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On the Spectrum of the $AdS_5 \times S^5$ String at large λ

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Abstract

We put forth a program for perturbatively quantizing the bosonic sector of the IIB superstring on $AdS_5 \times S^5$ in the large radius of curvature R (i.e. flat-space) limit, and in light-cone gauge. Using the quantization of the massive particle on $AdS_5 \times S^5$ as a guiding exercise, we read off the correct scaling of the particle coordinates in the large radius limit. In the corresponding large $\sqrt{\lambda} = R^2/\alpha'$ limit of the string the oscillator modes must be scaled differently from the zero modes. We compute the bosonic string Hamiltonian in this limit which gives the flat-space mass-squared operator at leading order, followed by a harmonic oscillator potential in the zero modes at subleading order. Using these ingredients we calculate the leading and sub-leading terms in the conformal dimension of the length-four Konishi state in the large-λ limit. Furthermore we work out the relevant terms of the energy expansion to next-to-next-to leading order, which are plagued by severe ordering ambiguities. This prevents us from determining the corrections to the spectrum at order $\lambda^{-1/4}$.

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1 Introduction

The AdS/CFT correspondence [1] [2, 3] has seen remarkable progress since its inception over a decade ago. At the forefront of these developments has been the study of the spectrum of scaling dimensions of local gauge invariant operators in planar $\mathcal{N} = 4$ super Yang-Mills theory (SYM) which should be equivalent to the energy spectrum of quantum type IIB superstrings in an $AdS_5 \times S^5$ background. These advances where made possible by the apparent integrability [4] of this maximally symmetric AdS_5/CFT_4 system (for reviews see [5] and the very recent [6]). Generally the gauge theory operators or string states fall into representations of the bosonic subgroup $SO(2, 4) \times SO(6)$ of the underlying symmetry supergroup $PSU(2, 2|4)$ which may be labelled by the Cartan charges $(E, S_1, S_2; J_1, J_2, J_3)$. Here E is the energy, whereas the S_i and J_i are angular momenta associated to AdS_5 and S^5 respectively. In limiting situations where subsets of the S_i and J_i tend to infinity, corresponding to gauge theory operators of large classical dimension or, respectively, long spinning strings, exact non-perturbative predictions for the spectrum can be made through the asymptotic Bethe-ansatz (ABA) equations [7] in their final form [8]. A particularly striking example is the large spin $S_1 \to \infty$ limit from which an exact prediction for the cusp anomalous dimension of $\mathcal{N} = 4$ SYM could be made and matched to high perturbative orders in the weak [9] and strong coupling [10] expansions. At strong coupling these asymptotically large Cartan charge regimes are accessible through a semiclassical quantization of the $AdS_5 \times S^5$ string in form of a fluctuation expansion around classical spinning string solutions [11] (for reviews see [12]). This method has been extensively studied and refined to various long-string limit scenarios in the literature (see e.g. [13]).

Recently, substantial progress towards understanding the spectrum of "short" gauge theory operators and short quantum strings was made, i.e. situations in which the charges J_i and S_i remain finite and the asymptotic Bethe equations fail. The prototype gauge theory operator in this scenario is the Konishi operator $\mathrm{Tr}(\phi_I \phi_I)$, where the ϕ_I denote the six real scalars of $\mathcal{N} = 4$ SYM. It is the shortest operator with a non-trivial scaling dimension, which has been computed at weak gauge coupling $\lambda \ll 1$ up to four orders of perturbation theory [14]. In fact one usually computes the anomalous scaling dimension of the Konishi multiplet operator $\mathrm{Tr}[Z, W]^2$, with Z and W complex scalars, differing from the scaling dimensions of the Konishi operator only in its tree-level (classical) part. From the integrable systems perspective important parallel progress was achieved by using two-dimensional field theory inspiration to generalize the ABA. On the one hand improvements of the ABA equations through Lüscher corrections capturing the wrapping effects at low orders in λ [15] were efficient enough to extend the string/integrability weak coupling prediction for the Konishi scaling dimensions to the five loop order [16]. On the other hand the most promising approach appears to be the generalization of the ABA description of states on a decompactified $R^{1,1}$ string-worldsheet to states on a cylinder $R \times S^1$ by the Thermodynamic Bethe Ansatz (TBA) and Y-system [17–19]. This approach led to numerical predictions for the scaling dimensions of the Konishi operator at strong coupling [18, 20].

It is generally believed that at large values of the 't Hooft coupling λ the energy of the Konishi state can be expanded in an asymptotic series in powers of $\lambda^{-1/4}$

$$E_{\text{Konishi}} = c_{-1}\,\lambda^{1/4} + c_0 + \frac{c_1}{\lambda^{1/4}} + \frac{c_2}{\lambda^{1/2}} + \frac{c_3}{\lambda^{3/4}} + \frac{c_4}{\lambda} + \dots \,. \tag{1.1}$$

Here the leading term arises from the lowest excited closed string state in flat-space [3] and takes the value $c_{-1} = 2$. It has also been shown to arise from the ABA [21] in the large λ limit. In fact such an expansion in $\lambda^{-1/4}$ was shown to arise from the exact solution of the $\mathfrak{su}(1|1)$-sector truncation of the full $AdS_5 \times S^5$ superstring model in [22], where in addition the values $c_{-1} = 2$ and $c_0 = 0$ were obtained for the first excited states from both the truncated $\mathfrak{su}(1|1)$-model and the ABA. The two independent numerical studies [18,20] based on the TBA/Y-system approach have reported the values[1]

$$c_{-1} = 2.000(2)\,, \qquad c_0 = 0\,, \quad c_1 = 2.000(3)\,, \qquad c_2 = -0.0(2)\,, \qquad c_3 \sim -\mathcal{O}(1)\,, \tag{1.2}$$

for the length four $\mathfrak{sl}(2)$-sector Konishi descendent $\mathrm{Tr}[D, Z]^2$.

Clearly this result poses a challenge for a strong-coupling quantum string analysis. Here the stumbling block is the full quantization of the IIB superstring in the $AdS_5 \times S^5$ background geometry, a problem which so far continues to elude solution, and without solution prevents one from describing string states which are not semi-classical (i.e. long) in nature. Nevertheless, semi-classical string methods have been adapted to this question in the work of Roiban and Tseytlin [23, 24] providing a puzzling conclusion. They found the coefficients

$$c_{-1} = 2\,, \qquad c_0 = \Delta_0 - 4\,, \qquad c_1 = 1\,, \qquad c_2 = 0\,. \tag{1.3}$$

Note that for the Konishi multiplet state in question $\mathrm{Tr}[Z, W]^2$, the tree-level scaling dimensions is $\Delta_0 = 4$ and hence there is consistency with the numerical result (1.2) for c_0. A clear disagreement by a puzzling factor of two, however, resides in the next-to-next-to-leading order contribution c_1.

However, different results than (1.3) were obtained by extrapolating the semi-classical quantization about spinning folded string solutions in AdS_3 [25]

$$c_{-1} = 2\,, \qquad c_0 = 1\,, \qquad c_1 = 6 - 8\log 2\,, \qquad c_2 = 0\,, \tag{1.4}$$

with non-integer c_1. Extrapolating a pulsating string solution to the short string limit yields yet a different analytical value for c_1 [26]. Hence the present status of the strong-coupling string side is certainly unsatisfactory.

What is needed is an efficient perturbative string quantization for the large λ limit. In this paper we take the first steps towards this goal and for the moment consider only the bosonic part of the string action. We will work in light-cone gauge by using the AdS_5 global time and an azimuthal angle from the S^5 to build the light-cone coordinates $x^{\pm}$ [22,27,28]. An alternative approach using a light-cone gauge entirely within AdS_5 has been studied in [29].

We begin in section 2 with an exercise: the quantization of the massive particle[2] in the large MR limit of $AdS_5 \times S^5$, with M the mass of the particle and R the curvature radius. Here the exact spectrum is known; the recovery of the correct answer provides us with a proof of concept of our self-consistent perturbative procedure and also crucially provides the correct scaling to

[1] We present an averaged value of the two reported results. The value $c_0 = 0$ and the form of the power law expansion has actually been used as an input for fitting the numerical data.

[2] The quantization of the massless $AdS_5 \times S^5$ superparticle in AdS light-cone gauge was discussed in [30].

apply to the phase space variables in order achieve the flat-space limit correctly. The scaling we employ is

$$\vec{x} \to \frac{\vec{x}}{\sqrt{MR}}, \qquad \vec{y} \to \frac{\vec{y}}{\sqrt{MR}}, \qquad \vec{p} \to \sqrt{MR}\,\vec{p}, \qquad \vec{q} \to \sqrt{MR}\,\vec{q}\,, \tag{1.5}$$

where $\vec{x}$ (and, correspondingly $\vec{p}$) are the four transverse coordinates (momenta) of AdS_5, and $\vec{y}$ ($\vec{q}$) those of S^5, see section 2 for details. This form of the scaling may be easily understood by looking at the action of the massive particle in dimensionless coordinates before any scaling on the coordinates performed

$$S_{\text{particle}} = -MR \int d\tau\,\sqrt{-\dot{x}^\mu\,\dot{x}^\nu\,g_{\mu\nu}(x)}\,. \tag{1.6}$$

The scaling (1.5) for the coordinates then yields a pertubatively accessible theory upon expanding the action in $1/\sqrt{MR}$, as it starts out with a quadratic action in fields independent of the coupling constant MR and all interaction terms then come with powers of the coupling constant $1/\sqrt{MR}$.

In section 3 we turn to the bosonic sector of the IIB superstring. Here we apply the same scaling (1.5) to the string zero-modes, but a *different* scaling to the internal string oscillations. The scaling of the oscillatory modes follows the same logic as above: The Nambu-Goto action for the string reads

$$S_{\text{string}} = -\frac{R^2}{\alpha'} \int d\tau\,d\sigma\,\sqrt{-\det\partial_r X^\mu\,\partial_s X^\nu\,g_{\mu\nu}(X)} \tag{1.7}$$

and hence the string coordinates should be scaled by the square-root of the string tension $\lambda^{1/4} \propto R/\sqrt{\alpha'}$. However, as we are interested in the perturbative spectrum of excited *massive* string states at large λ we should rescale string zero-modes differently, namely according to (1.5) with $M \sim 1/\sqrt{\alpha'}$. In summary we therefore scale

$$\begin{aligned}
&\vec{x} \to \frac{1}{\lambda^{1/8}}\vec{x}, \quad \vec{p} \to \lambda^{1/8}\vec{p}, \quad \tilde{\vec{X}} \to \frac{1}{\lambda^{1/4}}\tilde{\vec{X}}, \quad \tilde{\vec{P}} \to \lambda^{1/4}\tilde{\vec{P}},\\
&\vec{y} \to \frac{1}{\lambda^{1/8}}\vec{y}, \quad \vec{q} \to \lambda^{1/8}\vec{q}, \quad \tilde{\vec{Y}} \to \frac{1}{\lambda^{1/4}}\tilde{\vec{Y}}, \quad \tilde{\vec{Q}} \to \lambda^{1/4}\tilde{\vec{Q}}\,,
\end{aligned} \tag{1.8}$$

where $\tilde{\vec{X}}$ are the oscillatory modes and $\vec{x}$ the zero modes of the AdS_5 transverse coordinates, and similarly for the S^5 part of the geometry. We then quantize the string oscillatory modes in the large λ limit. Note that a priori this perturbative prescription points at an expansion in powers of $\lambda^{-1/8}$ opposed to the general expectation (1.1). We observe, however, that at least for the first three non-vanishing orders the expansion turns out to be effectively in $\lambda^{-1/4}$ as expected. To leading order one naturally obtains the flat string spectrum. At subleading order we find a potential for the zero modes which is a harmonic oscillator in the transverse AdS_5 directions and a free particle in the transverse S^5 directions, to wit

$$E^2 - q_\phi^2 = \sqrt{\lambda}\,M^2 + \lambda^{1/4}\,\left(\vec{p}^2 + M^2\vec{x}^2 + \vec{q}^2\right) + \lambda^{1/8}H_{1/8} + \lambda^0 H_0 + \ldots \tag{1.9}$$

where the operator E is the global AdS energy, q_ϕ is the quantum number for the momentum in the light-cone direction on S^5, and M^2 is the string oscillator mass-squared operator in units of α' (i.e. the number operator for non-zero modes). Indeed by replacing $\sqrt{\lambda}M^2 \to (MR)^2$, one obtains exactly the particle spectrum derived in section 2 out to order $\lambda^{1/4}$. We therefore see that to leading order we obtain a particle whose mass is determined by the number of string oscillations, which is what one would expect from the flat-space limit. We have also determined the Hamiltonians at orders $\lambda^{1/8}$ and λ^0, and they are included in the appendices. Importantly the Hamiltonian $\lambda^{1/8}H_{1/8}$ is shown to not contribute to the spectrum of E^2 down to order λ^0. We have not, however, been able to fix the normal ordering constants appearing in these higher-order Hamiltonians $H_{1/8}$ and H_0, and a full treatment would require the addition of the fermionic degrees of freedom, which is beyond the scope of the present paper.

It is interesting to use our result to give an answer for the strong-coupling expansion of the Konishi multiplet's conformal dimension. It is natural to assume that the string state with two oscillations in the transverse S^5 directions, and two units of q_ϕ should correspond to the length-four member of the Konishi multiplet

$$\mathcal{O}(x) = \mathrm{Tr}[Z(x), W(x)]^2, \tag{1.10}$$

where Z and W are two of the complex scalar fields of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We would therefore have that the eigenvalue[3] of M^2 is 4, coming from two transverse S^5 string oscillator excitations, while that of q_ϕ is 2. This gives

$$E = 2\lambda^{1/4} + \frac{1}{4}\left(\vec{p}^2 + 4\,\vec{x}^2 + \vec{q}^2\right)\lambda^0 + \dots \tag{1.11}$$

The natural ground state for the zero modes is a harmonic oscillator ground state for the AdS_5 transverse directions tensored with a plane wave of zero momentum in the S^5 transverse directions. If we had included the fermionic degrees of freedom, we expect that the harmonic oscillator would be generalized to the superharmonic oscillator, which would give a vanishing ground state energy. This is consistent with the results of [18, 20] of eqn. (1.2), where no λ^0 term is found in the expansion for E. Hence if we introduce the gauge theory operator- string theory state correspondence

$$\mathcal{O}(x) \qquad \Leftrightarrow \qquad \beta^W_{-1}\,\tilde{\beta}^W_{-1}\,|0\rangle \tag{1.12}$$

with $\beta^W_n := \beta^1_n + i\beta^2_n$ in the notation of section 3 and where $|0\rangle$ denotes the oscillator and zero-mode groundstate, we find agreement with (1.2) and (1.3) to order λ^0. Note also that in (1.4) as well as in the further 'short string' examples quoted in [26] there is a non-zero integer for c_0 which points at a zero-mode sector excitation in our picture.

It would be very interesting to continue the calculation to higher orders. In appendix A.1 we show that for the states of interest in (1.12) the $H_{1/8}$ term does not contribute. We provide the H_0 Hamiltonian in appendix A.2. Supplemented by the fermionic degrees of freedom, this term is sufficient to calculate the $\mathcal{O}(\lambda^{-1/4})$ contribution to the energy E, which would provide a verdict on the differing values of c_1 from the TBA/Y-system (1.2) [18, 20] versus semi-classical strings (1.3) [23] and (1.4) [25]. We leave this calculation to further research.

[3] In fact, the eigenvalue of M^2 should be 4 only once the fermionic modes have been added. These remove the normal ordering constants from the bosonic modes which would otherwise give $M^2 = 0$ for the level-two states.

2 Massive particle on $AdS_5 \times S^5$

Using a generalization of the methods in [31], the quantum spectrum of the massive particle on $AdS_5 \times S^5$ may be precisely obtained [32]

$$E_{J,n} = 2 + n + \sqrt{4 + J(J+4) + (MR)^2}, \tag{2.1}$$

where M is the mass of the particle and R the common radius of $AdS_5 \times S^5$, J is the total angular momentum on S^5, and $n \geq 0$ is a level number corresponding to excitations of the wavefunction in the AdS_5 directions. Taking the large MR limit, one obtains

$$E_{J,n} = (MR) + n + 2 + \frac{1}{2MR}\Big(J(J+4) + 4\Big) + \mathcal{O}((MR)^{-2}). \tag{2.2}$$

As a warm-up to the string case, we would like to reproduce this result via a direct quantization of the particle action, using coordinates compatible with the standard light-cone coordinates we will use in the next section. We therefore use the AdS_5 embedding coordinates X^A with metric $\eta_{AB} = (-1,-1,1,1,1,1)$ and index $A = (0', 0, i)$. We introduce polar coordinates for the $0 - 0'$ directions $X_0 = r\cos\theta$ and $X_{0'} = r\sin\theta$ and we denote the remaining coordinates as $X^i = x^i$. For the S^5 space, the embedding coordinates are $Y^{\hat{A}}$ with metric $\eta_{\hat{A}\hat{B}} = (1,1,1,1,1,1)$ and index $\hat{A} = (0', 0, i)$. Also for this space, we introduce polar coordinates for the $0 - 0'$ directions $Y_0 = s\cos\phi$ and $Y_{0'} = s\sin\phi$ and we denote the remaining coordinates as $Y^i = y^i$. The phase space form of the massive $AdS_5 \times S^5$ particle Lagrangian then becomes

$$\begin{aligned}\mathcal{L} = (\dot{r}\, p_r + \dot{\theta}\, p_\theta + \dot{\vec{x}} \cdot \vec{p} + \dot{s}\, q_s + \dot{\phi}\, q_\phi + \dot{\vec{y}} \cdot q) - \frac{e}{2}(-p_r^2 - \frac{p_\theta^2}{r^2} + \vec{p}^2 + q_s^2 + \frac{q_\phi^2}{s^2} + \vec{q}^2 + M^2) \\ + \frac{\lambda_1}{2}(\vec{x}^2 - r^2 + R^2) + \lambda_2\,(r\, p_r + \vec{x}\cdot\vec{p}) + \frac{\lambda_3}{2}(\vec{y}^2 + s^2 - R^2) + \lambda_4\,(s\, q_s + \vec{y}\cdot\vec{q})\end{aligned} \tag{2.3}$$

where the terms proportional to λ_2 and λ_4 are secondary constraints arising from the primary constraints $X^A X_A + R^2 = 0$ and $Y^{\hat{A}} Y_{\hat{A}} - R^2 = 0$. We can solve the Hamiltonian constraint for p_θ, which is the space-time energy whose spectrum we are interested in

$$p_\theta^2 = r^2\,[-p_r^2 + \vec{p}^2 + q_s^2 + \frac{q_\phi^2}{s^2} + \vec{q}^2 + M^2]\,. \tag{2.4}$$

The primary and secondary constraints can be solved for the variables r, p_r, s and q_s

$$r = \sqrt{R^2 + \vec{x}^2}\,, \qquad p_r = -\frac{\vec{x}\cdot\vec{p}}{\sqrt{R^2 + \vec{x}^2}} \qquad s = \sqrt{R^2 - \vec{y}^2}\,, \qquad q_s = -\frac{\vec{y}\cdot\vec{q}}{\sqrt{R^2 - \vec{y}^2}}\,. \tag{2.5}$$

In the following it will be useful to work in dimensionless quantities for the coordinates and momenta:

$$\tilde{x}_i = \frac{x_i}{R}\,, \qquad \tilde{p}_i = R\, p_i \qquad \tilde{y}_i = \frac{y_i}{R}\,, \qquad \tilde{q}_i = R\, q_i\,, \tag{2.6}$$

in terms of which the dimensionless energy squared takes the form

$$p_\theta^2 = (1+\tilde{\vec{x}}^2)\left[\tilde{\vec{p}}^2 + \tilde{\vec{q}}^2 + (MR)^2 - \frac{(\tilde{\vec{x}}\cdot\tilde{\vec{p}})^2}{1+\tilde{\vec{x}}^2} + \frac{(\tilde{\vec{y}}\cdot\tilde{\vec{q}})^2}{1-\tilde{\vec{y}}^2} + \frac{q_\phi^2}{1-\tilde{\vec{y}}^2}\right] . \tag{2.7}$$

We note the Dirac-brackets originating from the λ_1, λ_2, λ_3 and λ_4 constraints:

$$\{\theta, p_\theta\} = 1\,, \qquad \{\tilde{x}_i, \tilde{p}_j\}_D = \delta_{ij} + \tilde{x}_i\,\tilde{x}_j\,, \qquad \{\tilde{p}_i, \tilde{p}_j\}_D = \tilde{x}_i\,\tilde{p}_j - \tilde{x}_j\,\tilde{p}_i\,, \tag{2.8}$$

$$\{\phi, p_\phi\} = 1\,, \qquad \{\tilde{y}_i, \tilde{q}_j\}_D = \delta_{ij} - \tilde{y}_i\,\tilde{y}_j\,, \qquad \{\tilde{q}_i, \tilde{q}_j\}_D = -\tilde{y}_i\,\tilde{q}_j + \tilde{y}_j\,\tilde{q}_i\,. \tag{2.9}$$

These can be nicely mapped to a canonical system, which we shall in a slight abuse of notation again denote by (x_i, y_i, p_i, q_i) via

$$\tilde{x}_i = x_i\,, \qquad \tilde{p}_i = p_i + x_i\,(\vec{x}\cdot\vec{p}) \qquad \tilde{y}_i = y_i\,, \qquad \tilde{q}_i = q_i - y_i\,(\vec{y}\cdot\vec{q}), \tag{2.10}$$

with canonical brackets

$$\{x_i, p_j\} = \delta_{ij} \qquad \{y_i, q_j\} = \delta_{ij}. \tag{2.11}$$

Inserting the representation (2.10) into (2.7) yields the space-time energy

$$p_\theta^2 = (1+\vec{x}^2)\left((MR)^2 + \vec{p}^2 + (\vec{p}\cdot\vec{x})^2 + \vec{q}^2 - (\vec{q}\cdot\vec{y})^2 + \frac{q_\phi^2}{1-\vec{y}^2}\right) . \tag{2.12}$$

The S^5 factor of the wavefunction Ψ_J, in the coordinate system defined in (2.10) satisfies the following equation

$$\left(\vec{q}^2 - (\vec{q}\cdot\vec{y})^2 + \frac{q_\phi^2}{1-\vec{y}^2}\right)\Psi_J = J(J+4)\Psi_J \tag{2.13}$$

and this implies that the space-time energy can be written as

$$p_\theta^2 = (1+\vec{x}^2)\left((MR)^2 + \vec{p}^2 + (\vec{p}\cdot\vec{x})^2 + J(J+4)\right) . \tag{2.14}$$

Hence we see that the energy p_θ is bounded from below by MR.

The equation (2.14) shows that, as expected, the contribution of the S^5 excitations is implemented by a shift of the mass-squared: $M^2 \to M^2 + J(J+4)$.

To study the particle in the large radius approximation it is convenient to rescale the AdS variables in the following way

$$\vec{x} \to \frac{\vec{x}}{\sqrt{MR}} \qquad \vec{p} \to \sqrt{MR}\,\vec{p} \tag{2.15}$$

so that the squared energy can be expanded as

$$p_\theta^2 = (MR)^2 + (MR)\big(\vec{p}^2 + \vec{x}^2\big) + \vec{p}^2\,\vec{x}^2 + (\vec{p}\cdot\vec{x})^2 + J(J+4) + \mathcal{O}[(MR)^{-1}] \tag{2.16}$$

This gives

$$E = p_\theta = MR + \frac{1}{2}\big(\vec{p}^2 + \vec{x}^2\big) + \mathcal{O}[(MR)^{-1}] \tag{2.17}$$

In section 3 we will recover the $\mathcal{O}[(MR)^0]$ result above for the zero-modes of the bosonic string. Now, promoting the (2.11) to their quantum analogues, for the AdS coordinates we have

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \quad \hat{x}_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger), \quad \hat{p}_i = -\frac{i}{\sqrt{2}}(a_i - a_i^\dagger), \tag{2.18}$$

and therefore

$$\begin{aligned} E_n &= (MR) + a_i^\dagger a_i + \delta_{ii}/2 + \ldots \\ &= (MR) + \hat{n} + 2 + \ldots . \end{aligned} \tag{2.19}$$

which matches (2.2) at the leading two orders. We can go further and tackle the subleading terms in (2.16). Introducing constants d_i to capture ordering ambiguities originating from x-p self-contractions, we have

$$(p_i)^2\,(x_i)^2 + (p_i\,x_i)^2 \to (\hat{x}_i)^2\,(\hat{p}_i)^2 + (\hat{x}_i\,\hat{p}_i)^2 + i\,d_1\,(\hat{x}_i\,\hat{p}_i) + d_2\,\mathbf{1}\,. \tag{2.20}$$

Imposing the hermiticity of the operator $(\hat{p}_\theta)^2 = (\hat{p}_\theta^\dagger)^2$ fixes $d_1 = -6$. One finds

$$\begin{aligned} \hat{p}_\theta^2 &= (MR)^2 + MR\,\big(\,2\,\hat{n} + 4\,\big) \\ &\quad + \Big(\,12 + d_2 + 4\,\hat{n} + \hat{n}^2 - \frac{1}{2}\,[\,(a_i^\dagger\,a_i^\dagger)^2 + (a_i\,a_i)^2\,] + J(J+4)\Big) + \mathcal{O}[(MR)^{-1}] \end{aligned} \tag{2.21}$$

where $\hat{n} := a_i^\dagger\,a_i$. We first note that the non-diagonal term at order $\mathcal{O}[1]$ can be removed by a unitary transformation up to terms of order $\mathcal{O}[(MR)^{-1}]$:

$$\begin{aligned} \hat{p}_\theta^2 \to e^{\hat{V}/(MR)}\,\hat{p}_\theta^2\,e^{-\hat{V}/(MR)} &= (MR)^2 + MR\,\big(\,2\,\hat{n} + 4\,\big) \\ &\quad + \big(\,12 + d_2 + 4\,\hat{n} + \hat{n}^2 + J(J+4)\big) + \mathcal{O}[(MR)^{-1}]\,, \end{aligned} \tag{2.22}$$

with the anti-hermitian operator

$$\hat{V} = -\frac{1}{16}\,\Big[\,(a_i^\dagger\,a_i^\dagger)^2 - (a_i\,a_i)^2\,\Big]\,. \tag{2.23}$$

Thus we find the spectrum E_n of p_θ:

$$\begin{aligned} E_n^2 &= (MR)^2 + MR\,(2n+4) + 12 + d_2 + 4n + n^2 + J(J+4) + \mathcal{O}[(MR)^{-1}], \\ E_n &= MR + n + 2 + \frac{8 + d_2 + J(J+4)}{2MR} + \mathcal{O}[(MR)^{-2}]\ , \end{aligned}$$

which agrees with (2.2) for the choice $d_2 = -4$ of the normal ordering constant. Indeed this value of the ordering constant c_2 can be shown to be unambiguously determined by the closure of the $SO(2,4)$ quantum symmetry algebra of the $AdS_5 \times S^5$ particle problem.

3 Bosonic string on $AdS_5 \times S^5$

In the previous section we saw how a rescaling of the transverse particle coordinates and momenta

$$(x_i,\, y_i) \to \frac{1}{\sqrt{MR}}(x_i,\, y_i), \qquad (p_i,\, q_i) \to \sqrt{MR}\,(p_i,\, q_i) \tag{3.1}$$

led to a recovering of the quantum spectrum in the $MR \to \infty$ limit. In generalizing to the string we keep this scaling for the string zero-modes, while scaling the oscillating string modes with $\lambda^{1/4} \sim R/\sqrt{\alpha'}$. As described in the introduction, this is the scaling that produces a pertubatively accessible theory. What we will find is the flat-space string spectrum at leading order, and the particle spectrum found above at the first subleading order, where the mass of the particle is given by the flat space mass of the string.

Let us consider the coordinate system where the $AdS_5 \times S^5$ metric is

$$ds^2 = -\left(\frac{1+z^2/4}{1-z^2/4}\right)^2 dt^2 + \frac{d\vec{z}\cdot d\vec{z}}{(1-z^2/4)^2} + \left(\frac{1-y^2/4}{1+y^2/4}\right)^2 d\phi^2 + \frac{d\vec{y}\cdot d\vec{y}}{(1+y^2/4)^2}. \tag{3.2}$$

This can be obtained by a simple coordinate transformation of the coordinates in section 2

$$\vec{x} \to \frac{\vec{z}}{1-z^2/4} \quad , \quad \vec{y} \to \frac{\vec{y}}{1+y^2/4}. \tag{3.3}$$

We will take the convention where the action is

$$S = \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[P_T \dot{T} + P \cdot \dot{Z} + Q_\phi \dot{\phi} + Q \cdot \dot{Y} - \eta_1 \mathcal{S} - \eta_2 \mathcal{T}\right]. \tag{3.4}$$

The constraints which are enforced by the Lagrange multipliers η_i are

$$\mathcal{S} = 0 \quad , \quad \mathcal{T} = 0, \tag{3.5}$$

where

$$\mathcal{S} = P_T T' + Q_\phi \phi' + P \cdot Z' + \vec{Q} \cdot \vec{Y}', \tag{3.6}$$

$$\begin{aligned}\mathcal{T} = \tfrac{1}{\sqrt{\lambda}} &\left[-\left(\tfrac{1-\vec{Z}^2/4}{1+\vec{Z}^2/4}\right)^2 P_T^2 + (1-\vec{Z}^2/4)^2 \vec{P}^2 + \left(\tfrac{1+\vec{Y}^2/4}{1-\vec{Y}^2/4}\right)^2 Q_\phi^2 + (1+\vec{Y}^2/4)^2 \vec{Q}^2\right] \\ +\sqrt{\lambda} &\left[-\left(\tfrac{1+\vec{Z}^2/4}{1-\vec{Z}^2/4}\right)^2 T'^2 + \tfrac{(\vec{Z}')^2}{(1-\vec{Z}^2/4)^2} + \left(\tfrac{1-\vec{Y}^2/4}{1+\vec{Y}^2/4}\right)^2 (\phi')^2 + \tfrac{(\vec{Y}')^2}{(1+\vec{Y}^2/4)^2}\right].\end{aligned} \tag{3.7}$$

We now introduce light-cone coordinates using global time T and the azimuthal angle ϕ

$$X_- = \phi - T \quad , \quad X_+ = \frac{1}{2}(T+\phi). \tag{3.8}$$

$$P_- = Q_\phi + P_T \quad , \quad P_+ = \frac{1}{2}(Q_\phi - P_T). \tag{3.9}$$

$$T = X_+ - \frac{1}{2}X_- \quad , \quad \phi = X_+ + \frac{1}{2}X_-. \tag{3.10}$$

$$P_T = \frac{1}{2}P_- - P_+ \quad , \quad Q_\phi = \frac{1}{2}P_- + P_+. \tag{3.11}$$

Then, we impose the light-cone gauge conditions,

$$P_+ = p_+ \quad , \quad X_+ = x_+ + p_+\tau, \tag{3.12}$$

where x_+ and p_+ are (τ,σ)-independent. With these conditions, the constraints are

$$0 = p_+X'_- + P\cdot Z' + \vec{Q}\cdot\vec{Y}' \ \rightarrow \quad X'_- = -\frac{1}{p_+}\left(P\cdot Z' + \vec{Q}\cdot\vec{Y}'\right), \tag{3.13}$$

$$\begin{aligned} 0 =& \left[\left(\frac{1+\vec{Y}^2/4}{1-\vec{Y}^2/4}\right)^2 - \left(\frac{1-\vec{Z}^2/4}{1+\vec{Z}^2/4}\right)^2\right]\left(\frac{P_-^2}{4}+p_+^2\right) + \left[\left(\frac{1+\vec{Y}^2/4}{1-\vec{Y}^2/4}\right)^2 + \left(\frac{1-\vec{Z}^2/4}{1+\vec{Z}^2/4}\right)^2\right]P_-p_+ \\ &+ (1-Z^2/4)^2\vec{P}^2 + (1+\vec{Y}^2/4)^2\vec{Q}^2 + \lambda\left[\frac{(\vec{Z}')^2}{(1-\vec{Z}^2/4)^2} + \frac{(\vec{Y}')^2}{(1+\vec{Y}^2/4)^2}\right] \\ &+ \lambda\left[\left(\frac{1-\vec{Y}^2/4}{1+\vec{Y}^2/4}\right)^2 - \left(\frac{1+\vec{Z}^2/4}{1-\vec{Z}^2/4}\right)^2\right]\frac{1}{4p_+^2}\left(\vec{P}\cdot\vec{Z}' + \vec{Q}\cdot\vec{Y}'\right)^2, \end{aligned} \tag{3.14}$$

where, for convenience, the right-hand-side of (3.14) is $\mathcal{T}$ rescaled by a factor of $\sqrt{\lambda}$. Here, we have solved the constraint $\mathcal{S}=0$ as indicated and plugged the solution into the second constraint to obtain (3.14). In the following, we shall use lower case letters to denote the worldsheet averages of coordinates and momenta $x_\mu = \int_0^{2\pi}\frac{d\sigma}{2\pi}X_\mu(\sigma)$ and $p_\mu = \int_0^{2\pi}\frac{d\sigma}{2\pi}P_\mu(\sigma)$.

We must now solve (3.14) for the remaining variable $P_-(\sigma)$. This will be done perturbatively about the large λ limit, according to a scheme which we outlined in the introduction. Together with finding $P_-(\sigma)$, we will find expressions for the momenta p_+ and p_-.

We begin by recalling that p_+ and p_- are not independent, they are related by the second equation in (3.11),

$$q_\phi = \tfrac{1}{2}p_- + p_+ \,. \tag{3.15}$$

Here, q_ϕ is conjugate to the zero mode of the angle coordinate ϕ and its spectrum is integers. We will be interested in states where the magnitude of these integers is of order λ^0. We will see that, as a consequence, the leading terms in p_+ and p_- must be of order $\lambda^{1/4}$, and the asymptotic expansion of these quantities is generically in powers of $\lambda^{-1/8}$. From p_+ and p_-, we will deduce the spectrum of the squared string energy, p_T^2 where

$$p_T = \tfrac{1}{2}p_- - p_+ \quad , \quad p_T^2 = q_\phi^2 - 2p_+p_-, \tag{3.16}$$

where we have used Eq. (3.9).

To proceed, we will scale the fields as

$$\vec{Z}(\sigma,\tau) = \frac{1}{\lambda^{\frac{1}{8}}}\vec{z}(\tau) + \frac{1}{\lambda^{\frac{1}{4}}}\tilde{\vec{Z}}(\sigma,\tau) \quad , \quad \vec{P}(\sigma,\tau) = \lambda^{\frac{1}{8}}\vec{p}(\tau) + \lambda^{\frac{1}{4}}\tilde{\vec{P}}(\sigma,\tau)\,, \tag{3.17}$$

$$\vec{Y}(\sigma,\tau) = \frac{1}{\lambda^{\frac{1}{8}}}\vec{y}(\tau) + \frac{1}{\lambda^{\frac{1}{4}}}\tilde{\vec{Y}}(\sigma,\tau) \quad , \quad \vec{Q}(\sigma,\tau) = \lambda^{\frac{1}{8}}\vec{q}(\tau) + \lambda^{\frac{1}{4}}\tilde{\vec{Q}}(\sigma,\tau). \tag{3.18}$$

Here, we have separated the zero modes, $(\vec{p}, \vec{z}, \vec{q}, \vec{y})$ from the internal oscillations of the string which we denote as $(\tilde{\vec{P}}, \tilde{\vec{Z}}, \tilde{\vec{Q}}, \tilde{\vec{Y}})$ and which are constrained by

$$\int d\sigma \tilde{\vec{P}} = \int d\sigma \tilde{\vec{Z}} = \int d\sigma \tilde{\vec{Q}} = \int d\sigma \tilde{\vec{Y}} = 0.$$

The scaling of zero modes is consistent with that in Section 2. A priori this scaling suggest an expansion of the energy in powers of $\lambda^{-1/8}$.

We shall find that P_- and p_+ scale as $\lambda^{\frac{1}{4}}$ for large λ. Then, to the leading order in large λ, (3.14) becomes

$$-2P_-p_+ = \lambda^{\frac{1}{2}}\mathcal{M}^2(\sigma) + \ldots, \tag{3.19}$$

where the dots indicate terms of order less than $\lambda^{\frac{1}{2}}$ and the mass operator-density is

$$\mathcal{M}^2(\sigma) = \left[\tilde{\vec{P}}^2 + \tilde{\vec{Q}}^2 + (\tilde{\vec{Z}}')^2 + (\tilde{\vec{Y}}')^2\right]. \tag{3.20}$$

Then, remembering that $2p_+p_- = q_\phi^2 - p_T^2$, we get, after integrating (3.19) over σ,

$$p_T^2 = q_\phi^2 + \lambda^{\frac{1}{2}}M^2 + \mathcal{O}(\lambda^{\frac{1}{4}}), \tag{3.21}$$

where

$$M^2 \equiv \frac{1}{2\pi}\int_0^{2\pi} d\sigma \mathcal{M}^2(\sigma) = \frac{1}{2\pi}\int_0^{2\pi} d\sigma \left[\tilde{\vec{P}}^2 + \tilde{\vec{Q}}^2 + (\tilde{\vec{Z}}')^2 + (\tilde{\vec{Y}}')^2\right], \tag{3.22}$$

is the flat-space mass operator. For the states of interest to us, q_ϕ will be of order one. Also, we learn that

$$p_+ = p_+^{(0)}\lambda^{1/4} + p_+^{(2)}, \tag{3.23}$$

$$P_-(\sigma) = -\lambda^{\frac{1}{4}}\frac{\mathcal{M}^2(\sigma)}{2p_+^{(0)}} + \ldots, \tag{3.24}$$

$$p_- = -\lambda^{\frac{1}{4}}\frac{M^2}{2p_+^{(0)}} + \ldots. \tag{3.25}$$

where the three dots in each of the above formulae denote corrections of order at least $\lambda^{\frac{1}{8}}$. We leave the constants $p_+^{(0)}$ and $p_+^{(2)}$ of the p_+ expansion undetermined for the moment[4]. They will be shown to follow at each order in the $\lambda^{-1/8}$ expansion from a self-consistency analysis.

Now, we must solve the equation for P_- to the next order. There is an order $\lambda^{\frac{1}{8}}$ contribution which is arises by expanding to the next order. Because of the orthogonality conditions, $\int d\sigma \tilde{P}^i =$

[4]Note that a term of the form $\lambda^{\frac{1}{8}}p_+^{(1)}$ does not appear. This can be seen using equation (3.9) for $p_\pm$ and recalling that we are interested in states where q_ϕ is of order one.

0, $\int d\sigma \tilde{Q}^j = 0$, the result does not contribute to p_- or p_+, but it must be taken into account in $P_-(\sigma)$. In summary, so far, we have

$$P_-(\sigma) = -\lambda^{\frac{1}{4}} \frac{\mathcal{M}^2(\sigma)}{2p_+^{(0)}} - \lambda^{\frac{1}{8}} \left[\frac{1}{p_+^{(0)}} \left(\vec{p} \cdot \vec{\tilde{P}} + \vec{q} \cdot \vec{\tilde{Q}} \right) \right] + \dots, \tag{3.26}$$

$$p_- = -\lambda^{\frac{1}{4}} \frac{M^2}{2p_+^{(0)}} + \dots, \tag{3.27}$$

where the dots stand for terms of order λ^0 and higher. Now, we are ready to solve the next non-trivial order. For this, we expand (3.14) as

$$-2P_- p_+ = \lambda^{\frac{1}{2}} \mathcal{M}^2 + 2\lambda^{\frac{3}{8}} \left[\vec{p} \cdot \vec{\tilde{P}} + \vec{q} \cdot \vec{\tilde{Q}} \right] + \lambda^{\frac{1}{4}} \mathcal{H}_{1/4}(\sigma) + \dots \tag{3.28}$$

where we have introduced

$$\begin{aligned} \mathcal{H}_{1/4}(\sigma) =& \vec{p}^2 + \vec{q}^2 + \left(\vec{z}^2 + \vec{y}^2 \right) (p_+^{(0)})^2 - \frac{\vec{y}^2}{2} \left(\mathcal{M}_\alpha^2 + 2\vec{\tilde{Y}}'^2 \right) \\ &+ \frac{\vec{z}^2}{2} \left(\mathcal{M}_\beta^2 + 2\vec{\tilde{Z}}'^2 \right) + \frac{\vec{y}^2 + \vec{z}^2}{16(p_+^{(0)})^2} \left(\mathcal{M}^4 - 4\mathcal{C}^2 \right), \end{aligned} \tag{3.29}$$

with

$$\mathcal{M}_\alpha^2 \equiv \vec{\tilde{P}}^2 + \vec{\tilde{Z}}'^2, \quad \mathcal{M}_\beta^2 \equiv \vec{\tilde{Q}}^2 + \vec{\tilde{Y}}'^2, \quad \mathcal{M}^2 = \mathcal{M}_\alpha^2 + \mathcal{M}_\beta^2, \quad \mathcal{C} \equiv \vec{\tilde{P}} \cdot \vec{\tilde{Z}}' + \vec{\tilde{Q}} \cdot \vec{\tilde{Y}}'. \tag{3.30}$$

From this equation we learn that

$$\begin{aligned} P_-(\sigma) =& -\lambda^{\frac{1}{4}} \frac{\mathcal{M}^2}{2p_+^{(0)}} - \lambda^{\frac{1}{8}} \frac{1}{p_+^{(0)}} \left(\vec{p} \cdot \vec{\tilde{P}} + \vec{q} \cdot \vec{\tilde{Q}} \right) + p_+^{(2)} \frac{\mathcal{M}^2(\sigma)}{2(p_+^{(0)})^2} \\ &- \frac{1}{2p_+^{(0)}} \mathcal{H}_{1/4}(\sigma) + \dots, \end{aligned} \tag{3.31}$$

$$p_- = -\lambda^{\frac{1}{4}} \frac{M^2}{2p_+^{(0)}} + p_+^{(2)} \frac{M^2}{2(p_+^{(0)})^2} - \frac{1}{2p_+^{(0)}} H_{1/4} + \dots, \tag{3.32}$$

where

$$H_{1/4} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{H}_{1/4}(\sigma). \tag{3.33}$$

The three dots at the end of (3.31) represent terms of order at least $\lambda^{-\frac{1}{8}}$. The energy squared p_T^2 has a simpler expression and reads

$$p_T^2 = \lambda^{1/2} M^2 + \lambda^{1/4} H_{1/4} + \lambda^{1/8} H_{1/8} + \lambda^0 (H_0 + q_\phi^2) + \mathcal{O}(\lambda^{-1/8}) \tag{3.34}$$

The procedure which we are following here can be iterated to a systematic computation of the classical p_+, p_- and $P_-(\sigma)$ to any order. In the Appendix we work out the operators that are needed to compute p_T^2 to orders $\lambda^{1/8}$ and λ^0, $H_{1/8}$ and H_0, respectively.

3.1 Quantization and the string spectrum

The string coordinates and momenta obey the equal-time Poisson brackets

$$\left\{Z^i(\sigma,\tau), P^j(\sigma',\tau)\right\} = 2\pi\delta(\sigma-\sigma')\delta^{ij}, \quad \left\{Y'^i(\sigma,\tau), Q^{j'}(\sigma',\tau)\right\} = 2\pi\delta(\sigma-\sigma')\delta^{i'j'}. \tag{3.35}$$

We solve these and diagonalize the flat space mass operator M^2 with the oscillator expansion

$$\tilde{Z}^i(\sigma,\tau) = \frac{i}{\sqrt{2}}\sum_{n\neq 0}\left[\frac{\alpha_n^i(\tau)}{n}e^{-in\sigma} + \frac{\tilde{\alpha}_n^i(\tau)}{n}e^{in\sigma}\right], \tag{3.36}$$

$$\tilde{P}^i(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\neq 0}\left[\alpha_n^i(\tau)e^{-in\sigma} + \tilde{\alpha}_n^i(\tau)e^{in\sigma}\right], \tag{3.37}$$

$$\tilde{Y}^{i'}(\sigma,\tau) = \frac{i}{\sqrt{2}}\sum_{n\neq 0}\left[\frac{\beta_n^{i'}(\tau)}{n}e^{-in\sigma} + \frac{\tilde{\beta}_n^{i'}(\tau)}{n}e^{in\sigma}\right], \tag{3.38}$$

$$\tilde{Q}^{i'}(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\neq 0}\left[\beta_n^{i'}(\tau)e^{-in\sigma} + \tilde{\beta}_n^{i'}(\tau)e^{in\sigma}\right], \tag{3.39}$$

where the non-vanishing equal-time oscillator brackets are

$$\begin{gathered}\left\{z^i(\tau), p^j(\tau)\right\} = \delta^{ij}, \quad \left\{\alpha_m^i, \alpha_n^j\right\} = -im\delta_{m+n}\delta^{ij}, \quad \left\{\tilde{\alpha}_m^i, \tilde{\alpha}_n^j\right\} = -im\delta_{m+n}\delta^{ij}, \\ \left\{y^{i'}(\tau), q^{j'}(\tau)\right\} = \delta^{i'j'}, \quad \left\{\beta_m^{i'}, \beta_n^{j'}\right\} = -im\delta_{m+n}\delta^{i'j'}, \quad \left\{\tilde{\beta}_m^{i'}, \tilde{\beta}_n^{j'}\right\} = -im\delta_{m+n}\delta^{i'j'}.\end{gathered} \tag{3.40}$$

The Virasoro generators are defined in such a way to exclude any zero modes, i.e.

$$L_n \equiv \frac{1}{2}\sum_{\substack{m=-\infty \\ m\neq n,0}}^{\infty}\left(\vec{\alpha}_{n-m}\cdot\vec{\alpha}_m + \vec{\beta}_{n-m}\cdot\vec{\beta}_m\right). \tag{3.41}$$

With this convention,

$$M^2 = \sum_{n\neq 0}\left[\vec{\alpha}_{-n}\cdot\vec{\alpha}_n + \vec{\tilde{\alpha}}_{-n}\cdot\vec{\tilde{\alpha}}_n + \vec{\beta}_{-n}\cdot\vec{\beta}_n + \vec{\tilde{\beta}}_{-n}\cdot\vec{\tilde{\beta}}_n\right] = 2\left(L_0 + \tilde{L}_0\right). \tag{3.42}$$

Physical states are constrained by the level matching condition which is

$$\Phi = \sum_{n\neq 0}\left[\vec{\alpha}_{-n}\cdot\vec{\alpha}_n - \vec{\tilde{\alpha}}_{-n}\cdot\vec{\tilde{\alpha}}_n + \vec{\beta}_{-n}\cdot\vec{\beta}_n - \vec{\tilde{\beta}}_{-n}\cdot\vec{\tilde{\beta}}_n\right] = 2\left(L_0 - \tilde{L}_0\right) \sim 0. \tag{3.43}$$

In both the classical and the quantum theory, the expression (3.43) should vanish for physical states. Note that (3.43) is an exact expression that is independent of perturbation theory. It is obtained by plugging the oscillator expansion into the integral of the constraint $\mathcal{S} = 0$ over σ. In the quantum theory, level matching is imposed as a physical state condition where (3.43) annihilates physical states. One may check that $H_{1/4}, H_{1/8}$ and H_0 commute with the level matching condition constraint Φ as it should. In the quantum theory which we shall consider shortly, both L_0 and $\tilde{L}_0$ should be ambiguous up to a normal ordering constant. However, because of the discrete symmetry which interchanges these operators, it would be reasonable that the constant is the same for each operator and cancels in the difference $L_0 - \tilde{L}_0$.

We therefore have the perturbative structure of the squared space-time Hamiltonian p_T^2

$$p_T^2 = q_\phi^2 + \sqrt{\lambda} M^2 + \lambda^{1/4} H_{1/4} + \mathcal{O}(\lambda^{1/8}), \tag{3.44}$$

where, in terms of oscillators,

$$\begin{aligned} H_{1/4} =& \vec{p}^2 + \vec{q}^2 + (\vec{y}^2 + \vec{z}^2)(p_+^{(0)})^2 + \frac{(\vec{z}^2 - \vec{y}^2)}{2} M^2 \\ & - \vec{z}^2 \sum_{n \neq 0} \vec{\alpha}_n \cdot \vec{\tilde{\alpha}}_n + \vec{y}^2 \sum_{n \neq 0} \vec{\beta}_n \cdot \vec{\tilde{\beta}}_n + \frac{\vec{z}^2 + \vec{y}^2}{(p_+^{(0)})^2} \sum_n L_n \tilde{L}_n \,. \end{aligned} \tag{3.45}$$

Let us now quantize this system by promoting coordinates and modes to operators and replacing $\{.,,\} \to -i[.,]$. We note the standard commutators

$$\begin{aligned} &[\alpha_m^i, \alpha_n^j] = m\, \delta_{m+n}\, \delta^{ij}\,, \qquad [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\, \delta_{m+n}\, \delta^{ij}\,, \\ &[\beta_m^{i'}, \beta_n^{j'}] = m\, \delta_{m+n}\, \delta^{i'j'}\,, \qquad [\tilde{\beta}_m^{i'}, \tilde{\beta}_n^{j'}] = m\, \delta_{m+n}\, \delta^{i'j'}\,, \\ &[L_m, \alpha_n^i] = -n\, \alpha_{n+m}^i\,, \qquad [L_m, \beta_n^i] = -n\, \beta_{n+m}^i\,, \\ &[\tilde{L}_m, \tilde{\alpha}_n^i] = -n \tilde{\alpha}_{n+m}^i\,, \qquad [L_m, \tilde{\beta}_n^i] = -n\, \tilde{\beta}_{n+m}^i\,. \end{aligned} \tag{3.46}$$

It turns out that we can remove the last three terms in (3.45) through a unitary transformation[5],

$$\tilde{p}_T^2 := e^{i\hat{V}/\lambda^{1/4}}\, p_T^2\, e^{-i\hat{V}/\lambda^{1/4}} = p_T^2 + i\,\lambda^{1/4}\,[\hat{V}, M^2] + \mathcal{O}(\lambda^0)\,, \tag{3.47}$$

where $\tilde{p}_T^2$ and p_T^2 have identical spectrum. Choosing the Hermitian operator $\hat{V}$ to be

$$\hat{V} = -\frac{\vec{z}^2}{4} \sum_{n \neq 0} \frac{i}{n}\, \vec{\alpha}_n\, \vec{\tilde{\alpha}}_n + \frac{\vec{y}^2}{4} \sum_{n \neq 0} \frac{i}{n}\, \vec{\beta}_n\, \vec{\tilde{\beta}}_n + \frac{\vec{z}^2 + \vec{y}^2}{4(p_+^{(0)})^2} \sum_{n \neq 0} \frac{i}{n}\, L_n\, \tilde{L}_n\,, \tag{3.48}$$

one can rotate away all non-diagonal terms at order $\lambda^{1/4}$, as[6]

$$i\,[\hat{V}, M^2] = \vec{z}^2 \sum_{n \neq 0} \vec{\alpha}_n \cdot \vec{\tilde{\alpha}}_n - \vec{y}^2 \sum_{n \neq 0} \vec{\beta}_n \cdot \vec{\tilde{\beta}}_n - \frac{\vec{z}^2 + \vec{y}^2}{(p_+^{(0)})^2} \sum_{n \neq 0} L_n \tilde{L}_n\,. \tag{3.49}$$

[5] Since $\hat{V}$ commutes with the operator $L_0 - \tilde{L}_0$, it does not upset the level matching condition. It is easy to see that such a unitary transformation can be used to remove any monomial in operators whose integer world-sheet momentum labels do not add to zero. The unitary transformation does not remove the term proportional to $L_0 \tilde{L}_0$.

[6] The contribution of the unitary transformation at order λ^0 is evaluated in appendix A.3

We thus find

$$\begin{aligned}\tilde{p}_T^2 =& q_\phi^2 + \sqrt{\lambda}\, M^2 \\ &+ \lambda^{1/4}\left(\vec{p}^2 + \vec{q}^2 + (\vec{z}^2+\vec{y}^2)(p_+^{(0)})^2 + \frac{(\vec{z}^2-\vec{y}^2)}{2}M^2 + \frac{\vec{z}^2+\vec{y}^2}{(p_+^{(0)})^2}L_0\tilde{L}_0\right) + \mathcal{O}(\lambda^{1/8}).\end{aligned} \tag{3.50}$$

The operator $L_0\tilde{L}_0$ can be written as $L_0\tilde{L}_0 = \frac{1}{16}(M^4 - \Phi^2)$, where Φ is the level matching constraint. For physical states, obeying $\Phi|\text{phys}\rangle = 0$, we finally have

$$\begin{aligned}\tilde{p}_T^2 =& q_\phi^2 + \sqrt{\lambda}\, M^2 \\ &+ \lambda^{1/4}\left(\vec{p}^2 + \vec{q}^2 + \vec{z}^2\left(p_+^{(0)} + \frac{M^2}{4p_+^{(0)}}\right)^2 + \vec{y}^2\left(p_+^{(0)} - \frac{M^2}{4p_+^{(0)}}\right)^2\right) + \mathcal{O}(\lambda^{1/8})\,.\end{aligned} \tag{3.51}$$

This renders the Hamiltonian diagonal to this order and the spectrum can be written down. We are interested in states where q_ϕ is of order one. Therefore we must have that

$$q_\phi = p_- + 2p_+ = \mathcal{O}(1), \quad \text{and therefore} \quad p_+^{(0)} = \frac{M}{2}, \tag{3.52}$$

where we are considering M to be an eigenvalue. With this restriction, we find

$$\tilde{p}_T^2 = q_\phi^2 + \sqrt{\lambda}\, M^2 + \lambda^{1/4}\left(\vec{p}^2 + M^2\vec{z}^2 + \vec{q}^2\right) + \mathcal{O}(\lambda^{1/8})\,, \tag{3.53}$$

and identifying $(MR)^2 \to \lambda^{1/2}\, M^2$, we find that at $\mathcal{O}(\lambda^{1/4})$ we have recovered precisely the particle energy (2.16).

We work out the subleading terms $\lambda^{1/8}\, H_{1/8}$ and $\lambda^0\, H_0$ for $\tilde{p}_T^2$ of (3.53) in the appendix. There we also show that the $\lambda^{1/8}\, H_{1/8}$ term does not contribute to the spectrum of $\tilde{p}_T$ down to order λ^0. Unfortunately the in principle straightforward computation of correction to the spectrum (3.53) at first order perturbation theory $\langle\text{phys}|H_0|\text{phys}\rangle$, in the sense of (3.34), will depend on a large number of so far unfixed normal ordering constants.

4 Concluding remarks

In this paper we have outlined an approach to the quantization of the superstring in $AdS_5 \times S^5$ in the flat-space limit. The first step in taking the program further is to add the fermionic degrees of freedom. In principle this should be a straightforward application of the strategy employed here for the bosonic case. The reproduction of the superparticle spectrum along the lines of section 2 should inform the correct scaling of the fermionic fields, while the action itself is available for example from [27]. Another direction is to push the calculations performed here to higher orders; this will likely require knowledge of the fermionic terms in the higher-order Hamiltonian. The major stumbling block in going to higher orders is the proliferation of normal ordering constants. It would be nice to have a method of determining these, perhaps through comparison to known results for protected quantities or by matching against further numerical prediction for higher excited states from the TBA and Y-system approach. As mentioned in the introduction, the higher order Hamiltonian for which the bosonic contribution is provided in appendix A.2, will determine the $\mathcal{O}(\lambda^{-1/4})$ contribution to the energy E, once fermions have been added and normal ordering constants have been determined. Given the unitary transformation which has diagonalized the Hamiltonian at the previous order, only first order perturbation theory is needed to extract the answer. For bosonic external states, the contribution of fermionic terms is relegated to those bosonic terms they produce through self-contraction - i.e. normal ordering constant type terms. There may be other methods of determining these, perhaps through closure of the $PSU(2,2|4)$ quantum algebra.

Another puzzle is the interpretation of the zero-mode excitations. We have argued here that we should take the zero-modes in their ground-state in order to describe the length-four Konishi multiplet state. The result for the energy E at $\mathcal{O}(\lambda^0)$ is determined by the zero-mode Hamiltonian. By exciting the zero-modes above the ground state, one obtains non-zero results at $\mathcal{O}(\lambda^0)$. The dual gauge theory interpretation of these states is still wanting. Given our interpretation, we may compare our results at $\mathcal{O}(\lambda^0)$ to those of [23] where it is argued that the $\mathcal{O}(\lambda^0)$ term is $\Delta_0 - 4$, where Δ_0 is the bare dimension of the gauge theory operator. For the length-four Konishi multiplet state one has $\Delta_0 = 4$, and so the absence of a term at $\mathcal{O}(\lambda^0)$ argued in the introduction of this paper is consistent with the results of [23]. It may be that zero-mode excitations should be understood as the string-duals of the longer members of the Konishi multiplet, for which $\Delta_0 - 4 \geq 0$. It would be interesting to determine whether this is indeed the case. An obvious problem with this interpretation is the unboundedness of the number of zero-mode oscillator excitations.

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A Higher order terms

Considering the expansion of p_T^2 out to the λ^0 order, it results

$$p_T^2 = q_\phi^2 + \sqrt{\lambda} M^2 + \lambda^{1/4} H_{1/4} + \lambda^{1/8} H_{1/8} + \lambda^0 H_0 + \dots \tag{A.1}$$

where $H_{1/8}$ and H_0 can be computed following the same strategy described in section 3. Applying the unitary transformation (3.47) that diagonalizes the $\lambda^{1/4}$ term, we obtain

$$\begin{aligned}\tilde{p}_T^2 =& q_\phi^2 + \sqrt{\lambda}\, M^2 + \lambda^{1/4} \left(\vec{p}^2 + M^2 \vec{z}^2 + \vec{q}^2\right) + \lambda^{1/8} H_{1/8} \\ &+ \lambda^0 \left(H_0 + i[\hat{V}, H_{1/4}] - \frac{1}{2}[\hat{V}, [\hat{V}, M^2]] \right) + \dots \end{aligned}\tag{A.2}$$

In the remainder of this appendix we spell out explicitly $H_{1/8}$ and H_0 and the λ^0 contribution of the unitary transformation, i.e. $i[\hat{V}, H_{1/4}] - \frac{1}{2}[\hat{V}, [\hat{V}, M^2]]$.

A.1 $\lambda^{1/8}$ $H_{1/8}$ term

The $H_{1/8}$ operator is given by

$$H_{1/8} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\frac{\vec{z}^2 + \vec{y}^2}{4(p_+^{(0)})^2} \left[\mathcal{M}^2 (\vec{p} \cdot \vec{\tilde{P}} + \vec{q} \cdot \vec{\tilde{Q}}\,) - 2\,\mathcal{C}\,(\vec{p} \cdot \vec{\tilde{Z}}' + \vec{q} \cdot \vec{\tilde{Y}}') \right] + \vec{z} \cdot \vec{\tilde{Z}}\,(\mathcal{M}_\beta^2 + 2\vec{\tilde{Z}}'^2\,) \right.$$
$$\left. - \vec{y} \cdot \vec{\tilde{Y}}\,(\mathcal{M}_\alpha^2 + 2\vec{\tilde{Y}}'^2\,) + \frac{1}{8(p_+^{(0)})^2} (\vec{y} \cdot \vec{\tilde{Y}} + \vec{z} \cdot \vec{\tilde{Z}}) \left[\left(\mathcal{M}^2\right)^2 - 4\mathcal{C}^2 \right] \right]. \tag{A.3}$$

Every term in $H_{1/8}$ has an odd number of oscillators. It therefore maps a state with an even number of oscillators onto a state with an odd number of oscillators. For this reason, the operator $H_{1/8}$ has vanishing matrix elements between all of the states of interest, that are two-oscillator states of the form $\beta^i_{-1}\tilde{\beta}^j_{-1}|0\rangle$. Therefore, the first order perturbation theory correction due to $H_{1/8}$, which is of order $\lambda^{\frac{1}{8}}$, vanishes.

The leading contribution due to $H_{1/8}$ is therefore in second order perturbation theory. Given that the energy denominators in second order perturbation theory are always of order $\lambda^{\frac{1}{2}}$,[7] it results that this contribution is of order $\left((\lambda)^{\frac{1}{8}}\right)^2 \cdot \frac{1}{\lambda^{\frac{1}{2}}} \sim \lambda^{-\frac{1}{4}}$, which is suppressed compared with the λ^0 terms that are due to first order perturbation theory of H_0 and the unitary transformation term.

A.2 λ^0 H_0 term

Let us now work out the H_0 contributions. For convenience we split them into four parts according to the order of oscillatory modes, i.e.

$$H_0 = \int_0^{2\pi} \frac{d\sigma}{2\pi}\,\mathcal{H}_0 = \int_0^{2\pi} \frac{d\sigma}{2\pi}\,(\mathcal{H}_{0,0} + \mathcal{H}_{0,2} + \mathcal{H}_{0,4} + \mathcal{H}_{0,6}) = H_{0,0} + H_{0,2} + H_{0,4} + H_{0,6}\,. \tag{A.4}$$

Order zero in oscillatory modes:

$$\mathcal{H}_{0,0} = H_{0,0} = \frac{1}{2}\Big[\,(\vec{q} \cdot \vec{q} + 4\,p_+^{(2)} p_+^{(0)})\,\vec{z} \cdot \vec{z} + (4\,p_+^{(2)} p_+^{(0)} - \vec{p} \cdot \vec{p})\,\vec{y} \cdot \vec{y}\,\Big]. \tag{A.5}$$

[7]They are of order $\lambda^{\frac{1}{2}}$ because the difference in the level number between the states of interest and the states that are created from them by operating with $H_{1/8}$, is non-zero. In fact, it can be shown that any term with an odd number of oscillators which respects level matching, when operating on $\beta_{-1}\tilde{\beta}_{-1}|0\rangle$ creates states whose level numbers differ by at least one from the level number of $\beta_{-1}\tilde{\beta}_{-1}|0\rangle$.

Order two in oscillatory modes:

$$\mathcal{H}_{0,2} = \frac{\vec{z}^2+\vec{y}^2}{4(p_+^{(0)})^2}\Big[(\vec{p}\cdot\vec{\tilde{P}}+\vec{q}\cdot\vec{\tilde{Q}})^2-(\vec{p}\cdot\vec{\tilde{Z}}'+\vec{q}\cdot\vec{\tilde{Y}}')^2\Big]+\frac{(\vec{z}^2+\vec{y}^2)\,(\vec{p}^2+\vec{q}^2)}{8(p_+^{(0)})^2}\,\mathcal{M}^2$$
$$+(p_+^{(0)})^2\,(\vec{\tilde{Y}}^2+\vec{\tilde{Z}}^2)+2\,(\vec{q}\cdot\vec{\tilde{Q}}\,\vec{z}\cdot\vec{\tilde{Z}}-\vec{p}\cdot\vec{\tilde{P}}\,\vec{y}\cdot\vec{\tilde{Y}})-\frac{1}{2}\,\vec{y}^2\,\vec{z}^2\,(\vec{\tilde{Y}}'^2+\vec{\tilde{Z}}'^2)$$
$$\frac{(\vec{z}^2)^2}{8}\Big[-\frac{1}{2}\,\vec{\tilde{P}}^2+\vec{\tilde{Q}}^2+\vec{\tilde{Y}}'^2+\frac{9}{2}\,\vec{\tilde{Z}}'^2\Big]+\frac{(\vec{y}^2)^2}{8}\Big[-\frac{1}{2}\,\vec{\tilde{Q}}^2+\vec{\tilde{P}}^2+\vec{\tilde{Z}}'^2+\frac{9}{2}\,\vec{\tilde{Y}}'^2\Big]. \qquad (A.6)$$

Order four in oscillatory modes:

$$\mathcal{H}_{0,4} = \frac{1}{2(p_+^{(0)})^2}\Big[\mathcal{M}^2\,(\vec{p}\cdot\vec{\tilde{P}}+\vec{q}\cdot\vec{\tilde{Q}})-2\,\mathcal{C}\,(\vec{p}\cdot\vec{\tilde{Z}}'+\vec{q}\cdot\vec{\tilde{Y}}')\Big]\,(\vec{y}\cdot\vec{\tilde{Y}}+\vec{z}\cdot\vec{\tilde{Z}})$$
$$+\frac{1}{2}\Big[\vec{\tilde{Z}}^2\,(\mathcal{M}_\beta^2+2\vec{\tilde{Z}}'^2)-\vec{\tilde{Y}}^2\,(\mathcal{M}_\alpha^2+2\vec{\tilde{Y}}'^2)\Big]-(\vec{z}^2+\vec{y}^2)\,\frac{p_+^{(2)}}{8(p_+^{(0)})^3}\Big[(\mathcal{M}^2)^2-4\mathcal{C}^2\Big]$$
$$+\frac{(\vec{z}^2)^2}{4(p_+^{(0)})^2}\Big[\frac{1}{4}\,\mathcal{M}^2\,(\mathcal{M}_\beta^2+2\,\vec{\tilde{Z}}'^2)-\mathcal{C}^2\Big]-\frac{(\vec{y}^2)^2}{4(p_+^{(0)})^2}\Big[\frac{1}{4}\,\mathcal{M}^2\,(\mathcal{M}_\alpha^2+2\,\vec{\tilde{Y}}'^2)-\mathcal{C}^2\Big]$$
$$+\vec{z}^2\vec{y}^2\,\frac{\mathcal{M}^2}{16(p_+^{(0)})^2}\,(\mathcal{M}_\beta^2+2\vec{\tilde{Z}}'^2-\mathcal{M}_\alpha^2-2\,\vec{\tilde{Y}}'^2). \qquad (A.7)$$

Order six in oscillatory modes:

$$\mathcal{H}_{0,6} = \frac{1}{128(p_+^{(0)})^4}((\mathcal{M}^2)^2-4\mathcal{C}^2)\Big[8(p_+^{(0)})^2\,(\vec{\tilde{Y}}\cdot\vec{\tilde{Y}}+\vec{\tilde{Z}}\cdot\vec{\tilde{Z}})+\mathcal{M}^2\,(\vec{y}^2+\vec{z}^2)^2\Big]. \qquad (A.8)$$

A.2.1 Expressions in terms of oscillators

The parts of H_0 can be expressed in terms of oscillators.

Order two in oscillators:

$$
\begin{aligned}
H_{0,2} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{H}_{0,2} = \\
& \frac{\vec{z}^2 + \vec{y}^2}{2(p_+^{(0)})^2} \sum_{n \neq 0} \Big(\vec{p} \cdot \alpha_n \, \vec{p} \cdot \tilde{\alpha}_n + \vec{q} \cdot \beta_n \, \vec{q} \cdot \tilde{\beta}_n + \vec{q} \cdot \beta_n \, \vec{p} \cdot \tilde{\alpha}_n + \vec{p} \cdot \alpha_n \, \vec{q} \cdot \tilde{\beta}_n \Big) \\
& + \frac{1}{8(p_+^{(0)})^2} \, (\vec{z}^2 + \vec{y}^2) \, (\vec{p}^2 + \vec{q}^2) \, M^2 \\
& + \frac{1}{2} \, (p_+^{(0)})^2 \sum_{n \neq 0} \frac{1}{n^2} \Big(\alpha_n \cdot \alpha_{-n} + \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} - 2\alpha_n \cdot \tilde{\alpha}_n + \beta_n \cdot \beta_{-n} + \tilde{\beta}_n \cdot \tilde{\beta}_{-n} - 2\beta_n \cdot \tilde{\beta}_n \Big) \\
& + i \sum_{n \neq 0} \frac{1}{n} \Big(-\vec{q} \cdot \beta_n \, \vec{z} \cdot \alpha_{-n} + \vec{q} \cdot \beta_n \, \vec{z} \cdot \tilde{\alpha}_n + \vec{q} \cdot \tilde{\beta}_n \, \vec{z} \cdot \alpha_n - \vec{q} \cdot \tilde{\beta}_n \, \vec{z} \cdot \tilde{\alpha}_{-n} \\
& \qquad + \vec{p} \cdot \alpha_n \, \vec{y} \cdot \beta_{-n} - \vec{p} \cdot \alpha_n \, \vec{y} \cdot \tilde{\beta}_n - \vec{p} \cdot \tilde{\alpha}_n \, \vec{y} \cdot \beta_n + \vec{p} \cdot \tilde{\alpha}_n \, \vec{y} \cdot \tilde{\beta}_{-n} \Big) \\
& + \frac{(\vec{z}^2)^2 + (\vec{y}^2)^2}{8} M^2 - \frac{3}{8} (\vec{z}^2)^2 \sum_{n \neq 0} \alpha_n \cdot \tilde{\alpha}_n - \frac{3}{8} (\vec{y}^2)^2 \sum_{n \neq 0} \beta_n \cdot \tilde{\beta}_n \\
& + \frac{1}{4} \vec{z}^2 \left(\frac{1}{2} \vec{z}^2 - \vec{y}^2 \right) \sum_{n \neq 0} (\alpha_n \cdot \alpha_{-n} + \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} - 2\alpha_n \cdot \tilde{\alpha}_n) \\
& + \frac{1}{4} \vec{y}^2 \left(\frac{1}{2} \vec{y}^2 - \vec{z}^2 \right) \sum_{n \neq 0} \Big(\beta_n \cdot \beta_{-n} + \tilde{\beta}_n \cdot \tilde{\beta}_{-n} - 2\beta_n \cdot \tilde{\beta}_n \Big) \, .
\end{aligned}
\tag{A.9}
$$

Order six in oscillators:

$$
\begin{aligned}
H_{0,6} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{H}_{0,6} = \\
& - \frac{1}{2(p_+^{(0)})^2} \sum_{\substack{n+m+p+q=0 \\ p,q \neq 0}} \frac{1}{pq} \left(\alpha_p \cdot \alpha_q + \tilde{\alpha}_{-p} \cdot \tilde{\alpha}_{-q} - 2\alpha_p \cdot \tilde{\alpha}_{-q} \right) L_n \tilde{L}_{-m} \\
& - \frac{1}{2(p_+^{(0)})^2} \sum_{\substack{n+m+p+q=0 \\ p,q \neq 0}} \frac{1}{pq} \left(\beta_p \cdot \beta_q + \tilde{\beta}_{-p} \cdot \tilde{\beta}_{-q} - 2\beta_p \cdot \tilde{\beta}_{-q} \right) L_n \tilde{L}_{-m} \\
& + \frac{(\vec{z}^2 + \vec{y}^2)^2}{4(p_+^{(0)})^4} \sum_{n+m+p=0} L_n \tilde{L}_{-m} \left(L_p + \tilde{L}_{-p} \right) .
\end{aligned}
\tag{A.10}
$$

Order four in oscillators:

$$
\begin{aligned}
H_{0,4} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{H}_{0,2} = \\
& \frac{1}{(p_+^{(0)})^2} \sum_{\substack{n+m+p=0 \\ m,p\neq 0}} \frac{i}{p} \Big[L_n(\vec{p}\cdot\tilde{\alpha}_{-m} + \vec{q}\cdot\tilde{\beta}_{-m}) + \tilde{L}_{-n}(\vec{p}\cdot\alpha_m + \vec{q}\cdot\beta_m) \Big] \\
& \qquad\qquad\qquad \times \Big(\vec{z}\cdot\alpha_p - \vec{z}\cdot\tilde{\alpha}_{-p} + \vec{y}\cdot\beta_p - \vec{y}\cdot\tilde{\beta}_{-p} \Big) \\
& - \frac{1}{2} \sum_{\substack{n+m+p=0 \\ m,p\neq 0}} \frac{1}{mp} \left(\alpha_p\cdot\alpha_m + \tilde{\alpha}_{-p}\cdot\tilde{\alpha}_{-m} - 2\alpha_p\cdot\tilde{\alpha}_{-m} \right) \Big(L_n + \tilde{L}_{-n} \Big) \\
& + \frac{1}{2} \sum_{\substack{n+m+p+q=0 \\ n,m,p,q\neq 0}} \frac{1}{mp} \left(\alpha_p\cdot\alpha_m + \tilde{\alpha}_{-p}\cdot\tilde{\alpha}_{-m} - 2\alpha_p\cdot\tilde{\alpha}_{-m} \right) \alpha_n\cdot\tilde{\alpha}_{-q} \\
& + \frac{1}{2} \sum_{\substack{n+m+p=0 \\ m,p\neq 0}} \frac{1}{mp} \Big(\beta_p\cdot\beta_m + \tilde{\beta}_{-p}\cdot\tilde{\beta}_{-m} - 2\beta_p\cdot\tilde{\beta}_{-m} \Big) \Big(L_n + \tilde{L}_{-n} \Big) \\
& - \frac{1}{2} \sum_{\substack{n+m+p+q=0 \\ n,m,p,q\neq 0}} \frac{1}{mp} \Big(\beta_p\cdot\beta_m + \tilde{\beta}_{-p}\cdot\tilde{\beta}_{-m} - 2\beta_p\cdot\tilde{\beta}_{-m} \Big) \beta_n\cdot\tilde{\beta}_{-q} \\
& - 2p_+^{(2)} \frac{\vec{z}^2 + \vec{y}^2}{(p_+^{(0)})^3} \sum_n L_n\tilde{L}_n \\
& + \frac{(\vec{z}^2)^2 - (\vec{y}^2)^2}{(p_+^{(0)})^2} \sum_n L_n\tilde{L}_n - \frac{(\vec{z}^2)^2}{4(p_+^{(0)})^2} \sum_{\substack{n+m+p=0 \\ m,p\neq 0}} \Big(L_n + \tilde{L}_{-n} \Big) \alpha_m\cdot\tilde{\alpha}_{-p} \\
& + \frac{(\vec{y}^2)^2}{4(p_+^{(0)})^2} \sum_{\substack{n+m+p=0 \\ m,p\neq 0}} \Big(L_n + \tilde{L}_{-n} \Big) \beta_m\cdot\tilde{\beta}_{-p} \\
& - \frac{\vec{z}^2\vec{y}^2}{4(p_+^{(0)})^2} \sum_{\substack{n+m+p=0 \\ m,p\neq 0}} \Big(L_n + \tilde{L}_{-n} \Big) \Big(\alpha_m\cdot\tilde{\alpha}_{-p} - \beta_m\cdot\tilde{\beta}_{-p} \Big),
\end{aligned}
\tag{A.11}
$$

A.3 $\mathcal{O}(\lambda^0)$ contribution of the unitary transformation

We now compute the λ^0 order contribution of the unitary transformation (3.47) given by the term

$$
i[\hat{V}, H_{1/4}] - \frac{1}{2}[\hat{V}, [\hat{V}, M^2]]. \tag{A.12}
$$

Note that as there are no ordering ambiguities in the M^2 and $H_{1/4}$ operators, this $\mathcal{O}(\lambda^0)$ contribution from the unitary transformation does not suffer from ordering ambiguities.

The relevant commutators are

$$
\begin{aligned}
&[\vec{z}^2, \vec{p}^2] = 4i\vec{z}\cdot\vec{p} + 8, \quad [\vec{y}^2, \vec{q}^2] = 4i\vec{y}\cdot\vec{q} + 8, \\
&[\alpha_n\cdot\tilde{\alpha}_n, :L_0: + :\tilde{L}_0:] = 2\,n\,\alpha_n\cdot\tilde{\alpha}_n, \quad [\beta_n\cdot\tilde{\beta}_n, :L_0: + :\tilde{L}_0:] = 2\,n\,\beta_n\cdot\tilde{\beta}_n, \\
&[\alpha_n\cdot\tilde{\alpha}_n, L_m\tilde{L}_m] = n\,\alpha_{m+n}\cdot\tilde{\alpha}_n\tilde{L}_m + n\,L_m\,\tilde{\alpha}_{m+n}\cdot\alpha_n, \\
&[\beta_n\cdot\tilde{\beta}_n, L_m\tilde{L}_m] = n\,\beta_{m+n}\cdot\tilde{\beta}_n\tilde{L}_m + n\,L_m\,\tilde{\beta}_{m+n}\cdot\beta_n, \\
&[L_m\tilde{L}_m, :L_0: + :\tilde{L}_0:] = 2m\,L_m\tilde{L}_m, \\
&\left[\sum_{n\neq 0}\frac{i}{n}\alpha_n\cdot\tilde{\alpha}_n, \sum_{m\neq 0}\alpha_m\cdot\tilde{\alpha}_m\right] = i\sum_{n\neq 0}\left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_n\cdot\tilde{\alpha}_{-n}\right), \\
&\left[\sum_{n\neq 0}\frac{i}{n}\beta_n\cdot\tilde{\beta}_n, \sum_{m\neq 0}\beta_m\cdot\tilde{\beta}_m\right] = i\sum_{n\neq 0}\left(\beta_{-n}\cdot\beta_n + \tilde{\beta}_n\cdot\tilde{\beta}_{-n}\right), \\
&[L_m\tilde{L}_m, L_n\tilde{L}_n] = (m-n)\,\left(\tilde{L}_n\tilde{L}_m L_{m+n} + L_m L_n\tilde{L}_{m+n}\right) + \frac{c_V}{12}(m^3-m)(L_m L_n + \tilde{L}_n\tilde{L}_m)\delta_{m+n},
\end{aligned}
\tag{A.13}
$$

where we used

$$
\begin{aligned}
[L_m, L_n] &= (m-n)\,L_{m+n} + \frac{c_V}{12}(m^3-m)\delta_{m+n} \\
[\tilde{L}_m, \tilde{L}_n] &= (m-n)\,\tilde{L}_{m+n} + \frac{c_V}{12}(m^3-m)\delta_{m+n}
\end{aligned}
\tag{A.14}
$$

We find that

$$
\begin{aligned}
i[\hat{V}, H_{1/4}] - \frac{1}{2}[\hat{V}, [\hat{V}, M^2]] = \\
& (2 + i\vec{z}\cdot\vec{p})\sum_{n\neq 0}\frac{1}{n}\alpha_n\cdot\tilde{\alpha}_n - (2 + i\vec{y}\cdot\vec{q})\sum_{n\neq 0}\frac{1}{n}\beta_n\cdot\tilde{\beta}_n \\
& - \big(i(\vec{z}\cdot\vec{p} + \vec{y}\cdot\vec{q}) + 4\big)\,\frac{1}{(p_+^{(0)})^2}\sum_{n\neq 0}\frac{1}{n}L_n\tilde{L}_n \\
& + \frac{\vec{z}^2(\vec{z}^2 - \vec{y}^2)}{2}\sum_{n\neq 0}\alpha_n\cdot\tilde{\alpha}_n - \frac{\vec{y}^2(\vec{z}^2 - \vec{y}^2)}{2}\sum_{n\neq 0}\beta_n\cdot\tilde{\beta}_n - \frac{(\vec{z}^2)^2 - (\vec{y}^2)^2}{2(p_+^{(0)})^2}\sum_{n\neq 0}L_n\tilde{L}_n \\
& - \frac{(\vec{z}^2)^2}{8}\sum_{n\neq 0}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_n\cdot\tilde{\alpha}_{-n}) - \frac{(\vec{y}^2)^2}{8}\sum_{n\neq 0}\Big(\beta_{-n}\cdot\beta_n + \tilde{\beta}_n\cdot\tilde{\beta}_{-n}\Big) \\
& + \frac{\vec{z}^2(\vec{z}^2 + \vec{y}^2)}{8(p_+^{(0)})^2}\sum_{n\neq 0,\, m\neq 0}\Big(1 - \frac{n}{m}\Big)\Big(\alpha_{m+n}\cdot\tilde{\alpha}_n\,\tilde{L}_m + L_m\tilde{\alpha}_{m+n}\cdot\alpha_n\Big) \\
& - \frac{\vec{y}^2(\vec{z}^2 + \vec{y}^2)}{8(p_+^{(0)})^2}\sum_{n\neq 0,\, m\neq 0}\Big(1 - \frac{n}{m}\Big)\Big(\beta_{m+n}\cdot\tilde{\beta}_n\,\tilde{L}_m + L_m\tilde{\beta}_{m+n}\cdot\beta_n\Big) \qquad \text{(A.15)} \\
& - \frac{(\vec{z}^2 + \vec{y}^2)^2}{8(p_+^{(0)})^4}\sum_{n\neq 0,\, m\neq 0}\Big(1 - \frac{n}{m}\Big)\Big(\tilde{L}_n\tilde{L}_m L_{m+n} + L_m L_n\tilde{L}_{m+n}\Big) \\
& - \frac{c_V(\vec{z}^2 + \vec{y}^2)^2}{96(p_+^{(0)})^4}\sum_{n\neq 0}(n^2 - 1)(L_n L_{-n} + \tilde{L}_{-n}\tilde{L}_n) \\
& + \frac{\vec{z}^2(\vec{z}^2 + \vec{y}^2)}{4(p_+^{(0)})^2}\sum_{n\neq 0}((:L_0: + c_L)\alpha_n\cdot\tilde{\alpha}_n + \alpha_n\cdot\tilde{\alpha}_n(:\tilde{L}_0: + \tilde{c}_L)) \\
& - \frac{\vec{y}^2(\vec{z}^2 + \vec{y}^2)}{4(p_+^{(0)})^2}\sum_{n\neq 0}((:L_0: + c_L)\beta_n\cdot\tilde{\beta}_n + \beta_n\cdot\tilde{\beta}_n(:\tilde{L}_0: + \tilde{c}_L)) \\
& - \frac{(\vec{z}^2 + \vec{y}^2)^2}{4(p_+^{(0)})^4}\sum_{n\neq 0}((:L_0: + c_L)L_n\cdot\tilde{L}_n + L_n\cdot\tilde{L}_n(:\tilde{L}_0: + \tilde{c}_L))
\end{aligned}
$$

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