quad \tilde{Q}_{11} \rightarrow 0, \quad \tilde{Q}_{22} \rightarrow 0, \tag{27}$$

where the spinor index α has been suppressed. Therefore the restriction to $\mathcal{N} = 6$ is achieved by removing \tilde{Q}_{11} and \tilde{Q}_{22} . In the SYM language we lose Q_5 and Q_8 while

$$\tilde{Q}^+ = iQ_6 - Q_7, \quad \tilde{Q}^- = -iQ_6 - Q_7. \tag{28}$$

The algebras mentioned in this section largely constrain the structure of the S -matrices. In the case of four-particle amplitudes, the supersymmetry algebra constrains the S -matrix to one undetermined function of the coupling constant and the kinematic Mandelstam variables. In the notation of [2], one can decompose the scattering matrix S as $S = I + i\mathcal{T}$, where S is the scattering operator. The $SU(2|2)$ symmetry for the the $\mathcal{N} = 4$ SCS theory then allows one to parametrize the four-particle amplitudes in terms of ten independent functions of the Mandelstam variables

A, \dots, L as

$$\begin{aligned}
\langle \mathcal{T} | \phi_\alpha \phi_\beta \phi_\gamma \phi_\delta \rangle &= \left[\frac{1}{2}(A+B)\epsilon_{\alpha\delta}\epsilon_{\beta\gamma} + \frac{1}{2}(A-B)\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \right] \delta^3\left(\sum_i p_i\right), \\
\langle \mathcal{T} | \psi_{\dot{\alpha}} \psi_{\dot{\beta}} \psi_{\dot{\gamma}} \psi_{\dot{\delta}} \rangle &= \left[\frac{1}{2}(D+E)\epsilon_{\dot{\alpha}\dot{\delta}}\epsilon_{\dot{\beta}\dot{\gamma}} + \frac{1}{2}(D-E)\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}} \right] \delta^3\left(\sum_i p_i\right), \\
\langle \mathcal{T} | \phi_\alpha \psi_{\dot{\beta}} \phi_\gamma \psi_{\dot{\delta}} \rangle &= -G\epsilon_{\alpha\gamma}\epsilon_{\dot{\beta}\dot{\delta}} \delta^3\left(\sum_i p_i\right), \\
\langle \mathcal{T} | \psi_{\dot{\alpha}} \phi_\beta \psi_{\dot{\gamma}} \phi_\delta \rangle &= -L\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\beta\delta} \delta^3\left(\sum_i p_i\right), \\
\langle \mathcal{T} | \phi_\alpha \phi_\beta \psi_{\dot{\gamma}} \psi_{\dot{\delta}} \rangle &= -\frac{1}{2}C\epsilon_{\alpha\beta}\epsilon_{\dot{\gamma}\dot{\delta}} \delta^3\left(\sum_i p_i\right), \\
\langle \mathcal{T} | \phi_\alpha \psi_{\dot{\beta}} \psi_{\dot{\gamma}} \phi_\delta \rangle &= -H\epsilon_{\alpha\delta}\epsilon_{\dot{\beta}\dot{\gamma}} \delta^3\left(\sum_i p_i\right), \\
\langle \mathcal{T} | \psi_{\dot{\alpha}} \psi_{\dot{\beta}} \phi_\gamma \phi_\delta \rangle &= -\frac{1}{2}F\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\gamma\delta} \delta^3\left(\sum_i p_i\right), \\
\langle \mathcal{T} | \psi_{\dot{\alpha}} \phi_\beta \phi_\gamma \psi_{\dot{\delta}} \rangle &= -K\epsilon_{\dot{\alpha}\dot{\delta}}\epsilon_{\beta\gamma} \delta^3\left(\sum_i p_i\right),
\end{aligned} \tag{29}$$

where we have suppressed the extra index corresponding to the $g_{\mathcal{N}-4}$ representation, i.e. we have fixed the choice of + or - for the $\mathcal{N} = 6$ case or fixed the hatted and tilded indices to 1 or 2 in the $\mathcal{N} = 8$ case. All the functions can be expressed in terms of a single function (chosen to be A in [2]) using the constraining properties of the algebra alone. For instance, $D = -A \frac{\langle 34 \rangle}{\langle 12 \rangle}$, $G = +A \frac{\langle 41 \rangle}{\langle 12 \rangle}$, etc. [2]. Similar constraints for $\mathcal{N} = 5, 6$, and 8 SCS theories were also obtained in [2]. On the other hand, amplitudes for SYM theories with extended supersymmetry were constrained in a similar fashion using a real basis for the scalar and fermion fields in [1]. The map between the SYM and SCS on-shell degrees of freedom obtained earlier in this section implies that these constraints are common to both families of theories, as long as the underlying on-shell superalgebras can be mapped to each other, namely, in the cases of $\mathcal{N} = 4, 6$, and 8 supersymmetry.

4 Dimensional reduction to $d = 2$

In this final section we note that the three dimensional on-shell techniques can be easily reduced to two spacetime dimensions to make the $SO(\mathcal{N})$ invariance of the lower dimensional gauge theories manifest at the level of S matrices. We implement dimensional reduction by compactifying the “1” direction. The real three dimensional Majorana spinor (9) becomes

$$\tilde{u}(p) = \frac{1}{\sqrt{p_0}} \begin{pmatrix} p_2 \\ -p_0 \end{pmatrix} = \sqrt{p_0} \begin{pmatrix} \text{sgn}(p) \\ -1 \end{pmatrix}, \tag{30}$$

upon dimensional reduction. The sign refers to the two dimensional mass-shell condition $p_2 = \text{sgn}(p)p_0$. Under the action of the two dimensional ‘‘gamma-five’’ (which is γ^1 in the three dimensional notation), $\gamma^1 \tilde{u} = -\text{sgn}(p)\tilde{u}$. Now, the $D = 3$ polarization vector (7) can be simplified by choosing the auxiliary momentum k judiciously after reducing it to three dimensions. Choosing $k_2 = p_0$ and $k_0 = -p_2$ makes ϵ_0 and ϵ_2 vanish. Recalling that fixing k is tantamount to choosing a gauge, the above statement is nothing but an illustration of the fact that the two dimensional gluon can be gauge transformed away. Under the same gauge choice, $\epsilon_1 = -\text{sgn}(p)$. Using the dimensionally reduced versions of the three-dimensional quantities in the relation $\delta A_1 = \delta\phi^1 = \frac{1}{2}(\bar{\epsilon}_I \gamma_1 \lambda_I)$, and employing the mode expansions given above, we get

$$Q_A |a^1\rangle = \frac{\tilde{u}}{2} |\lambda_A\rangle, \quad (31)$$

in $D = 2$. The dimensional reductions of the other mode expansions are trivial, and they yield the $SO(\mathcal{N})$ covariant algebra

$$Q_A |a^B\rangle = \frac{\tilde{u}}{2} \rho_{AC}^B |\lambda_C\rangle, \quad Q_A |\lambda_B\rangle = -\frac{\tilde{u}}{2} \rho_{AB}^C |a^C\rangle, \quad (32)$$

in two dimensions as expected. This reduction makes the $SO(\mathcal{N})$ structure, and by the analysis presented earlier in the paper, the $SU(1|1)$, $SU(2|2)$, and $SU(2|2) \otimes SU(2|2)$ symmetries of the S -matrices of the dimensional reductions of three dimensional $\mathcal{N} = 2, 4$, and 8 SYM theories to two spacetime dimensions manifest.

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A Off-shell supersymmetry via ρ tensors

In this section we provide further details of how supersymmetry invariance of the action (1) depends on the specific properties of the ρ tensors (5)-(6). While most of the cancelations necessary to see the supersymmetry invariance of the action are obtained using the properties of the ρ tensors mentioned before, the cancelation of the variation of the Yukawa term imposes some further constraints. The variation of the Yukawa term produces

$$-\frac{i}{2} \int [\rho^i \rho^{[k} \rho^{l]}]_{CD} f^{amn} f^{apq} (\bar{\epsilon}_D \lambda_C^n) (\phi^k)^p (\phi^l)^q (\phi^i)^m. \quad (33)$$

The ρ tensors are multiplied in the expression above in the sense of matrix multiplication, where for each value of i , ρ^i is an $\mathcal{N} \times \mathcal{N}$ matrix. For this term to cancel against the variation of the ϕ^4 term, we shall need to reduce the term cubic in the ρ tensors to fewer factors of ρ . To see this in the case of $\mathcal{N} = 4$ supersymmetry one needs to utilize the fact that the tensors obey the $SO(3)$ algebra

$$\rho^i \rho^j = -\epsilon^{ijk} \rho^k, \quad (\rho^i)^2 = -I. \quad (34)$$

Using these identities, and carrying out the sum over the i, k, l indices in (33) produces two terms, T_1 and T_2 . T_1 corresponds to the case where $i = k$ or $i = l$

$$T_1 = -2i \int f^{abc} f^{amn} \rho_{CD}^i (\bar{\lambda}_C \epsilon_D) (\phi^j)^c (\phi^i)^m (\phi^j)^n, \quad (35)$$

and it cancels against the variation of the ϕ^4 term. T_2 corresponds to the case of $i \neq k \neq l$

$$T_2 = -i \int (\bar{\lambda}_C \epsilon_C)^n [f^{amn} f^{apl} + f^{apn} f^{alm} + f^{aln} f^{amp}] (\phi^1)^p (\phi^2)^l (\phi^3)^m. \quad (36)$$

The combination of structure constants in the square brackets vanishes due to the Jacobi identity, hence

$$T_2 = 0. \quad (37)$$

To see how the $\mathcal{N} = 4$ cancellation generalizes to the $\mathcal{N} = 8$ case we note the following properties. Let the indices i, j , and k be distinct. Then

$$\rho^{ij} \equiv \frac{1}{2} (\rho^i \rho^j - \rho^j \rho^i) = \rho^i \rho^j = -\rho^j \rho^i. \quad (38)$$

Consider

$$\rho^{ij} M = \rho^k, \implies M = (\rho^{ij})^{-1} \rho^k. \quad (39)$$

Since $(\rho^i)^2 = -\mathbb{1}$ (no sum over i), we see that $(\rho^i)^{-1} = -\rho^i = (\rho^i)^T$. Therefore

$$M = -\rho^i \rho^j \rho^k. \quad (40)$$

We would now like to prove that the first relation in (39) is cyclically invariant. So we consider

$$\rho^{ki} M = \rho^k \rho^i (-\rho^i \rho^j \rho^k) = \rho^k \rho^j \rho^k = -(\rho^k)^2 \rho^j = \rho^j, \quad (41)$$

and we have thus proven cyclical invariance. We then find that T_2 generalizes to

$$T_2 = -i \int (\bar{\lambda}_A (\rho^i \rho^j \rho^k)_{AB} \epsilon_B)^n [f^{amn} f^{apl} + f^{apn} f^{alm} + f^{aln} f^{amp}] (\phi^i)^p (\phi^j)^l (\phi^k)^m \quad (42)$$

which vanishes similarly; T_1 is produced in the same way as for the $\mathcal{N} = 4$ case.

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