Convolution operators and the discrete Laplacian

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A statement concerning conjoint work

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In this thesis, the results in Theorem 2.2.7, Proposition 3.2.2 and Example 3.2.5 are due to my supervisor.

Signature of Author

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Abstract

In this thesis, we obtain new results for convolution operators on homogeneous spaces and give applications to the Laplacian on a homogeneous graph. Some of these results have been published in joint papers [13, 14] with my supervisor.

Let Ω be a homogeneous space of a locally compact group G and let $T_\sigma : L^p(\Omega) \to L^p(\Omega)$ be a convolution operator induced by a measure σ on G, where $1 \leq p < \infty$. When σ is symmetric and absolutely continuous, we describe the L^2-spectrum of T_σ in terms of the Fourier transform of σ. An operator T is said to be hypercyclic if there is a vector $x \in L^p(\Omega)$ such that the orbit $\{x, Tx, \ldots, T^n x, \ldots\}$ is dense in $L^p(\Omega)$. Given a positive weight w on Ω, we consider the weighted convolution operator $T_{\sigma,w}(f) = wT_\sigma(f)$ on $L^p(\Omega)$ and study hypercyclic properties of $T_{\sigma,w}$. For a unit point mass σ, we show that $T_{\sigma,w}$ is hypercyclic under some condition on the weight w. This condition is also necessary in the discrete case, and is equivalent to hereditary hypercyclicity of the operator. The condition can be strengthened to characterise topologically mixing weighted translation operators on discrete spaces.

A weighted homogeneous graph is a homogeneous space Ω of a discrete group G and the Laplacian $\mathcal{L}$ on Ω can be viewed as a convolution operator. We can therefore apply the above result on L^2-spectrum to describe the spectrum of $\mathcal{L}$ in terms of irreducible representations of G. We compare the eigenvalues of $\mathcal{L}$ with eigenvalues of the Laplacian on a regular tree, and obtain a Dirichlet eigenvalue comparison theorem. We also prove a version of the Harnack inequality for a Schrödinger operator on an invariant homogeneous graph.

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Chapter 1

Introduction

Let G be a locally compact group and $1 \leq p < \infty$. Given a compact subgroup H of G, there is a G-invariant measure ν on the homogeneous space G/H and one can form the Lebesgue space $L^p(G/H)$ with respect to ν. In this thesis, we study convolution operators on the $L^p(G/H)$ spaces and their applications.

Convolution operators play an important role in harmonic analysis which, according to [31], is a study of unitary representations of locally compact groups, and the analysis of functions on such groups and their homogeneous spaces. We introduce convolution operators as well as some basic results and notation to begin Chapter 2. Let σ be a complex Borel measure on a locally compact group G. The convolution operator $T_\sigma : L^p(G/H) \to L^p(G/H)$ is defined by

$$T_\sigma(f) = f * \sigma \qquad (f \in L^p(G/H))$$

where the convolution

$$f * \sigma(Hx) = \int_G f(Hxy^{-1}) d\sigma(y)$$

exists ν-almost everywhere and is defined to be 0 otherwise. The main result in Chapter 2 is Theorem 2.2.7 where the L^2-spectrum of T_σ is completely determined in terms of irreducible representations of G, when σ is symmetric and absolutely continuous. Let $\widehat{G}$ be the dual space of G and let $\widehat{\sigma}$ be the Fourier transform of σ defined by

$$\widehat{\sigma}(\pi) = \int_G \pi(x^{-1}) d\sigma(x) \qquad (\pi \in \widehat{G}, x \in G).$$

Then the L^2-spectrum of T_σ is given by

$$\mathrm{Spec}(T_\sigma) \cup \{0\} = \bigcup \{\mathrm{Spec}(\widehat{\sigma}(\pi)) : \pi \in \widehat{G}_r, \ker \pi \supset \ker \rho_{_H}\} \cup \{0\}$$

where $\widehat{G}_r$ is the reduced dual and $\rho_{_H}$ is the regular representation induced by H. In particular, $\mathrm{Spec}(T_\sigma) \cup \{0\} = \bigcup \{\mathrm{Spec}(\widehat{\sigma}(\pi)) : \pi \in \widehat{G}_r\} \cup \{0\}$ if the compact subgroup H of G is the identity group $\{e\}$. If G is discrete, then $\mathrm{Spec}(T_\sigma) = \bigcup \{\mathrm{Spec}(\widehat{\sigma}(\pi)) : \pi \in \widehat{G}_r, \ker \pi \supset \ker \rho_{_H}\}$.

In Chapter 3, we study the question of hypercyclicity of convolution operators. Let X be a Banach space. An operator $T : X \to X$ is said to be *hypercyclic* if there is a vector $x \in X$ such that the orbit $\{x, Tx, \ldots, T^n x, \ldots\}$ is dense in X, in which case, x is called a *hypercyclic vector* for T. Hypercyclicity arises from the invariant subset problem. Indeed, each non-zero vector of X is hypercyclic for T if, and only if, T has no non-trivial closed invariant subset in X. Hypercyclicity is also equivalent to topological transitivity on X, which is one of the ingredients for chaotic dynamic systems. In the last two decades, hypercyclicity has been studied intensively. Following the recent study in [19], it is natural to ask when a convolution operator T_σ is hypercyclic. We note that T_σ is never hypercyclic if $\|\sigma\|=1$. However, a weighted convolution operator can be hypercyclic. Given a positive weight w on G/H, we consider the weighted convolution operator $T_{\sigma,w}(f) = wT_\sigma(f)$ on $L^p(G/H)$ and study hypercyclicity of $T_{\sigma,w}$. For a unit point mass $\sigma = \delta_a$ $(a \in G)$, we write $T_{a,w}$ for $T_{\delta_a,w}$ and show that $T_{a,w}$ is hypercyclic under some condition on the weight w. If the group is discrete, this condition is

also necessary, and is equivalent to hereditary hypercyclicity of the operator. In Theorem 3.2.8, we prove the following result.
Let $a \in G$ which is not a torsion element. Let $w \in \ell^\infty(G)$ and $1 \leq p < \infty$. Then the following conditions are equivalent.

(i) $T_{a,w}$ is hypercyclic.

(ii) $T_{a,w}$ is hereditarily hypercyclic.

(iii) Both sequences (depending on a)

$$w_n = \prod_{s=1}^{n} w * \delta_{a^{-1}}^{s} \quad \mathit{and} \quad \overline{w}_n = \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit subsequences (w_{n_k}) and $(\overline{w}_{n_k})$ which converge to 0 pointwise in G.

By strengthening condition (iii) above and analogous arguments, we describe topologically mixing weighted convolution operators in Theorem 3.2.14: the following conditions are equivalent.

(i) $T_{a,w}$ is topologically mixing.

(ii) Both sequences (depending on a)

$$w_n = \prod_{s=1}^{n} w * \delta_{a^{-1}}^{s} \quad \mathit{and} \quad \overline{w}_n = \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

converge to 0 pointwise in G.

We also obtain a characterization of supercyclic weighted convolution operators in a similar way.

These results extend works in [25, 42, 47] for bilateral weighted shifts on $\ell^p(\mathbb{Z})$. We also give a sufficient condition for a bilateral weighted shift on $\ell^p(\mathbb{Z})$ to be frequently hypercyclic, and an example of a quasi-nilpotent hypercyclic operator.

We apply our results to the Laplacian $\mathcal{L}$, in Chapter 4, on a weighted homogeneous graph (V, K), where V is the vertex set and K is the edge generating set. In this case, V is represented as a coset space G/H of G by a finite subgroup H. Let the weight w be given by a measure μ on G which is symmetric and constant on each set xHy $(x, y \in G)$. Then, in Section 4.1, we describe the spectrum of $\mathcal{L}$ in terms of irreducible representations of G in Corollary 4.1.1:

$$\mathrm{Spec}(\mathcal{L}) = 1 - \bigcup \left\{ \mathrm{Spec}\left(\sum_{a \in K} \mu(a)|K|^{-1}\pi(a) \right) : \pi \in \widehat{G}_r, \ker\pi \supset \ker\rho_{_H} \right\}.$$

We prove an eigenvalue comparison theorem in Section 4.2 which extends a result in [50]. Let T_d be a regular tree. Choose $v_0 \in V$ and $x_0 \in T_d$. Let λ_1 and ν_1 be the first eigenvalues of the Laplacians with Dirichlet boundaries on balls $B(v_0, R)$ and $B(x_0, R)$ respectively. Then we prove in Theorem 4.2.7 that

(i) **condition A** implies

$$\lambda_1(B(v_0, R)) \leq \nu_1(B(x_0, R));$$

(ii) **condition B** implies

$$\lambda_1(B(v_0, R)) \geq \nu_1(B(x_0, R)).$$

By this comparison theorem, one can estimate the spectrum Spec($\mathcal{L}$) of $\mathcal{L}$ for an infinite graph (V, K). In Section 4.3, we characterise the invariance of a connected homogeneous graph in terms of group structures, and show that all positive $\mathcal{L}$-harmonic functions on an invariant connected homogeneous graph are constant. In Theorem 4.3.3, we prove a version of Harnack inequality for a Schrödinger operator which is stated below.

Let (V, K) be a possibly infinite invariant homogeneous graph. Let $\varphi \geq 0$ be a function on V and let f be a real function on V satisfying

$$\mathcal{L}f + \varphi f = \lambda f \qquad (\lambda > 0).$$

Then on any finite subgraph with vertex set S satisfying $SK \subset S$, we have

$$\sum_{a \in K} w_a[f(v) - f(va)]^2 + \alpha\lambda f^2(v) \leq \left(\frac{\alpha^2\lambda}{\alpha - 2} + \frac{4}{(\alpha - 2)\lambda} \sup_S \varphi\right) \sup_S f^2$$

for $v \in S$ and $\alpha > 2$.

The above inequality for $\mathcal{L} + \varphi$ extends a Harnack inequality for $\mathcal{L}$ in [23]. This inequality can be applied to derive a lower bound for the first eigenvalue of $\mathcal{L}$ on a finite weighted invariant graph (V, K). Indeed, we show that

$$\lambda_1 \geq \frac{k}{8D^2}$$

where D is the diameter of (V, K) and k is a constant depending on K and the weight. Finally we conclude with a version of Harnack inequality for Dirichlet eigenfunctions on a finite convex subgraph of an invariant homogeneous graph (V, K), extending the result of [24].

Chapter 2

Convolution operators on homogeneous spaces

In this chapter, we study some properties of convolution operators on homogeneous spaces of locally compact groups G. Given a measure σ on G, we define a convolution operator T_σ on $L^p(G/H)$ for $1 \leq p \leq \infty$. For an absolutely continuous symmetric measure σ, we develop a device to study the L^2-spectrum of T_σ by identifying T_σ as an element in a quotient of the group C^*-algebra $C^*(G)$. This enables us to describe the spectrum of T_σ in terms of the Fourier transform of the measure σ. This result will be used later to describe the spectrum of a discrete Laplacian on a weighted homogeneous graph.

2.1 Locally compact groups

We first recall in this section some basic definitions and results in locally compact groups and homogeneous spaces for future reference.

Let G be a group. We denote by e the identity of G throughout. A group G

is called a *topological group* if it is a topological space and satisfies the following continuity properties:

(i) the map $(x, y) \mapsto xy$ from $G \times G$ to G is continuous;

(ii) the map $x \mapsto x^{-1}$ from G to G is continuous.

A topological group G is called a *locally compact group* if its topology is Hausdorff and each point $x \in G$ has a relatively compact neighbourhood. For example, every discrete group and the Euclidean space $\mathbb{R}^d$ with coordinatewise addition and the usual topology are locally compact.

A locally compact group G is *second countable* if its topology has a countable base in which case G is metrizable and separable. We refer to [36, p.125] for a proof.

In this thesis, we study operators on homogeneous spaces of locally compact groups G. Given a closed subgroup H of G, the right coset space

$$G/H = \{Hx : x \in G\}$$

is a prototype of a homogeneous space of G. To introduce the concept of a homogeneous space of a locally compact group G, we first define an action of G on a topological space.

Let Ω be a locally compact Hausdorff space. A continuous map

$$(v, x) \in \Omega \times G \mapsto vx \in \Omega$$

is called a (right) *action* of G on Ω if

(i) $v \mapsto vx$ is a homeomorphism of Ω for each $x \in G$;

(ii) $(vx)y = v(xy)$ for all $x, y \in G$ and $v \in \Omega$.

We say that G *acts transitively* on Ω, or the action is *transitive*, if for every $u, v \in \Omega$, there exists $x \in G$ such that $vx = u$. For instance, G acts on the coset space G/H transitively by right multiplication. We call Ω a *transitive G-space* if G acts transitively on Ω in which case, for any fixed $v_0 \in \Omega$, the subgroup

$$H = \{x \in G : v_0 x = v_0\}$$

is closed, called an *isotropy subgroup* of G, and there exists a continuous bijection $\Psi : G/H \to \Omega$ defined by

$$\Psi(Hx) = v_0 x \qquad (x \in G).$$

In general, Ω need not be homeomorphic to G/H (cf. [31]). If Ω is homeomorphic to G/H, we call Ω a *homogeneous space* of G. In particular, G/H is a homogeneous space of G. Actually, a transitive G-space is a homogeneous space if G is σ-compact. The proof of the following result can be found in [31, Proposition 2.44].

Proposition 2.1.1 *Let G be σ-compact and act transitively on a locally compact Hausdorff space Ω. Then Ω is homeomorphic to, and hence identifies with, G/H.*

Let G be a locally compact group and let $\mathcal{B}$ be the σ-algebra of Borel subsets of G. A measure on $\mathcal{B}$ is called a *Borel measure* on G. Let $x \in G$. The *right translation* μ_x of a Borel measure μ on G by x is defined by

$$\mu_x(E) = \mu(Ex) \text{ for every Borel set } E \subset G.$$

A *right invariant* measure on G is a Borel measure satisfying $\mu(Ex) = \mu(E)$ for every Borel set $E \subset G$ and every $x \in G$. Similarly a *left invariant* measure on G is a Borel measure such that $\mu(xE) = \mu(E)$ for every Borel set $E \subset G$ and every

$x \in G$.

A *right Haar measure* is a nonzero right invariant Borel measure μ on a locally compact group G. For instance, the measure $\frac{dx}{|x|}$ on the multiplicative group $\mathbb{R} \setminus \{0\}$ and the counting measure on a discrete group are both right and left invariant.

We note that each right Haar measure μ is associated to a left invariant measure σ defined by $\sigma(E) = \mu(E^{-1})$ for every Borel set $E \subset G$, where we have

$$\sigma(xE) = \mu(E^{-1}x^{-1}) = \mu(E^{-1}) = \sigma(E) \qquad (x \in G).$$

The existence of a Haar measure on a locally compact group is of fundamental importance in harmonic analysis.

Theorem 2.1.2 *Every locally compact group G possesses a right Haar measure which is unique up to a positive constant multiple.*

Proof. See, for example, [31, Theorem 2.10, 2.20]. □

Throughout the thesis, we will denote by λ a right Haar measure on a locally compact group G, and assume that λ is σ-finite. A right Haar measure λ need not be left invariant, however, G admits a function $\Delta_G : G \to (0, \infty)$, called the *modular function*, which is a continuous homomorphism from G to the multiplicative group of positive real numbers such that

$$d\lambda(yx) = \Delta_G(y) d\lambda(x),$$

$$d\lambda(x^{-1}) = \Delta_G(x) d\lambda(x).$$

The group G is called *unimodular* if $\Delta_G \equiv 1$, that is, if a right Haar measure is also a left invariant. We note that all compact groups, abelian groups and discrete groups are unimodular. For a discrete group, we choose λ to be the counting measure.

Example 2.1.3 Let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : (x, y) \in (0, \infty) \times \mathbb{R} \right\}$$

be the affine group of $\mathbb{R}$. We denote an element in G by (x, y). A right Haar measure of G is given by $\frac{dxdy}{x}$ which is not left invariant. Therefore G is not unimodular. Indeed, given $f(x, y) = \frac{x\exp(-x)}{1+y^2}$, we have

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{0}^{\infty} f(x, y)\frac{dxdy}{x} &= \int_{-\infty}^{\infty} \frac{dy}{1 + y^2}\int_{0}^{\infty} \exp(-x)dx = \pi \\ &= \int_{-\infty}^{\infty}\int_{0}^{\infty} f((x, y)(2, 0))\frac{dxdy}{x} \end{aligned}$$

which is not equal to

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} f((2, 0)(x, y))\frac{dxdy}{x} = \int_{-\infty}^{\infty} \frac{dy}{1 + 4y^2}\int_{0}^{\infty} 2\exp(-2x)dx = \frac{\pi}{2}.$$

Henceforth we fix a right Haar measure λ on G. Given a subgroup H of G, we will always denote by $q : G \to G/H$ the quotient map in the sequel. Let H be a compact subgroup of G. Let $C_c(G/H)$ be the space of continuous functions on G/H with compact support. The Borel measure ν on G/H defined by $\nu = \lambda \circ q^{-1}$ [31, p.58] satisfies

$$\int_G f d\lambda = \int_{G/H} Qf d\nu = \int_{G/H}\int_H f(\xi x)d\xi d\nu(Hx) \qquad (f \in C_c(G))$$

where $Q : C_c(G) \to C_c(G/H)$ is defined by $(Qf)(Hx) = \int_H f(\xi x)d\xi$, $d\xi$ being normalized Haar measure on H. For $1 \leq p \leq \infty$, let $L^p(G/H)$ be the complex Lebesgue space of G/H with respect to ν, and write $L^p(G)$ when $H = \{e\}$, also $\ell^p(G)$ for a discrete group G. We note that $L^1(G)$ has an involution

$$f^*(x) = \overline{f(x^{-1})}\Delta_G(x^{-1}) \qquad (x \in G).$$

Let $C_0(G)$ be the Banach space of complex continuous functions on G vanishing at infinity. The dual $C_0(G)^*$ identifies with the space $M(G)$ of complex regular Borel measures on G. Each $\sigma \in M(G)$ has a finite total variation $|\sigma|$ and $M(G)$ is a Banach algebra in the total variation norm and the convolution product:

$$\|\sigma\| = |\sigma|(G), \qquad \int_G f d(\sigma * \mu) = \int_G \int_G f(xy) d\sigma(x) d\mu(y)$$

for $\sigma, \mu \in M(G)$ and all $f \in C_0(G)$. We have

$$\left| \int_G \int_G f(xy) d\sigma(x) d\mu(y) \right| \leq \|f\|_\infty \|\sigma\| \|\mu\|$$

and

$$\|\sigma * \mu\| \leq \|\sigma\| \|\mu\|.$$

Given Borel functions f and g on G, we define their convolution, whenever it exists, by

$$(f * g)(x) = \int_G f(xy^{-1}) g(y) d\lambda(y).$$

We also define

$$(f * \sigma)(x) = \int_G f(xy^{-1}) d\sigma(y),$$

$$(\sigma * f)(x) = \int_G f(y^{-1}x) \Delta_G(y^{-1}) d\sigma(y)$$

whenever they exist. We note that $f \in L^p(G)$ and $\sigma \in M(G)$ imply $f * \sigma \in L^p(G)$ $(1 \leq p \leq \infty)$.

A measure $\sigma \in M(G)$ is called *absolutely continuous* if its total variation $|\sigma|$ is absolutely continuous with respect to the Haar measure λ, in which case σ has a density $f \in L^1(G)$ so that $\sigma = f \cdot \lambda$. We call σ *symmetric* if $d\sigma(x) = d\sigma(x^{-1})$. For each $a \in G$, we denote by δ_a the point mass at a and by σ^n the n-fold convolution $\sigma * \cdots * \sigma$. The n-fold convolution $f * \cdots * f$ is denoted by f^n. We define $\sigma^0 = \delta_e$. The unit mass δ_e is the identity in the Banach algebra $M(G)$.

2.2 Convolution operators on L^p spaces

In this section, we give a description of the L^2-spectrum of a convolution operator on the homogeneous space G/H of a locally compact group G by a compact subgroup H. The description of the spectrum of $T_\sigma : L^2(G/H) \to L^2(G/H)$ in Theorem 2.2.7 has appeared in [13].

We note that locally compact groups often have a good supply of compact subgroups. Indeed, every finite subgroup is trivially compact. The circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is a compact subgroup of the multiplicative group $\mathbb{C} \setminus \{0\}$.

Following [19, p.76], we define two natural continuous linear maps $J : L^p(G/H) \to L^p(G)$ and $Q : L^p(G) \to L^p(G/H)$, where $1 \leq p \leq \infty$, by

$$J(f) = f \circ q, \quad Qg(Hx) = \int_H g(\xi x) d\xi \quad (f \in L^p(G/H), g \in L^p(G))$$

with $d\xi$ being the normalized Haar measure on the compact group H. For $1 \leq p < \infty$, the map J is an isomeric embedding by the change-of-variable formula

$$\int_{G/H} |f(y)|^p d\nu(y) = \int_G |f \circ q(x)|^p d\lambda(x)$$

and Q is a contraction because Jensen's inequality gives

$$\begin{aligned} \|Q(g)\|_p^p &= \int_{G/H} |Qg(Hx)|^p d\nu(Hx) \\ &= \int_G \left| \int_H g(\xi x) d\xi \right|^p d\lambda(x) \\ &\leq \int_G \int_H |g(\xi x)|^p d\xi d\lambda(x) \\ &= \int_H \int_G |g(x)|^p \Delta_G(\xi^{-1}) d\lambda(x) d\xi = \|g\|_p^p \end{aligned}$$

since

$$\int_H \Delta_G(\xi^{-1}) d\xi = \int_H \Delta_G|_H(\xi^{-1}) d\xi = \int_H 1 d\xi = 1.$$

We also have $\|f\|_\infty = \|J(f)\|_\infty$ and $\|Q(g)\|_\infty \leq \|g\|_\infty$. Further, Q is surjective since we have

$$(QJ)f(Hx) = \int_H (f \circ q)(\xi x)d\xi = \int_H f(Hx)d\xi = f(Hx)$$

for all $f \in L^p(G/H)$. Let $P := JQ$ on $L^p(G)$. Then P is a norm-one projection given by

$$(Pf)(x) = \int_H f(\xi x)d\xi.$$

Given a linear operator $T : L^p(G) \to L^p(G)$, we can define an induced operator $\Phi(T) : L^p(G/H) \to L^p(G/H)$ by the following commutative diagram:

$$\begin{tikzcd}
L^p(G) \arrow[r, "T"] & L^p(G) \arrow[d, "Q"] \\
L^p(G/H) \arrow[u, "J"] \arrow[r, "\Phi(T)"] & L^p(G/H)
\end{tikzcd}$$

where

$$\Phi(T) = Q \circ T \circ J.$$

For a Banach space X, we will denote by $B(X)$ be the Banach algebra of bounded linear operators on X. The above construction gives a linear map

$$\Phi : B(L^p(G)) \to B(L^p(G/H)). \tag{2.1}$$

Let $\sigma \in M(G)$ and, let $g \in L^p(G)$ where $1 \leq p < \infty$. The convolution

$$g * \sigma(x) = \int_G g(xy^{-1})d\sigma(y)$$

exists λ-almost everywhere in G and is defined to be 0 otherwise. For $p = \infty$, $g * \sigma$ exists outside a λ-null set N and we define $g * \sigma = 0$ on N. We have

$g * \sigma \in L^\infty(G)$ with $\|g * \sigma\|_\infty \leq \|g\|_\infty \|\sigma\|$, whenever $g \in L^\infty(G)$. For $1 \leq p < \infty$, we have

$$\begin{aligned}
\left(\int_G |g * \sigma(x)|^p d\lambda(x)\right)^{1/p} &= \left(\int_G \left|\int_G g(xy^{-1}) d\sigma(y)\right|^p d\lambda(x)\right)^{1/p} \\
&\leq \left(\int_G \left(\int_G |g(xy^{-1})| \; d|\sigma|(y)\right)^p d\lambda(x)\right)^{1/p} \\
&\leq \int_G \left(\int_G |g(xy^{-1})|^p d\lambda(x)\right)^{1/p} d|\sigma|(y).
\end{aligned}$$

The last inequality above can be obtained by the computation below. Let $H(x) = \int_G |g(xy^{-1})| \; d|\sigma|(y)$. Then

$$\begin{aligned}
\int_G H^p(x) d\lambda(x) &= \int_G \left(\int_G |g(xy^{-1})| \; d|\sigma|(y)\right) H^{p-1}(x) d\lambda(x) \\
&= \int_G \left(\int_G |g(xy^{-1})| H^{p-1}(x) d\lambda(x)\right) d|\sigma|(y) \\
&\leq \int_G \left(\int_G |g(xy^{-1})|^p d\lambda(x)\right)^{\frac{1}{p}} \left(\int_G H^p(x) d\lambda(x)\right)^{\frac{p-1}{p}} d|\sigma|(y).
\end{aligned}$$

Therefore we have $g * \sigma \in L^p(G)$ and $\|g * \sigma\|_p \leq \|g\|_p \|\sigma\|$.

Given $\sigma \in M(G)$, we can define, by the above remarks, the convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$ by

$$T_\sigma(g) = g * \sigma \qquad (g \in L^p(G)).$$

We note that T_σ induces the operator $\Phi(T_\sigma) = Q \circ T_\sigma \circ J$ on $L^p(G/H)$, where

$$\Phi(T_\sigma) f(Hx) = \int_G f(Hxy^{-1}) d\sigma(y) \qquad (f \in L^p(G/H)). \tag{2.2}$$

If confusion is unlikely, we will also write T_σ for $\Phi(T_\sigma)$ and call it a *convolution operator on* $L^p(G/H)$ defined by $\sigma \in M(G)$. We also write $f * \sigma$ when $f \in L^p(G/H)$ to mean $\Phi(T_\sigma)f$. We note that $T_{\sigma^n} = T_\sigma^n$ for the n-fold convolution σ^n. This follows from the associativity of convolution.

Lemma 2.2.1 *The norm-one projection $P = JQ$ commutes with all the convolution operators T_σ ($\sigma \in M(G)$). Moreover, $\Phi(T_\sigma)Q = QT_\sigma$ and $J\Phi(T_\sigma) = T_\sigma J$.*

Proof.

$$\begin{aligned} P(T_\sigma f)(x) &= \int_H T_\sigma f(\xi x)d\xi = \int_H \int_G f(\xi x y^{-1})d\sigma(y)d\xi \\ &= \int_G \int_H f(\xi x y^{-1})d\xi d\sigma(y) = \int_G Pf(xy^{-1})d\sigma(y) = T_\sigma(Pf)(x). \end{aligned}$$

Hence we have $\Phi(T_\sigma)Q = QT_\sigma P = QPT_\sigma = QT_\sigma$ and $J\Phi(T_\sigma) = PT_\sigma J = T_\sigma PJ = T_\sigma J$. □

When considering the convolution operator $T_\sigma : L^p(G) \to L^p(G)$, we denote by $\operatorname{Spec}(T_\sigma, L^p(G))$ the spectrum of T_σ and by $\Lambda(T_\sigma, L^p(G))$ the set of eigenvalues of T_σ respectively. If p is understood, we write $\operatorname{Spec}(T_\sigma)$ for $\operatorname{Spec}(T_\sigma, L^p(G))$ and $\Lambda(T_\sigma)$ for $\Lambda(T_\sigma, L^p(G))$. Using the equalities in Lemma 2.2.1, we can compare the spectrum of T_σ with the spectrum of $\Phi(T_\sigma)$ as in [19, Lemma 3.4.4].

Lemma 2.2.2 *Let $\sigma \in M(G)$ and $T_\sigma : L^p(G) \to L^p(G)$ be the induced convolution operator. Then*

(i) $\operatorname{Spec}(\Phi(T_\sigma)) \subset \operatorname{Spec}(T_\sigma)$;

(ii) $\Lambda(\Phi(T_\sigma)) \subset \Lambda(T_\sigma)$.

Proof. Let $S \in B(L^p(G))$ and $\mu \in M(G)$. If $T_\mu S = I$ then $\Phi(T_\mu)(QSJ) = QT_\mu SJ = QJ = I$. Similarly, $(QSJ)\Phi(T_\mu) = QST_\mu = QJ = I$ if $ST_\mu = I$. If $\Phi(T_\mu)f = 0$ ($f \neq 0$), then $0 = J\Phi(T_\mu) = T_\mu(Jf)$. Let $\mu = \sigma - \alpha\delta_e$ ($\alpha \in \mathbb{C}$). Then $T_\mu = T_\sigma - \alpha I$ which implies (i) and (ii). □

Following convention, we always denote the conjugate exponent $\frac{p}{p-1}$ of $p \in [1, \infty]$ by q. The confusion of notation with the quotient map $q : G \to G/H$ is unlikely. Let

$$\langle \cdot, \cdot \rangle : L^p(G/H) \times L^q(G/H) \longrightarrow \mathbb{C}$$

be the duality. Let $\widetilde{\sigma}$ be the measure $d\widetilde{\sigma}(x) = d\sigma(x^{-1})$. For $f \in L^p(G/H)$ $(1 \leq p < \infty)$ and $g \in L^q(G/H)$, we have

$$\begin{aligned}
\langle T_\sigma f, g \rangle &= \int_{G/H} (T_\sigma f)(Hx) g(Hx) d\nu(Hx) \\
&= \int_{G/H} \int_G f(Hxy) g(Hx) d\sigma(y^{-1}) d\nu(Hx) \\
&= \int_{G/H} f(Ht) \int_G g(Hty^{-1}) d\sigma(y^{-1}) d\nu(Ht) \\
&= \int_{G/H} f(Ht) T_{\widetilde{\sigma}} g(Ht) d\nu(Ht) = \langle f, T_{\widetilde{\sigma}}(g) \rangle.
\end{aligned}$$

Therefore the dual map $T_\sigma^* : L^q(G/H) \longrightarrow L^q(G/H)$ $(1 < q \leq \infty)$ is given by $T_\sigma^*(g) = T_{\widetilde{\sigma}}(g)$ for $g \in L^q(G/H)$.

Our objective is to describe the spectrum of $T_\sigma : L^2(G/H) \to L^2(G/H)$ for an absolutely continuous symmetric measure σ. For this, we develop a device to identify T_σ as an element in a quotient of the group C*-algebra $C^*(G)$ which then enables us to use spectral theory of C*-algebras to achieve the task.

We first recall a *representation* π of an involutive Banach algebra A on a Hilbert space $\mathcal{H}_\pi$ is a $*$-algebra homomorphism $\pi : A \to B(\mathcal{H}_\pi)$, in other words, π is a linear map from A into $B(\mathcal{H}_\pi)$ satisfying

$$\pi(ab) = \pi(a)\pi(b) \qquad \text{and} \qquad \pi(a^*) = \pi(a)^*$$

for all $a, b \in A$. We note that π is continuous and contractive: $\|\pi(a)\| \leq \|a\|$ for all $a \in A$ [29, 1.3.7].

For the remaining of this section, we let A be a C*-algebra. Two representations $\pi : A \to B(\mathcal{H}_\pi)$ and $\tau : A \to B(\mathcal{H}_\tau)$ are said to be (*unitarily*) *equivalent*, in symbols: $\pi \simeq \tau$, if there is a surjective linear isometry $u : \mathcal{H}_\pi \to \mathcal{H}_\tau$ such that

$u\pi(a) = \tau(a)u$ $(a \in A)$. We denote by $[\pi]$ the equivalent class of π with respect to the unitary equivalence.

Let $\pi : A \to B(\mathcal{H}_\pi)$ be a representation of A. A closed subspace $E \subset \mathcal{H}_\pi$ is called *invariant* under $\pi(A)$ if $\pi(A)(E) \subset E$. A representation $\pi : A \to B(\mathcal{H}_\pi)$ is said to be *irreducible* if $\pi(A)$ has no invariant subspace other than $\{0\}$ and $\mathcal{H}_\pi$.

Let $\widehat{A}$ be the space of all equivalence classes of irreducible representations $\pi : A \longrightarrow B(H_\pi)$ of A [29, 3.1.5]. We call $\widehat{A}$ the *spectrum* of the C*-algebra A. As usual, we write π for $[\pi] \in \widehat{A}$ if no confusion is likely.

For a locally compact group G, a *continuous unitary representation* of G is a homomorphism π from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on a Hilbert space $\mathcal{H}_\pi$ and π is continuous with respect to the strong operator topology of $B(\mathcal{H}_\pi)$. In other words, a continuous unitary representation is a map $\pi : G \to U(\mathcal{H}_\pi)$ such that for all $x, y \in G$,

$$\pi(xy) = \pi(x)\pi(y), \qquad \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$$

and the mapping $x \mapsto \pi(x)h$ is continuous from G to $\mathcal{H}_\pi$ for every $h \in \mathcal{H}_\pi$. An irreducible representation of a locally compact group G can be defined in a similar way as that of a C*-algebra A above.

We recall that the group C^*-algebra $C^*(G)$ of G is the completion of $L^1(G)$ with respect to the norm

$$\|f\|_c = \sup_\pi\{\|\pi(f)\|\}$$

where the supremum is taken over all representations $\pi : L^1(G) \longrightarrow B(\mathcal{H}_\pi)$. If G is discrete, then $C^*(G)$ contains an identity.

Let $\rho : C^*(G) \to B(L^2(G))$ be the right regular representation given by (continuous extension of)

$$\rho(f)h = h * f \quad (f \in L^1(G), h \in L^2(G)) \tag{2.3}$$

which is an extension of the right regular representation $a \in G \mapsto \rho(a) \in B(L^2(G))$ of G, where $\rho(a)h = h * \delta_a$. The reduced group C*-algebra $C_r^*(G)$ is the norm closure $\overline{\rho(L^1(G))} = \rho(C^*(G))$ of $\rho(L^1(G))$ in $B(L^2(G))$.

We define a unitary representation $\tau : G \to B(L^2(G/H))$ by right translation:

$$\tau(a)h(Hx) = h(Hxa^{-1}) \quad (a, x \in G, h \in L^2(G/H)).$$

We can extend τ to a representation $\rho_{H} : C^*(G) \to B(L^2(G/H))$ in the usual way (cf. [43, p.229]), that is, τ induces a representation of $L^1(G)$ by integration:

$$\rho_{H}(f) = \int_G f(x)\tau(x)d\lambda(x) \qquad (x \in G, f \in L^1(G) \subset C^*(G)).$$

We interpret this operator-valued integral in the week sense [31, p.73]. That is, for any $h \in L^2(G/H)$, we define $\rho_{H}(f)h$ by specifying its inner product with an arbitrary $g \in L^2(G/H)$, and the letter is given by

$$\langle \rho_{H}(f)h, g\rangle = \int_G f(x)\langle \tau(x)h, g\rangle d\lambda(x).$$

Lemma 2.2.3 *Let $\rho : C^*(G) \to B(L^2(G))$ be the right regular representation defined in* (2.3) *and let $\Phi : B(L^2(G)) \to B(L^2(G/H))$ be the mapping defined in* (2.1) *for $p = 2$. Then, the diagram*

$$\begin{tikzcd}
C^*(G) \arrow[rr, "\rho_H"] \arrow[dr, "\rho"'] & & B(L^2(G/H)) \\
& B(L^2(G)) \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

is commutative.

Proof. For $f \in L^1(G)$ and $g \in L^2(G/H)$, we have

$$\Phi(\rho f)(g) = Q(\rho f)J(g) = Q(\rho f(g \circ q)) = Q((g \circ q) * f)$$

and

$$\begin{aligned} Q((g \circ q) * f)(Hx) &= \int_H (g \circ q) * f(\xi x) d\xi \\ &= \int_H \int_G (g \circ q)(\xi x y^{-1}) f(y) d\lambda(y) d\xi \\ &= \int_H \int_G g(Hxy^{-1}) f(y) d\lambda(y) d\xi \\ &= \int_G g(Hxy^{-1}) f(y) d\lambda(y) \\ &= \rho_{_H}(f)(g)(Hx). \end{aligned}$$

Hence $\Phi(\rho f) = \rho_{_H}(f)$. $\square$

The surjective contraction Φ factors the representation ρ_H through the right regular representation of the group C*-algebra $C^*(G)$.

Lemma 2.2.4 *Let $\sigma \in M(G)$ be absolutely continuous with $\sigma = f \cdot \lambda$ and $f \in L^1(G)$. Then $\rho_{_H}(f) = T_\sigma \in B(L^2(G/H))$.*

Proof. We have

$$\rho_{_H}(f)h = \int_G (h * \delta_x) f(x) d\lambda(x) \in L^2(G/H) \qquad (h \in L^2(G/H))$$

and

$$\begin{aligned} \rho_{_H}(f)h(Hy) &= \int_G (h * \delta_x)(Hy) f(x) d\lambda(x) \\ &= \int_G h(Hyx^{-1}) f(x) d\lambda(x) \\ &= (h * f)(Hy) = T_\sigma(h)(Hy). \end{aligned}$$

□

Let $\widehat{G}$ be the dual space of G, consisting of (equivalence classes of) continuous irreducible unitary representations of G. If G is abelian, then $\widehat{G}$ is the character group of G.

The spectrum $\widehat{C^*(G)}$ identifies with $\widehat{G}$ [29, 13.9.3] where each $\pi \in \widehat{G}$ is identified as the irreducible representation of $C^*(G)$ satisfying

$$\pi(f) = \int_G f(x)\pi(x)d\lambda(x) \qquad (f \in L^1(G) \subset C^*(G)).$$

The spectrum $\widehat{C_r^*(G)}$ identifies with the following closed subset of $\widehat{G}$, the *reduced dual* of G:

$$\widehat{G}_r = \{\pi \in \widehat{G} : \ker \pi \supset \ker \rho\}$$

(cf. [29, 18.3]). We note that $\widehat{G}_r = \widehat{G}$ if G is abelian or compact.

We define the *Fourier transform* $\widehat{\sigma}$ of a measure $\sigma \in M(G)$ by

$$\widehat{\sigma}(\pi) = \int_G \pi(x^{-1})d\sigma(x) \in B(H_\pi) \qquad (\pi \in \widehat{G})$$

with its spectrum denoted by $\mathrm{Spec}(\widehat{\sigma}(\pi))$.

The spectrum $\mathrm{Spec}(a)$ of a self-adjoint element a in a C*-algebra A with identity is given by

$$\mathrm{Spec}(a) = \bigcup_{\pi \in \widehat{A}} \mathrm{Spec}(\pi(a))$$

where $\mathrm{Spec}(\pi(a))$ is the spectrum of $\pi(a)$ in $B(H_\pi)$ (cf. [29, 3.3.5]). In fact, the above equality holds in the following situation.

Lemma 2.2.5 *Let A be a C*-algebra with identity, and let $a \in A$ satisfy*

$$\alpha \in \mathrm{Spec}(a) \Leftrightarrow a - \alpha 1 \textit{ has no left inverse in } A. \tag{2.4}$$

Then

$$\mathrm{Spec}(a) = \bigcup_{\pi \in \widehat{A}} \mathrm{Spec}(\pi(a)).$$

Proof. Let $a - \alpha 1$ be invertible. Then

$$\pi(a - \alpha 1)\pi((a - \alpha 1)^{-1}) = \pi(1) = I$$

for all $\pi \in \widehat{A}$. Hence $\pi(a) - \alpha I$ is invertible in $\pi(A)$. This implies

$$\mathrm{Spec}(a) \supset \bigcup_{\pi \in \widehat{A}} \mathrm{Spec}(\pi(a)).$$

Conversely, let $\alpha \in \mathrm{Spec}(a)$. Then $a - \alpha 1$ has no left inverse in A. This implies $A(a - \alpha 1)$ is a proper left ideal in A and is therefore contained in a maximal left ideal L in A. By [29, 2.9.5], there exists a pure state φ such that

$$L = N_\varphi := \{a \in A : \varphi(a^*a) = 0\}.$$

Let $\pi_\varphi : A \to B(H_\varphi)$ be the GNS-representation induced by φ. Suppose $\pi_\varphi(a - \alpha 1)$ is invertible in $\pi_\varphi(A)$. Then there exists some $x \in A$ such that

$$I = \pi_\varphi(1) = \pi_\varphi(x)\pi_\varphi(a - \alpha 1) = \pi_\varphi(x(a - \alpha 1)).$$

Since $a - \alpha 1 \in L = N_\varphi$, we have $x(a - \alpha 1) \in N_\varphi$ and therefore $\pi_\varphi(x(a - \alpha 1)) \neq I$ which is a contradiction. Hence $\pi_\varphi(a - \alpha 1)$ is not invertible in $\pi_\varphi(A)$ which implies $\alpha \in \mathrm{Spec}(\pi_\varphi(a))$ with $\pi_\varphi \in \widehat{A}$. □

If A is without identity, we adjoin an identity to A as usual to obtain $A_1 = A \oplus \mathbb{C}$, then we have the identification $\widehat{A_1} = \widehat{A} \cup \{\omega\}$ where ω is the one-dimensional irreducible representation of A_1 annihilating A (cf. [29, 3.2.4]). In this case, for $a \in A$ satisfying (2.4) in A_1, we have the quasi-spectrum

$$\mathrm{Spec}'(a) = \mathrm{Spec}_{A_1}(a) = \bigcup_{\pi \in \widehat{A_1}} \mathrm{Spec}(\pi(a)) = \bigcup_{\pi \in \widehat{A}} \mathrm{Spec}(\pi(a)) \cup \{0\}.$$

Given $\sigma \in M(G)$, the Hilbert space adjoint of $T_\sigma : L^2(G/H) \to L^2(G/H)$ is the operator $T_{\tilde{\overline{\sigma}}}$ where $\overline{\sigma}$ is the complex conjugate of σ. Hence, if σ is symmetric and real-valued, then T_σ is self-adjoint. If $\sigma \in M(G)$ is only symmetric, then the convolution operator T_σ satisfies (2.4) which has been shown in [19, Lemma 3.3.38], as stated below.

Lemma 2.2.6 *Let $\sigma \in M(G)$ be symmetric. Then for $\alpha \in \mathbb{C}$, we have $\alpha \in \operatorname{Spec}(T_\sigma)$ if, and only if, $T_\sigma - \alpha I$ has no left inverse in $B(L^2(G/H))$.*

Theorem 2.2.7 *Let $\sigma \in M(G)$ be symmetric and absolutely continuous and let $\operatorname{Spec}(T_\sigma)$ be the spectrum of the convolution operator $T_\sigma : L^2(G/H) \longrightarrow L^2(G/H)$. Then we have*

$$\operatorname{Spec}(T_\sigma) \cup \{0\} = \bigcup \{\operatorname{Spec}(\widehat{\sigma}(\pi)) : \pi \in \widehat{G}_r, \ker \pi \supset \ker \rho_{_H}\} \cup \{0\}.$$

In particular, $\operatorname{Spec}(T_\sigma) \cup \{0\} = \bigcup \{\operatorname{Spec}(\widehat{\sigma}(\pi)) : \pi \in \widehat{G}_r\} \cup \{0\}$ if $H = \{e\}$. If G is discrete, then $\operatorname{Spec}(T_\sigma) = \bigcup \{\operatorname{Spec}(\widehat{\sigma}(\pi)) : \pi \in \widehat{G}_r, \ker \pi \supset \ker \rho_{_H}\}$.

Proof. Let $\sigma = f \cdot \lambda$ with $f \in L^1(G)$. By Lemma 2.2.4, we have $T_\sigma = \rho_{_H}(f) \in \rho_{_H}(C^*(G)) \cong C^*(G)/\ker \rho_{_H}$. We consider the quasi-spectrum $\operatorname{Spec}'(\rho_{_H}(f))$ of $\rho_{_H}(f)$ in $\rho_{_H}(C^*(G))$ which may not have an identity.

Let $\operatorname{Spec}'(T_\sigma)$ be the quasi-spectrum of the convolution operator T_σ in $B(L^2(G/H))$ which satisfies (2.4) by Lemma 2.2.6. Then we have

$$\begin{aligned}
\operatorname{Spec}(T_\sigma) \cup \{0\} &= \operatorname{Spec}'(T_\sigma) = \operatorname{Spec}'(\rho_{_H}(f)) = \operatorname{Spec}'(f + \ker \rho_{_H}) \\
&= \bigcup \{\operatorname{Spec}(\pi(f + \ker \rho_{_H})) : \pi \in \widehat{C^*(G)/\ker \rho_{_H}}\} \cup \{0\} \\
&= \bigcup \{\operatorname{Spec}(\pi(f)) : \pi \in \widehat{C^*(G)}, \ker \pi \supset \ker \rho_{_H}\} \cup \{0\} \\
&= \bigcup \{\operatorname{Spec}(\pi(f)) : \pi \in \widehat{G}, \ker \pi \supset \ker \rho_{_H}\} \cup \{0\} \\
&= \bigcup \{\operatorname{Spec}(\pi(f)) : \pi \in \widehat{G_r}, \ker \pi \supset \ker \rho_{_H}\} \cup \{0\}
\end{aligned}$$

where, by Lemma 2.2.3, $\ker \rho_{_H} \supset \ker \rho$ which gives the last equality, and

$$\pi(f) = \int_G \pi(x) f(x) d\lambda(x) = \int_G \pi(x) d\sigma(x) = \widehat{\sigma}(\pi)$$

by symmetry of σ. This proves the first assertion.

If G is discrete, then $C^*(G)$ has an identity and one can dispense with the quasi-spectrum and remove $\{0\}$. □

Corollary 2.2.8 *If H is a normal subgroup of G in Theorem 2.2.7, then*

$$\mathrm{Spec}(T_\sigma) \cup \{0\} = \bigcup \{\mathrm{Spec}(\widehat{\sigma}(\pi)) : \pi \in \widehat{G}_r, \pi(H) = \pi\{e\}\} \cup \{0\}.$$

Proof. By composing with the quotient map $q : G \longrightarrow G/H$, the dual space $\widehat{G/H}$ identifies with $\{\pi \in \widehat{G} : \pi(H) = \pi\{e\}\}$, and also $\rho_{_H} = \rho_{_{G/H}} \circ q$ where $\rho_{_{G/H}}$ is the right regular representation of the group G/H. It follows that the reduced dual $\widehat{G/H}_r$ identifies with $\{\pi \in \widehat{G}_r : \pi(H) = \pi\{e\}\}$. □

Remark 2.2.9 It is known that if G is abelian and σ is absolutely continuous, then the L^p-spectrum of $T_\sigma : L^p(G) \to L^p(G)$ is given by

$$\mathrm{Spec}(T_\sigma, L^p(G)) = \overline{\widehat{\sigma}(\widehat{G})}.$$

For $p = 2$, the result can be deduced directly from the Plancherel theorem without absolute continuity of σ (see, for instance, [19]). Without absolute continuity, the result is false for $p \neq 2$ (see, for example, [52]). Theorem 2.2.7 gives a description of $\mathrm{Spec}(T_\sigma, L^2(G))$ for non-abelian groups G.

We will make use of the results in Theorem 2.2.7 to describe the spectrum of a discrete Laplacian on a homogeneous graph in Chapter 4. We first give some examples below.

Example 2.2.10 Let $G = \mathbb{R}$ which is an abelian group, and let

$$d\sigma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} d\lambda(x).$$

Then $\widehat{\mathbb{R}} = \{\chi_t : x \mapsto e^{itx}, t \in \mathbb{R}\}$ and

$$\widehat{\sigma}(\chi_t) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-itx-\frac{x^2}{2}} dx = e^{-\frac{t^2}{2}}.$$

Hence, for $1 \leq p \leq \infty$, we have

$$\mathrm{Spec}(T_\sigma, L^p(\mathbb{R})) = \overline{\widehat{\sigma}(\widehat{\mathbb{R}})} = \overline{\left\{e^{-\frac{t^2}{2}} : t \in \mathbb{R}\right\}} = [0, 1].$$

Example 2.2.11 Let

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

be the Heisenberg group which is neither abelian nor compact. For convenience, an element in G is written as (x, y, z). By [31, 6.51], we have

$$\widehat{G}_r = \{\pi_{a,b} : a, b \in \mathbb{R}\} \cup \{\pi_t : t \in \mathbb{R} \setminus \{0\}\}$$

where

$$\pi_{a,b} : (x, y, z) \in G \mapsto e^{2\pi i(ax+by)} \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

is a character and

$$\pi_t : G \to B(L^2(\mathbb{R}))$$

is given by

$$\pi_t(x, y, z)f(w) = e^{2\pi it(yw+z)} f(w + x) \qquad (f \in L^2(\mathbb{R})).$$

Let

$$d\sigma(x, y, z) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp -\frac{1}{2} \left(x^2 + y^2 + (z - \frac{xy}{2})^2\right) d\lambda(x, y, z).$$

Then

$$\begin{aligned}\widehat{\sigma}(\pi_{a,b}) &= \int_G \pi_{a,b}(-x,-y,xy-z)d\sigma(x,y,z)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(2\pi)^{\frac{3}{2}}}e^{-\frac{x^2+y^2+(z-\frac{xy}{2})^2}{2}}e^{2\pi i(-ax-by)}dxdydz\\ &= e^{-2\pi^2(a^2+b^2)} \qquad (a,b\in\mathbb{R})\end{aligned}$$

which implies

$$\bigcup_{a,b\in\mathbb{R}} \text{Spec } \widehat{\sigma}(\pi_{a,b}) = \{e^{-2\pi^2(a^2+b^2)} : a,b\in\mathbb{R}\} = (0,1].$$

We note that σ is symmetric since

$$d\sigma(x,y,z) = d\sigma(-x,-y,xy-z).$$

Hence we have

$$(0,1]\subset \text{Spec}(T_\sigma, L^2(G))\subset[-1,1]$$

by applying Theorem 2.2.7 and $\|T_\sigma\|\leq 1$.

Example 2.2.12 Let

$$SU(2)=\left\{\begin{pmatrix} a & -\overline{b}\\ b & \overline{a}\end{pmatrix} : a,b\in\mathbb{C}, |a|^2+|b|^2=1\right\}$$

be the group of unitary transformations of $\mathbb{C}^2$ with determinant 1 which is a compact group. For any unit mass δ_t on $SU(2)$, it is easy to see that 1 is an eigenvalue of the convolution operator T_{δ_t}. Indeed, let $f : SU(2)\to\mathbb{C}$ be the constant function $f\equiv\alpha\in\mathbb{C}$. Then $f\in L^p(G)$ and

$$f*\delta_t(x) = \int_{SU(2)} f(xy^{-1})d\delta_t(y) = \alpha = f(x).$$

The group $SU(2)$ has a one-parameter subgroup

$$F(\theta)=\begin{pmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix} \qquad (\theta\in\mathbb{R})$$

and $\widehat{SU(2)} = \{\pi_m : m \in \mathbb{N} \cup \{0\}\}$ (cf.[31, p.142]) where

$$\pi_m(F(\theta))(z^j w^{m-j}) = e^{i(2j-m)\theta} z^j w^{m-j} \qquad (z, w \in \mathbb{C}, 0 \leq j \leq m)$$

where $z^j w^{m-j}$ is an orthogonal basis of the space of homogeneous polynomials of degree m:

$$P_m(z, w) = \sum_0^m c_j z^j w^{m-j}.$$

Let σ be the unit mass $\delta_{F(\pi)}$. Then we have $-1 \in \mathrm{Spec}(T_\sigma, L^p(G))$ since

$$\widehat{\delta}_{F(\pi)}(\pi_5) = e^{-i(2j-5)\pi} = -1 \qquad (0 \leq j \leq 5)$$

by letting $m = 5$ and [19, Proposition 3.3.43].

We will study iteration of a convolution operator $T_\sigma : L^p(G/H) \to L^p(G/H)$ in Chapter 3. For the n-th iterate $T_\sigma^n : L^2(G/H) \to L^2(G/H)$, we can apply the results in Theorem 2.2.7 to describe its spectrum by the fact that $T_\sigma^n = T_{\sigma^n}$ and the following corollary.

Lemma 2.2.13 *If $\sigma = f \cdot \lambda$ and $\tau = g \cdot \lambda$ for $f, g \in L^1(G)$, then $\sigma * \tau = (f * g) \cdot \lambda$. Further, for $\sigma, \tau \in M(G)$, we have $\widetilde{\sigma * \tau} = \widetilde{\tau} * \widetilde{\sigma}$ where $\widetilde{\sigma}(x) = \sigma(x^{-1})$ $(x \in G)$.*

Proof. Let $\phi \in C_0(G)$. Then

$$\begin{aligned}
\int_G \phi(x) d(\sigma * \tau)(x) &= \int_G \int_G \phi(xy) d\sigma(x) d\tau(y) \\
&= \int_G \int_G \phi(xy) f(x) g(y) d\lambda(x) d\lambda(y) \\
&= \int_G \int_G \phi(z) f(zy^{-1}) g(y) d\lambda(y) d\lambda(z) \\
&= \int_G \phi(z)(f * g)(z) d\lambda(z).
\end{aligned}$$

This implies $\sigma * \tau = (f * g) \cdot \lambda$. Besides, we have

$$\begin{aligned}
\int_G \phi(x) d\widetilde{\sigma * \tau}(x) &= \int_G \phi(x^{-1}) d\sigma * \tau(x) \\
&= \int_G \int_G \phi(y^{-1}x^{-1}) d\sigma(x) d\tau(y) \\
&= \int_G \int_G \phi(y^{-1}x^{-1}) d\widetilde{\tau}(y^{-1}) d\widetilde{\sigma}(x^{-1}) \\
&= \int_G \phi(x) d\tilde{\tau} * \tilde{\sigma}(x).
\end{aligned}$$

□

By Lemma 2.2.13, we have the following simple consequence.

Corollary 2.2.14 *Let* $\sigma \in M(G)$ *and let* $n \in \mathbb{N}$.

(i) *If* $\sigma = f \cdot \lambda$ *with* $f \in L^1(G)$, *then* $\sigma^n = f^n \cdot \lambda$.

(ii) *If* σ *is symmetric, then* σ^n *is symmetric.*

Chapter 3

Hypercyclicity of convolution operators

In this chapter, we study hypercyclicity of convolution operators on homogeneous spaces. For $1 \leq p < \infty$, let $T_{a,w} : L^p(G/H) \to L^p(G/H)$ be a weighted translation operator defined by the unit point mass δ_a and a weight w on G/H. We will characterise hypercyclic weighted translation operators in terms of their weights. Indeed, we give a sufficient condition for a weighted translation operator $T_{a,w}$ to be hypercyclic, in terms of w. This condition is also necessary if G is discrete. By strengthening the condition and using analogous arguments, we characterise topologically mixing weighted translation operators $T_{a,w}$ on $L^p(G/H)$. Supercyclic weighted translation operators on homogeneous spaces are also characterised in a similar way. We derive a sufficient condition for bilateral weighted shifts to be frequently hypercyclic. We conclude this chapter with some hypercyclicity results on scalar multiples of weighted translation operators. Some results in this chapter have been published in [14].

Hypercyclic operators have been studied by many authors since the seminal work of Birkhoff [10] and MacLane [39]. We refer to [34, 35] for recent surveys and to [8, 15] for some recent works on hypercyclicity of sequences of operators. The related theory of topologically mixing, supercyclic, frequent hypercyclic operators and hypercyclic semigroups have been developed in [4, 6, 12, 25, 27, 28, 42]. In Section 3.1, we recall some relevant results on hypercyclicity for bounded linear operators on Banach spaces. The main results on hypercyclic weighted translation operators will be discussed in Section 3.2.

3.1 Hypercyclic criterion

We begin with some definitions and a discussion of the hypercyclic criterion. Although hypercyclic phenomena have been studied in Fréchet spaces, we restrict our attention to complex Banach spaces in this thesis.

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and positive integers respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given a bounded linear self-map T on a complex Banach space X, we denote its iterates by

$$T^0 = I, \quad \ldots, \quad T^{n+1} = T^n \circ T, \quad \ldots \qquad (n = 0, 1, \ldots).$$

The operator T is said to be *hypercyclic* if there is a vector $x \in X$ such that the orbit $\{x, Tx, \ldots, T^n x, \ldots\}$ is dense in X in which case x is called a *hypercyclic vector* for T. By definition, hypercyclicity can only occur in separable spaces. Indeed, a Banach space admits a hypercyclic operator if, and only if, it is separable and infinite-dimensional [1, 7]. We record a simple, but useful result for hypercyclic vectors from [32].

Proposition 3.1.1 *Let T be a bounded linear self-map on a Banach space X. If T has a hypercyclic vector, then it has a dense G_δ set of hypercyclic vectors.*

Proof. Fix a countable dense subset $\{y_k\}$ of X. For positive integers N, j and k, let

$$F(N, j, k) = \{x \in X : \|T^n x - y_j\| < \frac{1}{k} \text{ for some } n \geq N\}.$$

Each of these sets is open by the continuity of T. Moreover, each $F(N, j, k)$ is dense in X since, if x is a hypercyclic vector, then so is every member of the dense orbit $\{T^n x : n \geq 0\}$. The set of hypercyclic vectors for T is the intersection of these sets. It is therefore a dense G_δ subset of X. This completes the proof. $\square$

To characterise hypercyclic convolution operators, we will make use of the following form of the hypercyclic criterion in [9], derived from the original one due to Kitai [37], Gethner and Shapiro [32] independently.

Theorem 3.1.2 *Let T be a bounded linear self-map on a Banach space X. Then T is hypercyclic if it satisfies the following criterion: (T^n) admits a subsequence (T^{n_k}) such that*

(i) *(T^{n_k}) converges to zero pointwise on a dense subset of X;*

(ii) *there is a dense subset Y of X, and a sequence of maps $S_{n_k} : Y \to X$ such that (S_{n_k}) tends to zero pointwise on Y and $(T^{n_k} S_{n_k})$ tends to the identity pointwise on Y.*

In the above criterion, if $n_k = k$, then T is said to satisfy the *hypercyclic criterion for the full sequence.* This criterion has led to the following question.

Question 1: Does every hypercyclic operator satisfy the above hypercyclic criterion? In other words, is the hypercyclic criterion also a necessary condition for hypercyclicity?

The next question arises from the fact that there are hypercyclic operators T_1 and T_2 on a Hilbert space H such that the direct sum $T_1 \oplus T_2$ on $H \oplus H$ is not hypercyclic [46].

Question 2: Let T be a hypercyclic operator. Does it follow that the operator $T \oplus T$ is hypercyclic?

Bès and Peris [9] have settled Question 2 and showed that Question 1 and Question 2 are equivalent. We recall that an operator T *hereditarily hypercyclic* with respect to some sequence (n_k) if every subsequence (T^{m_k}) of (T^{n_k}) admits a vector $x \in X$ for which the orbit $\{T^{m_k}x\}_{k=1}^{\infty}$ is dense in X. It turns out that the two questions are equivalent to the problem whether every hypercyclic operator is hereditarily hypercyclic with respect to some (n_k). We give a precise formulation below.

Theorem 3.1.3 *Let T be a bounded linear operator on a Banach space X. Then the following conditions are equivalent:*

(i) *T satisfies the hypercyclic criterion;*

(ii) *$T \oplus T$ is hypercyclic;*

(iii) *T is hereditarily hypercyclic with respect to some sequence (n_k).*

Proof. See [9, Theorem 2.3]. □

Recently, negative answers to Question 1 have been given in [5, 26]. There exist hypercyclic operators on Banach spaces which fail the hypercyclic criterion. An operator T on a Banach space X is called *chaotic* if T is hypercyclic and has a dense set of periodic points in X, where a point $x \in X$ is *periodic* if $T^n x = x$

for some $n \in \mathbb{N}$ [33]. Using the above theorem, it has been shown in [9] that every chaotic operator on X satisfies the hypercyclic criterion.

Besides the sufficient conditions for hypercyclicity in Theorem 3.1.3, it is known that [34] T is hypercyclic if and only if T is *topologically transitive*, that is, given any nonempty open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. A *topologically mixing* operator T satisfies a stronger condition: there exists $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n > N$. We have the following simple results motivated by hypercyclicity.

Lemma 3.1.4 *Let $T : X \to X$ and $S : Y \to Y$ be two bounded operators on Banach spaces X and Y respectively.*

(i) *Let T be invertible. Then T is topologically mixing if, and only if, T^{-1} is topologically mixing.*

(ii) *Both T and S are topologically mixing if, and only if, $T \oplus S$ is topologically mixing.*

(iii) *Let T be topologically mixing. Then T satisfies the hypercyclic criterion.*

Proof. (i) Let U and V be nonempty open subsets of X. Then for all $n \in \mathbb{N}$, we have

$$T^n(U) \cap V \neq \emptyset \qquad \Leftrightarrow \qquad U \cap T^{-n}(V) \neq \emptyset.$$

(ii) Let both T and S be topologically mixing. Let U_1, V_1 be nonempty open subsets of X, and let U_2, V_2 be nonempty open subsets of Y. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$T^n(U_1) \cap V_1 \neq \emptyset \qquad \text{and} \qquad S^m(U_2) \cap V_2 \neq \emptyset$$

for all $n > N_1$ and $m > N_2$. Choose some $N > N_1, N_2$. Then for all $n > N$,

$$T^n(U_1) \cap V_1 \neq \emptyset \qquad \text{and} \qquad S^n(U_2) \cap V_2 \neq \emptyset$$

which implies $T \oplus S$ is topologically mixing. The converse is obvious.
(iii) Let $Y = X$ and $S = T$ in (ii). Then T is topologically mixing if, and only if, $T \oplus T$ is topologically mixing which implies that T satisfies the hypercyclic criterion by Theorem 3.1.3. $\square$

We note that (iii) above has been obtained by another approach in [15, Theorem 2.7]. The converse of (iii) holds if the sequence (n_k) in the hypercyclic criterion satisfies the *syndetic condition*, that is, $\sup_k\{n_{k+1} - n_k\} < \infty$, which has been proved in [25, Theorem 1.1] and the result is stated below.

Theorem 3.1.5 *Let an operator T satisfy the hypercyclic criterion for a syndetic sequence. Then T is topologically mixing.*

We note that an operator satisfies the hypercyclic criterion for a syndetic sequence if, and only if, it does so for the full sequence. We refer to [15, Corollary 2.8] for a proof.

Hypercyclicity was motivated by the concept of cyclicity in operator theory. A vector $x \in X$ is called *cyclic* if the linear span of its orbit $\{x, Tx, \dots, T^n x, \dots\}$ is dense in X. Accordingly, x is called *supercyclic* if the set

$$\{tT^n x : t \in \mathbb{C}, n \in \mathbb{N}_0\} = \bigcup_{n \in \mathbb{N}_0} \mathbb{C}T^n x$$

is dense in X in which case T also has a dense set of supercyclic vectors [42]. However, supercyclic and hypercyclic operators have a much richer structure than cyclic operators. For instance, we recall a result of [1] which asserts that if T is supercyclic, then so is T^n for every $n \geq 1$. This is not true for cyclic operators in general; for example, any power $n \geq 2$ of the forward shift on $\ell^p(\mathbb{N}_0)$ is not cyclic. Indeed, let the forward shift $T : \ell^p(\mathbb{N}_0) \to \ell^p(\mathbb{N}_0)$ be defined by $Te_n = e_{n+1}$ $(n \geq 0)$ for the canonical basis $\{e_n : n \in \mathbb{N}_0\}$. We have

$$\{e_0, Te_0, \ldots, T^n e_0, \ldots\} = \{e_n : n \in \mathbb{N}_0\}$$

whose linear span is dense in $\ell^p(\mathbb{N}_0)$. In contrast, for any $j \in \mathbb{N}_0$, the linear span of

$$\{e_j, T^2 e_j, \ldots, T^{2n} e_j, \ldots\} = \{e_{2n+j} : n \in \mathbb{N}_0\}$$

is not dense. A sufficient condition, a supercyclic criterion, for supercyclicity has been given in [42, Theorem 2.2], which is stated below.

Theorem 3.1.6 *Let (α_n) be a sequence of nonzero complex numbers. Let T be a bounded linear self-map on a Banach space X and satisfy the following criterion:*

(i) *$(\alpha_n T^n)$ admits a subsequence $(\alpha_{n_k} T^{n_k})$ converging to zero pointwise on a dense subset of X;*

(ii) *there is a dense subset Y of X, and a map $S : Y \to Y$ such that $(\frac{1}{\alpha_{n_k}} S^{n_k})$ tends to zero pointwise on Y and TS is the identity on Y.*

Then T is supercyclic and there is a supercyclic vector $x \in X$ such that $\{\alpha_{n_k} T^{n_k} x\}_{k \geq 1}$ is dense in X.

In the above criterion, if there is a supercyclic vector $x \in X$ such that $\{\alpha_{n_k} T^{n_k} x\}_{k \geq 1}$ is dense in X, then there is a dense subset D of supercyclic vectors satisfying $\{\alpha_{n_k} T^{n_k} x\}_{k \geq 1}$ is dense in X for all $x \in D$ [42]. If an operator T satisfies this criterion, we will say that T satisfies the *supercyclic criterion for the sequence* (α_{n_k}) or say that T is *supercyclic with respect to* (α_{n_k}). Hypercyclic criterion can be seen as a special case if we take $\alpha_n = 1$ for each n although in general, supercyclic operators need not be hypercyclic. The criterion will be used to study supercyclic convolution operators in the next section.

Recently, frequent hypercyclicity has been introduced in [4]. A vector x in a Banach space X is a hypercyclic vector for an operator $T \in B(X)$ if its orbit meets every nonempty open subset U of X. Bayart and Grivaux [4] call a vector $x \in X$ *frequently hypercyclic* if its orbit meets every such set U 'often' in the sense of positive lower density. A strictly increasing sequence (n_k) of positive integers is of *positive lower density* if

$$\sup_{k\geq 1} \frac{n_k}{k} < \infty.$$

A vector $x \in X$ is *frequently hypercyclic* for an operator T on a Banach space X if for every nonempty open subset U of X, there is a strictly increasing sequence (n_k) of positive integers and some $C > 0$ such that

$$n_k \leq Ck \qquad \text{and} \qquad T^{n_k}x \in U \qquad \text{for all } k \in \mathbb{N}.$$

The following frequently hypercyclic criterion has been proved in [4, Theorem 2.1].

Theorem 3.1.7 *Let T be a bounded operator on a Banach space X. Let there be a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that*

(i) *the series $\sum_n \|T^n x\|$ converges for all $x \in X_0$;*

(ii) *the series $\sum_n \|S^n x\|$ converges for all $x \in X_0$;*

(iii) *$TSx = x$ for all $x \in X_0$.*

Then T is frequently hypercyclic.

The above criterion is stronger than the hypercyclic criterion. Indeed, if an operator T satisfies the frequently hypercyclic criterion, then $T^n x \to 0$ and $S^n x \to 0$

for all $x \in X_0$. This implies that T is topologically mixing by Theorem 3.1.5. In fact, by [12, Remark 2.2], T is also chaotic.

We now give some well-known examples of hypercyclic operators in Banach spaces. Let $B : \ell^p(\mathbb{N}_0) \to \ell^p(\mathbb{N}_0)$ $(1 \leq p < \infty)$ be the unilateral backward shift defined by $B(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. Rolewicz [45] was the first to study hypercyclic operators on Banach spaces and showed that a scalar multiple λB is hypercyclic for any complex number λ with $|\lambda| > 1$. In fact, λB satisfies the hypercyclic criterion. If we define $S : \ell^p(\mathbb{N}_0) \to \ell^p(\mathbb{N}_0)$ by $S(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots)$, then λB satisfies the hypercyclic criterion for the full sequence with respect to the sequence $(\frac{1}{\lambda^n} S^n)$. Moreover, λB is frequently hypercyclic [4]. We note that B itself is supercyclic but not hypercyclic.

Shift operators and their generalizations have remained a main source of examples of hypercyclic operators. Hypercyclicity of generalized backward shifts on Banach spaces have been considered in [33, Theorem 3.6]. One of the most useful examples is bilateral weighted shifts. Given a positive bounded weight sequence $(a_n)_{n \in \mathbb{Z}}$ and the canonical basis $\{e_n : n \in \mathbb{Z}\}$ for $\ell^p(\mathbb{Z})$, hypercyclicity of a bilateral weighted shift $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ $(1 \leq p < \infty)$ defined by

$$Te_n = a_n e_{n+1} \qquad (a_n > 0) \tag{3.1}$$

has been characterised by Salas [47, Theorem 2.1] in terms of the weight (a_n).

Theorem 3.1.8 *Let T be a bilateral weighted shift defined by the weight (a_n). Then T is hypercyclic if and only if given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that for all $|j| \leq q$*

$$\prod_{s=0}^{n-1} a_{j+s} < \varepsilon \qquad \textit{and} \qquad \prod_{s=1}^{n} a_{j-s} > \frac{1}{\varepsilon}.$$

The above weight condition has been modified in [42] to characterise supercyclic bilateral weighted shifts. Costakis and Samarino [25, Theorem 1.2] have used Theorem 3.1.5 to characterise topologically mixing bilateral weighted shifts, with a stronger weight condition.

Another classic example is unilateral weighted backward shifts. Given a positive bounded weight sequence $(w_n)_{n\in\mathbb{N}_0}$ and the canonical basis $\{e_n : n \in \mathbb{N}_0\}$ of $\ell^p(\mathbb{N}_0)$, the unilateral weighted backward shift $B_w : \ell^p(\mathbb{N}_0) \to \ell^p(\mathbb{N}_0)$ is given by

$$B_w e_n = w_n e_{n-1} \quad \text{for} \quad n \geq 1 \qquad \text{and} \qquad B_w e_0 = 0. \tag{3.2}$$

A characterization of hypercyclic unilateral weighted backward shifts in terms of (w_n) has also been given in [47, Theorem 2.8]. For frequently hypercyclic unilateral weighted backward shifts, Bayart and Grivaux [4] have shown that if the series

$$\sum_{n\geq 1} \frac{1}{(w_1 w_2 \dots w_n)^p}$$

is convergent, then B_w is frequently hypercyclic. We will give a similar result for frequently hypercyclic bilateral weighted shifts.

Motivated by the above examples and following a recent study of convolution operators on groups and homogeneous spaces in [19], it is natural to consider the question of hypercyclicity for these operators. Hypercyclicity of convolution operators on spaces of ultradifferentiable functions has been studied in [11]. Although Birkhoff's seminal result [10] shows the hypercyclicity of the translation operator on the space of entire functions, in contrast, a translation operator, or a convolution operator by a measure of unit mass, on L^p spaces of locally compact groups is never hypercyclic.

In the next section, we give a sufficient condition for a weighted translation operator on the L^p space of a homogeneous space to be hypercyclic. This condition is also necessary in the discrete case which subsumes the result of Salas in Theorem 3.1.8, and further, it is equivalent to hereditary hypercyclicity of the weighted translation operator. By strengthening the condition and analogous arguments, we also characterise topologically mixing weighted translation operators which extends the result in [25, Theorem 1.2]. Supercyclic weighted translation operators on discrete homogeneous spaces can be described completely as well in terms of their weights.

3.2 Weighted translation operators

We now study hypercyclicity of a weighted convolution operator $T_{a,w}$, defined by a unit point mass δ_a with $a \in G$ and a weight w, on a homogeneous space of a group G. A convolution operator T_{δ_a} defined by δ_a is just a translation operator by a.

In the sequel, G will be a locally compact second countable group with identity e and a right invariant Haar measure λ which is the counting measure if G is discrete. We note that G is a union of a nested sequence

$$G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$$

of compact sets with G_n contained in the interior of G_{n+1}. Let H be a compact subgroup of G. We consider the right coset space G/H and the Lebesgue spaces $L^p(G/H)$ $(1 \leq p < \infty)$ with respect to the G-invariant measure $\nu = \lambda \circ q^{-1}$ on G/H, as in Section 2.2. Given $\sigma \in M(G)$, as in (2.2) and below, we consider the convolution operator $T_\sigma : L^p(G/H) \longrightarrow L^p(G/H)$ given by

$$(T_\sigma f)(Hx) = (f * \sigma)(Hx) = \int_G f(Hxy^{-1})d\sigma(y) \qquad (f \in L^p(G/H))$$

ν-almost everywhere. As in Section 2.2, we have $\|T_\sigma\| \leq \|\sigma\|$ and hence T_σ is not hypercyclic if $\|\sigma\| \leq 1$.

A continuous function $w : G \to (0, \infty)$ is called a *weight for G/H* if it satisfies

$$w(hx) = w(x) \qquad (x \in G, h \in H) \tag{3.3}$$

so that $w'(Hx) := w(x)$ is a well-defined function on G/H. If such a weight w is in $L^\infty(G)$, we can define a *weighted convolution operator*

$$T_{\sigma,w} : f \in L^p(G/H) \mapsto T_{\sigma,w} f \in L^p(G/H)$$

where

$$T_{\sigma,w} f(Hx) = w(x)(f * \sigma)(Hx) \qquad (f \in L^p(G/H)).$$

Thus $T_{\sigma,w} = M_{w'}T_\sigma$ where $M_{w'}$ is the multiplication by w'. The operator $T_{\sigma,w}$ is not hypercyclic if $\|\sigma\|\|w\|_\infty \leq 1$. One can also consider the weighted convolution operator $\widetilde{T}_{\sigma,w} : L^p(G) \longrightarrow L^p(G)$ with $\widetilde{T}_{\sigma,w} f = w(f * \sigma)$. It is a '*lift*' of $T_{\sigma,w}$ in the following commutative diagram:

$$\begin{tikzcd}
L^p(G) \arrow[r, "\widetilde{T}_{\sigma,w}"] & L^p(G) \arrow[d, "Q"] \\
L^p(G/H) \arrow[u, "J"] \arrow[r, "T_{\sigma,w}"] & L^p(G/H).
\end{tikzcd}$$

We have $T_{\sigma,w} = Q \circ \widetilde{T}_{\sigma,w} \circ J$ and $T_{\sigma,w}Q = Q\widetilde{T}_{\sigma,w}$ as in Section 2.2. These equalities enable us to prove the following simple lemma.

Lemma 3.2.1 *Let $1 \leq p < \infty$ and let $\widetilde{T}_{\sigma,w}$ and $T_{\sigma,w}$ be the weighted convolution operators on $L^p(G)$ and $L^p(G/H)$ respectively.*

(i) *If* $\widetilde{T}_{\sigma,w}$ *is (frequently) hypercyclic, then* $T_{\sigma,w}$ *is (frequently) hypercyclic.*

(ii) *If* $\widetilde{T}_{\sigma,w}$ *is chaotic, then* $T_{\sigma,w}$ *is chaotic.*

(iii) *If* $\widetilde{T}_{\sigma,w}$ *is topologically mixing, then* $T_{\sigma,w}$ *is topologically mixing.*

Proof. Since Q is surjective, each $h \in L^p(G/H)$ equals Qf for some $f \in L^p(G)$. (i) If $\widetilde{T}_{\sigma,w}$ possesses a (frequently) hypercyclic vector $g \in L^p(G)$, then Qg is a (frequently) hypercyclic vector for $T_{\sigma,w}$ on $L^p(G/H)$. This follows from $\|Q\| \leq 1$ and the fact that for all $n \in \mathbb{N}$, we have

$$\|T^n_{\sigma,w}(Qg) - h\| = \|Q\widetilde{T}^n_{\sigma,w}g - Qf\| = \|Q(\widetilde{T}^n_{\sigma,w}g - f)\| \leqslant \|\widetilde{T}^n_{\sigma,w}g - f\|.$$

(ii) Let $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ be the sets of periodic points for $\widetilde{T}_{\sigma,w}$ and $T_{\sigma,w}$ respectively. If $g \in \widetilde{\mathcal{P}}$, then $Qg \in \mathcal{P}$ by

$$Qg = Q(\widetilde{T}^n_{\sigma,w}g) = T^n_{\sigma,w}(Qg)$$

for some $n \in \mathbb{N}$. Let $\widetilde{T}_{\sigma,w}$ be chaotic. Then $T_{\sigma,w}$ is chaotic since for any $h \in L^p(G/H)$ with $h = Qf$ and $\varepsilon > 0$, there exists $g \in \widetilde{\mathcal{P}}$ such that

$$\|Qg - h\| = \|Qg - Qf\| \leq \|g - f\| < \varepsilon.$$

(iii) Let $\widetilde{T}_{\sigma,w}$ be topologically mixing. Then for two any non-empty open sets $U', V' \subset L^p(G/H)$, there exist two non-empty open sets $U, V \subset L^p(G)$, a sequence (g_n) in $L^p(G)$ and $N \in \mathbb{N}$ such that

$$U' = QU, \; V' = QV \quad \text{and} \quad g_n \in \widetilde{T}^n_{\sigma,w}U \cap V$$

for all $n > N$. This implies

$$Qg_n \in Q(\widetilde{T}^n_{\sigma,w}U) = T^n_{\sigma,w}(QU) \quad \text{and} \quad Qg_n \in QV.$$

Hence we have $T^n_{\sigma,w}U' \cap V' \neq \emptyset$ for all $n > N$. □

Given a weight $w \in L^\infty(G)$ for G/H, the weighted convolution operator $T_{\delta_a,w}$ is written simply $T_{a,w}$ which is a weighted translation operator. If we also have $w^{-1} \in L^\infty(G)$, then the weighted convolution operator $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$ is the inverse of $T_{a,w}$. We write $S_{a,w}$ for $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$ to simplify notation. Thus, for each $f \in L^p(G/H)$, we have

$$\begin{aligned} T_{a,w}f(Hx) &= w(x)f(Hxa^{-1}), \\ S_{a,w}f(Hx) &= \frac{1}{w(xa)}f(Hxa). \end{aligned}$$

Without the assumption of $w^{-1} \in L^\infty(G)$, one can still define the operator $S_{a,w}$ on the subspace $C_c(G/H) \subset L^p(G/H)$ and we will use the same notation for this map since no confusion is likely. The same remark applies to $T_{a,w}$ if $w \notin L^\infty(G)$.

By a similar computation as in (2.3), the dual map $T^*_{\sigma,w} : L^q(G/H) \longrightarrow L^q(G/H)$ is given by $T^*_{\sigma,w}(g) = T_{\tilde{\sigma}}(wg)$ for $g \in L^q(G/H)$. In particular, if $\sigma = \delta_a$, we have

$$T^*_{a,w}(g) = T_{\delta_{a^{-1}}}(wg) = T_{a^{-1},w*\delta_{a^{-1}}}(g) \qquad (g \in L^q(G/H))$$

and $T^*_{a,w}$ is a weighted convolution operator on $L^q(G/H)$.

We note that the translation operator T_a is not hypercyclic. However if one considers the weighted translation operator $T_{a,w}$, then hypercyclicity can occur for certain weights. Indeed, we are going to describe these weights for the homogeneous space G/H, and show, for a discrete group G, these are the only weights making $T_{a,w}$ hereditarily hypercyclic.

Proposition 3.2.2 *Let G be a locally compact second countable group with $a \in G$. Let $w : G \to (0, \infty)$ be a weight for G/H satisfying $w \in L^\infty(G)$. Let $1 \leq p < \infty$ and $T_{a,w}$ be the weighted convolution operator on $L^p(G/H)$ defined above. Then condition* (ii) *below implies* (i).

(i) $T_{a,w}$ *is hereditarily hypercyclic.*

(ii) *Both sequences* (*depending on* a)

$$w_n := \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \quad \text{and} \quad \overline{w}_n := \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit respectively subsequences (w_{n_k}) *and* $(\overline{w}_{n_k})$ *which converge pointwise to* 0 λ*-a.e. and are uniformly bounded on each non-null compact subset* K *of* G.

Proof. Let (w_{n_k}) and $(\overline{w}_{n_k})$ be subsequences of (w_n) and $(\overline{w}_n)$ respectively, satisfying (ii). We show $T_{a,w}$ satisfies the hypercyclic criterion.

We make use of the sequence of maps $S_{a,w}^{n_k} : C_c(G/H) \longrightarrow L^p(G/H)$. Let $f \in C_c(G/H) \setminus \{0\}$ with compact support $\operatorname{supp} f$. Then we have $T_{a,w}^{n_k}(S_{a,w}^{n_k} f) = f$. We show that $\|T_{a,w}^{n_k} f\|_p \to 0$ as $n_k \to \infty$. There exists a compact set $K \subset G$ with $q(K) = \operatorname{supp} f$ (see, for instance, [31, 2.46]). It follows that $q^{-1}(\operatorname{supp} f) = q^{-1}(q(K)) = HK$ which is compact and non-null. Let (w_{n_k}) be bounded on HK by M say. Let $\varepsilon > 0$ and choose, by Egoroff's theorem, a Borel set $E \subset HK$ such that $\lambda(HK \setminus E) < \frac{\varepsilon}{M^p \|f\|_\infty^p}$ and $(w_{n_k}^p)$ converges to 0 uniformly on E. There exists $N \in \mathbb{N}$ such that $w_{n_k}^p < \frac{\varepsilon}{\|f\|_p^p}$ on E for $n_k > N$. We have, by change of variables,

$$\begin{aligned}
\|T_{a,w}^{n_k} f\|_p^p &= \int_{G/H} |T_{a,w}^{n_k} f(Hx)|^p d\nu(Hx) \\
&= \int_{(\operatorname{supp}\ f)a^{n_k}} |w(x)w(xa^{-1}) \cdots w(xa^{-(n_k-1)})|^p |f(Hxa^{-n_k})|^p d\nu(Hx) \\
&= \int_{HKa^{n_k}} |w(x)w(xa^{-1}) \cdots w(xa^{-(n_k-1)})|^p |f(Hxa^{-n_k})|^p d\lambda(x) \\
&= \int_{HK} |w(xa^{n_k})w(xa^{n_k-1}) \cdots w(xa)|^p |f(Hx)|^p d\lambda(x) \\
&= \int_E w_{n_k}(x)^p |f(Hx)|^p d\lambda(x) + \int_{HK\setminus E} w_{n_k}(x)^p |f(Hx)|^p d\lambda(x) \\
&\leq \frac{\varepsilon}{\|f\|_p^p} \|f\|_p^p + M^p \|f\|_\infty^p \lambda(HK \setminus E) < 2\varepsilon
\end{aligned}$$

for $n_k > N$. Similar arguments using the sequence $(\overline{w}_{n_k})$ yield

$$\|S_{a,w}^{n_k} f\|_p^p = \int_{HKa^{-n_k}} \frac{1}{|w(xa)w(xa^2)\cdots w(xa^{n_k})|^p} |f(Hxa^{n_k})|^p d\lambda(x) \longrightarrow 0.$$

Hence $T_{a,w}^{n_k}$ satisfies the hypercyclic criterion in Theorem 3.1.2 since $C_c(G/H)$ is dense in $L^p(G/H)$. Therefore $T_{a,w}$ is hereditarily hypercyclic by Theorem 3.1.3. □

Remark 3.2.3 If we have $w^{-1} \in L^\infty(G)$ instead of $w \in L^\infty(G)$, then condition (ii) implies that $S_{a,w}$ is hereditarily hypercyclic on $L^p(G/H)$, by switching the role of $T_{a,w}$ and $S_{a,w}$ in the above proof.

We note that, if $a = e$, then condition (ii) in Proposition 3.2.2 fails and in fact, we have $\overline{w}_n = w_n^{-1} = w^{-n}$ in this case. Also, a pointwise convergence sequence of continuous functions need not be uniformly bounded on a compact set. For example, the sequence $w_n(x) = 2n^2 x e^{-n^2 x^2}$ is not uniformly bounded on $[0, 1]$.

There are examples of hypercyclic operators with hypercyclic dual [44, 46]. The following result shows that $T_{a,w}$ and its dual $T_{a,w}^*$ can both be hypercyclic for certain weights w.

Corollary 3.2.4 *The dual $T_{a,w}^*$ of a weighted translation operator $T_{a,w} : L^p(G/H) \longrightarrow L^p(G/H)$ is hypercyclic if the weight $w * \delta_{a^{-1}}$ satisfies condition* (ii) *of Proposition 3.2.2 with a^{-1} in place of a.*

Example 3.2.5 Fix $t \in (0, 1)$. We define a weight $w : \mathbb{R} \to (0, \infty)$ for $(\mathbb{R}, +)$ by

$$w(x) = \begin{cases} t & \text{if } 1 \leq x \\ t^x & \text{if } -1 \leq x \leq 1 \\ \frac{1}{t} & \text{if } x \leq -1. \end{cases}$$

Then w and w^{-1} are bounded and continuous on $\mathbb{R}$, with w satisfying condition (ii) in Proposition 3.2.2 if $a > 0$. Indeed, let $K = [b, c]$ say. Pick $n_0 \in \mathbb{N}$ such that $b + n_0 a > 1$. Since w is decreasing, we have

$$\begin{aligned} 0 &< w_n(x) = w(x+a)w(x+2a)\cdots w(x+na) \\ &\leq w(b+a)w(b+2a)\cdots w(b+na) \\ &\leq w(b+a)w(b+2a)\cdots w(b+n_0 a) \qquad (x \in K,\ n \geq n_0). \end{aligned}$$

It follows that (w_n) is uniformly bounded on K by some constant M. For each $x \in [b, c]$, we have $w(x+s) = t$ for all $s \geq n_0 a$. This implies, for $n > n_0$,

$$\begin{aligned} w_n(x) &= w(x+a)w(x+2a)\cdots w(x+n_0 a)w(x+(n_0+1)a)\cdots w(x+na) \\ &\leq Mt^{n-n_0} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence (w_n) converges to 0 uniformly on $[b, c]$. For the sequence $(\overline{w}_n)$, we have

$$\begin{aligned} \overline{w}_n(x) &= \frac{1}{w(x)w(x-a)\cdots w(x-(n-1)a)} \\ &\leq \frac{1}{w(c)w(c-a)\cdots w(c-(n-1)a)} \\ &\leq \frac{1}{w(c)w(c-a)\cdots w(c-(n_1-1)a)} \qquad (x \in K,\ n \geq n_1) \end{aligned}$$

where n_1 is chosen so that $c-(n_1-1)a < -1$. It follows that $(\overline{w}_n) \to 0$ uniformly on $[b, c]$ too.

In fact, the above example is a special case of the following lemma.

Lemma 3.2.6 *Let w be a weight for a locally compact second countable group G. Let $a \in G$, and let $(w_n), (\overline{w}_n)$ be as in Proposition 3.2.2. The following conditions are equivalent.*

(i) *Given $\varepsilon > 0$, a compact set $D \subset G$ and $N \in \mathbb{N}$, there exists $m > N$ satisfying $w_m(x) < \varepsilon$ and $\overline{w}_m(x) < \varepsilon$ for all $x \in D$.*

(ii) *Both sequences* (w_n) *and* $(\overline{w}_n)$ *admit subsequences* (w_{n_k}) *and* $(\overline{w}_{n_k})$ *which converge uniformly to* 0 *on each compact subset* K *of* G.

If G *is discrete,* D *can be replaced by a singleton.*

Proof. We show (i) $\Rightarrow$ (ii). Since G is a union $\bigcup_{k=1}^{\infty} G_k$ of nested compact sets G_k with G_k contained in the interior of G_{k+1}, it suffices to prove convergence on G_j for each $j \in \mathbb{N}$.

Let $\varepsilon = \frac{1}{2}$ and $D = G_1$. Then there exists n_1 such that $w_{n_1}(x) < \frac{1}{2}$ and $\overline{w}_{n_1}(x) < \frac{1}{2}$ for all $x \in G_1$. Inductively, for each $k > 1$, there exists $n_k > n_{k-1}$ such that $w_{n_k}(x) < \frac{1}{2^k}$ and $\overline{w}_{n_k}(x) < \frac{1}{2^k}$ for all $x \in G_k$.

Now let $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ with $k_0 > j$ and $\frac{1}{2^{k_0}} < \varepsilon$. Then, for all $k > k_0$, we have

$$w_{n_k}(x) < \frac{1}{2^k} < \frac{1}{2^{k_0}} < \varepsilon \quad \text{and} \quad \overline{w}_{n_k}(x) < \varepsilon$$

on $G_k \supset G_j$. Hence (w_{n_k}) and $(\overline{w}_{n_k})$ converge uniformly to 0 on G_j. □

We now consider discrete groups and derive necessary and sufficient conditions for a weighted translation operator to be hypercyclic. A *torsion element* of a group G is an element of finite order.

Lemma 3.2.7 *Let* G *be a discrete group and* $a \in G$. *Then* a *is not a torsion element if, and only if, for any finite subset* $D \subset G$, *there exists* $N \in \mathbb{N}$ *such that* $D \cap Da^{\pm n} = \emptyset$ *for* $n > N$.

Proof. Given that a is not a torsion element, we observe that, for every $d \in D$, there exists N_d such that $da^n \notin D$ for $n > N_d$. Otherwise, there is some $d \in D$ such that $da^{n_j} \in D$ for a strictly increasing sequence (n_j) in $\mathbb{N}$. Since D is finite, we must have $da^{n_j} = da^{n_k}$ for some $n_j \neq n_k$ which contradicts the fact that a

is not a torsion element. Let $N = max\{N_d : d \in D\}$. Then $D \cap Da^n = \emptyset$ for $n > N$. The condition $D \cap Da^{-n} = \emptyset$ can be proved similarly.

On the other hand, if $a \in G$ is a torsion element with order m, then for any finite subset $D \subset G$, there exist infinitely many n's such that $D \cap Da^n \neq \emptyset$. Indeed, $D \cap Da^n = D \neq \emptyset$ for $n \in m\mathbb{Z}$. □

Theorem 3.2.8 *Let G be a discrete group and H a finite subgroup. Let $a \in G$ which is not a torsion element. Let $w : G \to (0, \infty)$ be a weight for G/H such that $w \in \ell^\infty(G)$. Let $1 \le p < \infty$ and $T_{a,w}$ be the weighted convolution operator on $\ell^p(G/H)$ defined by a and w. The following conditions are equivalent.*

(i) *$T_{a,w}$ is hypercyclic.*

(ii) *$T_{a,w}$ is hereditarily hypercyclic.*

(iii) *Both sequences (depending on a)*

$$w_n = \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \qquad \text{and} \qquad \overline{w}_n = \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit subsequences (w_{n_k}) and $(\overline{w}_{n_k})$ which converge to 0 pointwise in G.

In particular, if G is torsion free, then the above conditions are equivalent for all $a \in G \setminus \{e\}$.

Proof. By Proposition 3.2.2, (iii) implies (ii) since a compact subset of a discrete group is finite. Since (ii) implies (i), we only need to show (i) implies (iii).

Let $T_{a,w}$ be hypercyclic. Let $\varepsilon > 0$ and $z \in G$. Fix $N \in \mathbb{N}$. Let $\chi_z \in \ell^p(G/H)$ be the characteristic function

$$\chi_z(Hx) = \begin{cases} 1 & \text{if } x \in Hz \\ 0 & \text{otherwise.} \end{cases}$$

Choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$. Since the set of hypercyclic vectors for $T_{a,w}$ is dense, there exist a hypercyclic vector $f \in \ell^p(G/H)$ for $T_{a,w}$ and some $m > N$ such that

$$\|f - \chi_z\|_p < \delta \quad \text{and} \quad \|T_{a,w}^m f - \chi_z\|_p < \delta.$$

By Lemma 3.2.7, we may choose m sufficiently large so that $Hz \cap Hza^{\pm m} = \emptyset$. Since

$$\|f - \chi_z\|_p^p = \sum_{Hx \in G/H} |f(Hx) - \chi_z(Hx)|^p \nu(Hx) = \sum_{x \in G} |f(Hx) - \chi_z(Hx)|^p < \delta^p$$

where $\nu(Hx) = \lambda(H)$, we have

$$|f(Hx) - \chi_z(Hx)| < \delta \qquad (x \in G).$$

This gives

$$\begin{aligned} |f(Hx)| &> 1 - \delta \qquad && \text{for } x \in Hz, \\ |f(Hx)| &< \delta && \text{for } x \notin Hz. \end{aligned}$$

From $\|T_{a,w}^m f - \chi_z\|_p < \delta$, we also deduce that

$$\left| w(x) w(xa^{-1}) \cdots w(xa^{-(m-1)}) f(Hxa^{-m}) - \chi_z(Hx) \right| < \delta \qquad (x \in G). \tag{3.4}$$

In particular,

$$\overline{w}_m(z)^{-1} |f(Hza^{-m})| > 1 - \delta.$$

Since $Hz \cap Hza^{-m} = \emptyset$, we have

$$\overline{w}_m(z) < \frac{|f(Hza^{-m})|}{1 - \delta} < \frac{\delta}{1 - \delta} < \varepsilon.$$

From (3.4), we have

$$\left| w(xa^m) w(xa^{m-1}) \cdots w(xa) f(Hx) - \chi_z(Hxa^m) \right| < \delta \qquad (x \in G)$$

and hence, as $Hz \cap Hza^m = \emptyset$, one obtains

$$w_m(z) |f(Hz)| < \delta.$$

It follows that

$$w_m(z) < \frac{\delta}{|f(Hz)|} < \frac{\delta}{1-\delta} < \varepsilon.$$

This proves that (w_n) and $(\overline{w}_n)$ satisfy condition (i) in Lemma 3.2.6 for each point $z \in G$, and hence admit subsequences (w_{n_k}) and $(\overline{w}_{n_k})$ which converge pointwise to 0 on G. □

Remark 3.2.9 The above result implies that if $T_{a,w} : \ell^p(G/H) \longrightarrow \ell^p(G/H)$ is hypercyclic for some $p \in [1, \infty)$, then it is so for all $p \in [1, \infty)$. As in Remark 3.2.3, if $w^{-1} \in \ell^\infty(G)$, Theorem 3.2.8 applies to $S_{a,w}$.

Corollary 3.2.10 *Let* $a \in G$ *and* $w \in \ell^\infty(G)$ *be as in Theorem 3.2.8 for the homogeneous space* G/H*. Then* $T_{a,w} : \ell^p(G/H) \longrightarrow \ell^p(G/H)$ *is hypercyclic if, and only if, the lift* $\widetilde{T}_{a,w} : \ell^p(G) \longrightarrow \ell^p(G)$ *is hypercyclic.*

Proof. Let $H = \{e\}$ in Theorem 3.2.8. Then $\widetilde{T}_{a,w}$ is hypercyclic if, and only if, the condition (iii) in Theorem 3.2.8 is satisfied. □

Example 3.2.11 The weighted shift with weight sequence (a_n) studied in [47] is the weighted convolution operator $S_{a,w}$ on $\ell^2(\mathbb{Z})$ with $H = \{0\}$, $a = -1 \in \mathbb{Z}$ and the weight $w(n) = a_n^{-1}$. By Remark 3.2.9 and Lemma 3.2.6, $S_{a,w}$ is hypercyclic if and only if given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that for all $|j| \le q$, we have

$$\prod_{s=1}^{n} w(j-s) = w_n(j) < \varepsilon \quad \text{and} \quad \prod_{s=0}^{n-1} w(j+s) = \overline{w}_n(j)^{-1} > \frac{1}{\varepsilon}$$

which is the condition in Theorem 3.1.8.

In the remaining section, we let $p \in [1, \infty)$ be fixed, but arbitrary. We now consider topological mixing for translation operators. Using similar arguments as in the proof of Theorem 3.2.8, one can also characterise topologically mixing weighted translation operators on $\ell^p(G/H)$ which extends a result in [25, Theorem 1.2] for $\ell^2(\mathbb{Z})$.

Proposition 3.2.12 *Let $T_{a,w} : L^p(G/H) \to L^p(G/H)$ be the operator defined in Proposition 3.2.2. Then condition* (ii) *below implies* (i).

(i) *$T_{a,w}$ is topologically mixing.*

(ii) *Both sequences (depending on a)*

$$w_n := \prod_{s=1}^{n} w * \delta^s_{a^{-1}} \quad \text{and} \quad \overline{w}_n := \left(\prod_{s=0}^{n-1} w * \delta^s_a\right)^{-1}$$

converge pointwise to 0 *λ-a.e. and are uniformly bounded on each non-null compact subset K of G.*

Proof. Using similar arguments as in the proof of Proposition 3.2.2, we have that T satisfies the hypercyclic criterion for the full sequence which is syndetic. Therefore we have (ii) implies (i) by Theorem 3.1.5. □

Corollary 3.2.13 *The dual $T^*_{a,w}$ of a weighted translation operator $T_{a,w} : L^p(G/H) \longrightarrow L^p(G/H)$ is topologically mixing if the weight $w * \delta_{a^{-1}}$ satisfies condition* (ii) *for a^{-1} in Proposition 3.2.12.*

We characterise topologically mixing weighted translation operators on discrete groups.

Theorem 3.2.14 *Let G, H, a and w be as in Theorem 3.2.8, and let $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ be the operator defined in Theorem 3.2.8. Then the following conditions are equivalent.*

(i) *$T_{a,w}$ is topologically mixing.*

(ii) *Both sequences (depending on a)*

$$w_n = \prod_{s=1}^{n} w * \delta^s_{a^{-1}} \quad \text{and} \quad \overline{w}_n = \left(\prod_{s=0}^{n-1} w * \delta^s_a\right)^{-1}$$

converge to 0 *pointwise in G.*

If G is torsion free, then the above conditions are equivalent for all $a \in G \setminus \{e\}$.

Proof. We see from Proposition 3.2.12 that condition (ii) implies (i). For the converse, let $\varepsilon > 0$ and fix $z \in G$ with the characteristic function $\chi_z \in \ell^p(G/H)$ as defined in the proof of Theorem 3.2.8. Choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$ and let $U = \{f \in \ell^p(G/H) : \|f - \chi_z\| < \delta\}$. By the topologically mixing assumption, there exists $N \in \mathbb{N}$ such that

$$T^n_{a,w}(U) \cap U \neq \emptyset \qquad (n > N).$$

We can therefore pick, for each $n > N$, a function $f_n \in U$ with $T^n_{a,w} f_n \in U$ which gives

$$\|f_n - \chi_z\|_p < \delta \quad \text{and} \quad \|T^n_{a,w} f_n - \chi_z\|_p < \delta.$$

Using this for each f_n and repeating the arguments in the proof of Theorem 3.2.8, we arrive at

$$\overline{w}_n(z) < \varepsilon \quad \text{and} \quad w_n(z) < \varepsilon$$

for all $n > N$, proving that (w_n) and $(\overline{w}_n)$ converge to 0 pointwise in G. ☐

Corollary 3.2.15 *Let $a \in G$ and $w \in \ell^\infty(G)$ be as in Theorem 3.2.8 for the homogeneous space G/H. Then $T_{a,w} : \ell^p(G/H) \longrightarrow \ell^p(G/H)$ is topologically mixing if, and only if, the lift $\widetilde{T}_{a,w} : \ell^p(G) \longrightarrow \ell^p(G)$ is topologically mixing.*

Proof. By Theorem 3.2.14. ☐

Remark 3.2.16 The weighted translation operator $T_{a,w}$ above and its dual $T^*_{a,w}$ can never be simultaneously topologically mixing since $T^*_{a,w} = T_{a^{-1}, w*\delta_{a^{-1}}}$ and for $a^{-1} \in G$, the two sequences for the weight $w * \delta_{a^{-1}}$ in condition (ii) above are given by

$$(w * \delta_{a^{-1}})_n = \prod_{s=1}^{n} (w * \delta_{a^{-1}}) * \delta^s_a = \overline{w}_n^{-1}$$

and $(\overline{w * \delta_{a^{-1}}})_n = w_n^{-1}$.

Modifying the weight condition and using similar arguments as in the proof of Theorem 3.2.8, one can also characterise supercyclic weighted translation operators on $\ell^p(G/H)$ which extends a result in [42, Proposition 2.8] for $\ell^2(\mathbb{Z})$.

Proposition 3.2.17 *Let $T_{a,w} : L^p(G/H) \to L^p(G/H)$ be the operator defined in Proposition 3.2.2 and (α_n) a sequence of nonzero complex numbers. Then condition* (ii) *below implies* (i).

(i) *$T_{a,w}$ is supercyclic with respect to (α_{n_k}).*

(ii) *Both sequences (depending on a)*

$$w_n := |\alpha_n| \prod_{s=1}^{n} w * \delta_{a^{-1}}^{s} \quad \textit{and} \quad \overline{w}_n := \left(|\alpha_n| \prod_{s=0}^{n-1} w * \delta_{a}^{s} \right)^{-1}$$

admit respectively subsequences (w_{n_k}) and $(\overline{w}_{n_k})$ which converge pointwise to 0 *λ-a.e. and are uniformly bounded on each non-null compact subset K of G.*

Proof. Using similar arguments as in the proof of Proposition 3.2.2, we have $\|\alpha_{n_k} T_{a,w}^{n_k} f\|_p \to 0$ and $\|\frac{1}{\alpha_{n_k}} S_{a,w}^{n_k} f\|_p \to 0$ for $f \in C_c(G/H) \setminus \{0\}$. ☐

Theorem 3.2.18 *Let $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ be the operator defined in Theorem 3.2.8 and (α_n) a sequence of nonzero complex numbers. The following conditions are equivalent.*

(i) *$T_{a,w}$ is supercyclic with respect to (α_{n_k}).*

(ii) *Both sequences (depending on a)*

$$w_n = |\alpha_n| \prod_{s=1}^{n} w * \delta_{a^{-1}}^{s} \quad \textit{and} \quad \overline{w}_n = \left(|\alpha_n| \prod_{s=0}^{n-1} w * \delta_{a}^{s} \right)^{-1}$$

admit subsequences (w_{n_k}) and $(\overline{w}_{n_k})$ which converge to 0 *pointwise in G.*

In particular, if G is torsion free, then the above conditions are equivalent for all $a \in G \setminus \{e\}$.

Proof. By Proposition 3.2.17, condition (ii) implies (i). For the converse, let $\varepsilon > 0$ and fix $z \in G$ with the characteristic function $\chi_z \in \ell^p(G/H)$ as defined in the proof of Theorem 3.2.8. Choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$. Since $T_{a,w}$ satisfies the supercyclic criterion for (α_{n_k}), there exist a supercyclic vector $f \in \ell^p(G/H)$ and some $m > N$ such that

$$\|f - \chi_z\|_p < \delta \quad \text{and} \quad \|\alpha_m T_{a,w}^m f - \chi_z\|_p < \delta.$$

Repeating the arguments in the proof of Theorem 3.2.8, we obtain condition (ii).
□

We now give a sufficient condition for a bilateral weighted shift to be frequently hypercyclic. Let $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ be defined by

$$Te_j = a_j e_{j+1}$$

and let $Se_j = \frac{1}{a_{j-1}} e_{j-1}$, where (e_j) is the canonical basis and both (a_j) and $(\frac{1}{a_j})$ are bounded sequences of positive real numbers.

Lemma 3.2.19 *Let $T, S : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ be bilateral weighted shifts with positive bounded weight sequences (a_j) and $(\frac{1}{a_j})$ respectively. Then T and S are frequently hypercyclic if given $q \in \mathbb{N}$, both series*

$$\sum_{n\geq 1} \left(\prod_{s=0}^{n-1} a_{j+s} \right) \qquad \textit{and} \qquad \sum_{n\geq 1} \left(\prod_{s=1}^{n} a_{j-s} \right)^{-1}$$

are convergent for all $|j| \leq q$.

Proof. Given $q \in \mathbb{N}$, consider $\{e_j : -q \leq j \leq q\}$. Then

$$T^n e_j = \left(\prod_{s=0}^{n-1} a_{j+s} \right) e_{j+n}, \qquad S^n e_j = \left(\prod_{s=1}^{n} a_{j-s} \right)^{-1} e_{j-n}$$

and $TSe_j = STe_j = e_j$. These imply that

$$\sum_{n\geq 1} \|T^n e_j\| = \sum_{n\geq 1} \left(\prod_{s=0}^{n-1} a_{j+s} \right)$$

and

$$\sum_{n\geq 1} \|S^n e_j\| = \sum_{n\geq 1} \left(\prod_{s=1}^{n} a_{j-s} \right)^{-1}$$

are convergent for all $|j| \leq q$. Hence T and S are frequently hypercyclic by Theorem 3.1.7. □

For weighted translation operators, we have the following result.

Lemma 3.2.20 *Let $T_{a,w} : L^p(G/H) \to L^p(G/H)$ be the operator defined in Proposition 3.2.2. Then condition* (ii) *below implies* (i).

(i) *$T_{a,w}$ is frequently hypercyclic.*

(ii) *There exist constant C_1, C_2 and r_1, r_2 with $0 < r_1, r_2 < 1$ such that both sequences (depending on a)*

$$w_n := \prod_{s=1}^{n} w * \delta_{a^{-1}}^s < C_1 r_1^n \quad \text{and} \quad \overline{w}_n := \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1} < C_2 r_2^n$$

on each non-null compact subset K of G.

Proof. Let $f \in C_c(G/H) \setminus \{0\}$ with compact support $\operatorname{supp} f$. As in the proof of Proposition 3.2.2, we have

$$\|T_{a,w}^n f\|_p^p = \int_{HK} |w(xa^n)w(xa^{n-1}) \cdots w(xa)|^p |f(Hx)|^p d\lambda(x) < (C_1 r_1^n \|f\|_p)^p.$$

Hence $\sum_n \|T_{a,w}^n f\|_p < \sum_n C_1 r_1^n \|f\|_p$. Similar arguments using the sequence $(\overline{w}_n)$ yield $\sum_n \|S_{a,w}^n f\|_p < \sum_n C_2 r_2^n \|f\|_p$. Hence $T_{a,w}$ satisfies the frequently hypercyclic criterion and therefore is frequently hypercyclic. □

Finally we conclude with a simple example of a hypercyclic operator $I+T$ on $\ell^2(\mathbb{Z})$ where T is quasi-nilpotent but not a weighted shift.

Example 3.2.21 Let $(u_n)_{n\in\mathbb{Z}}$ be the canonical basis of $\ell^2(\mathbb{Z})$. Let $(a_n)_{n\in\mathbb{N}}$ be a positive sequence. Define a linear operator $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$Tu_0 = 0, \qquad Tu_n = \begin{cases} a_{2n-1}u_{1-n} & \text{if} \quad n \geq 1 \\ a_{-2n}u_{-n} & \text{if} \quad n \leq -1. \end{cases}$$

Then $I+T$ is hereditarily hypercyclic on $\ell^2(\mathbb{Z})$ and this action may be expressed as follows:

$$(I+T)\begin{pmatrix} \cdot \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 0 & 0 & 0 & a_4 & \cdot \\ \cdot & 0 & 1 & 0 & a_2 & 0 & \cdot \\ \cdot & 0 & 0 & 1 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & a_1 & 1 & 0 & \cdot \\ \cdot & 0 & a_3 & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \cdot \end{pmatrix}.$$

In fact, $I+T$ is unitarily equivalent to a hereditarily hypercyclic operator $I+S : \ell^2(\mathbb{N}_0) \longrightarrow \ell^2(\mathbb{N}_0)$ where S is a positively weighted shift on $\ell^2(\mathbb{N}_0)$.

To see this, let (e_m) be the canonical basis of $\ell^2(\mathbb{N}_0)$ and $h : \mathbb{Z} \to \mathbb{N}_0$ the bijection

$$h(n) = \begin{cases} 2n-1 & \text{if} \quad n \geq 1 \\ -2n & \text{if} \quad n \leq 0. \end{cases}$$

Then h induces a unitary operator $U : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{N}_0)$ with $U(u_n) = e_{h(n)}$. Let $S = UTU^{-1}$. Then S is the weighted shift

$$Se_0 = 0, \qquad Se_m = a_m e_{m-1} \qquad (m \geq 1)$$

on $\ell^2(\mathbb{N}_0)$. By [47, Theorem 3.3], S is hypercyclic. In fact, S is hereditarily hypercyclic by [9, Corollary 2.9]. It follows that $I+T$ is hereditarily hypercyclic.

Finally, T is quasi-nilpotent if the sequence (a_n) decreases to 0. Indeed, let $x = (x_j) \in \ell^2(\mathbb{Z})$ and $\|x\| \leq 1$. Then

$$\begin{aligned}
\|S^k x\|^{\frac{2}{k}} &= \left\| \sum_{j=0}^{\infty} x_j (S^k e_j) \right\|^{\frac{2}{k}} = \left\| \sum_{j=k}^{\infty} x_j (S^k e_j) \right\|^{\frac{2}{k}} \\
&= \left\| \sum_{j=k}^{\infty} x_j (a_j a_{j-1} \cdot \cdot a_{j-(k-1)}) e_{j-k} \right\|^{\frac{2}{k}} \\
&= \left(\sum_{j=k}^{\infty} x_j^2 (a_{j-(k-1)} a_{j-(k-2)} \cdot \cdot a_j)^2 \right)^{\frac{1}{k}} \\
&= \left(x_k^2 (a_1 a_2 \cdot \cdot a_k)^2 + x_{k+1}^2 (a_2 a_3 \cdot \cdot a_{k+1})^2 + \cdots \right)^{\frac{1}{k}} \\
&\leq (a_1 a_2 \cdot \cdot a_k)^{\frac{2}{k}} \to 0
\end{aligned}$$

since $\|x\| \leq 1$ and the sequence $a_1 a_2 \cdot\cdot a_k$, $a_2 a_3 \cdot\cdot a_{k+1}$, $a_3 a_4 \cdot\cdot a_{k+2}$, $\cdots$ is decreasing.

3.3 Rotation and scalar multiples of operators

In this section, we study complex scalar multiples of weighted translation operators and determine when they are hypercyclic, topologically mixing and supercyclic. We show that these properties are preserved by rotations (that is, by multiplication by unit modulus scalars). Moreover, we show that a scalar multiple of a hypercyclic weighted translation operator is supercyclic.

In [38], León-Saavedra and Müller study rotations of hypercyclic operators and show that if an operator $T : X \to X$ is hypercyclic on a Banach space X, then βT is hypercyclic for $|\beta| = 1$. They construct a hypercyclic bilateral weighted shift T on $\ell^2(\mathbb{Z})$ such that βT is not hypercyclic for all $|\beta| \neq 1$. This gives a negative answer to the following question.

Question 1: Let $T : X \to X$ be a bounded operator on a Banach space X and $\beta \in \mathbb{C}$. Does T being hypercyclic imply that βT is also hypercyclic?

Recently, Badea, Grivaux and Müller posed the following question in [2] and gave a negative answer.

Question 2: Let $T : X \to X$ be a bounded operator on a Banach space X. Suppose there are numbers $0 < t_1 < t_2$ such that $t_1 T$ and $t_2 T$ are hypercyclic. Is it true that tT is hypercyclic for every $t \in [t_1, t_2]$?

We will consider the above questions in the setting of weighted translation operators. We study complex multiples of weighted translation operators on a discrete group G. We note that, for $\beta T_{a,w} = T_{a,\beta w}$ to be hypercyclic, we must have $|\beta| > \frac{1}{\|w\|_\infty}$, for otherwise, we have $\|\beta T_{a,w}\| \leq |\beta| \|w\|_\infty \leq 1$.

Since both topologically mixing and hypercyclic operators can be regarded as special cases of supercyclic operators, we consider supercyclicity first. From now on, let the weight $w \in \ell^\infty(G)$ and $\beta \in \mathbb{C}$ satisfy $\|w\|_\infty > 1$ and $|\beta| > \frac{1}{\|w\|_\infty}$. We also let $1 \leq p < \infty$ and (α_n) be a sequence of nonzero complex numbers throughout this section.

Theorem 3.3.1 *Given a weighted translation operator* $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$, *the following conditions are equivalent.*

(i) $\beta T_{a,w}$ *is supercyclic with respect to* (α_{n_k}).

(ii) *Both sequences (depending on* a *and* (α_{n_k}) *)*

$$w_{n,\beta} := |\alpha_n \beta^n| \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \quad \text{and} \quad \overline{w}_{n,\beta} := \left(|\alpha_n \beta^n| \prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit subsequences $(w_{n_k,\beta})$ *and* $(\overline{w}_{n_k,\beta})$ *which converge to* 0 *pointwise on* G.

Proof. Repeat the same arguments as in the proof of Theorem 3.2.8, replacing the weight w there by βw. ☐

Corollary 3.3.2 *Let* $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ *be a weighted translation operator and let* $|\beta| = 1$. *Then the following conditions are equivalent.*

(i) $T_{a,w}$ *is supercyclic with respect to* (α_{n_k}).

(ii) $\beta T_{a,w}$ *is supercyclic with respect to* (α_{n_k}).

Proof. Put $|\beta| = 1$ in Theorem 3.3.1. ☐

Letting $\alpha_n = 1$ in Theorem 3.3.1, we have the following result.

Corollary 3.3.3 *Let* $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ *be a weighted translation operator and* $\beta \in \mathbb{C}$. *Then the following conditions are equivalent.*

(i) $T_{a,w}$ *is supercyclic with respect to* (β^{n_k}) *for some increasing sequence* (n_k) *in* $\mathbb{N}$.

(ii) $\beta T_{a,w}$ *is hypercyclic.*

(iii) *Both sequences (depending on a)*

$$w_{n,\beta} := |\beta|^n \prod_{s=1}^{n} w * \delta_{a^{-1}}^{s} \quad \textit{and} \quad \overline{w}_{n,\beta} := \left(|\beta|^n \prod_{s=0}^{n-1} w * \delta_{a}^{s} \right)^{-1}$$

admit subsequences $(w_{n_k,\beta})$ *and* $(\overline{w}_{n_k,\beta})$ *which converge to* 0 *pointwise in* G.

Corollary 3.3.4 *Let* $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ *be a weighted translation operator, and let* $|\beta| = 1$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$. *Then the following conditions are equivalent.*

(i) $T_{a,w}$ *is hypercyclic.*

(ii) $\beta T_{a,w}$ *is hypercyclic.*

(iii) $\gamma T_{a,w}$ *is supercyclic with respect to* $(\frac{1}{\gamma^{n_k}})$

Proof. Put $|\beta| = 1$ in Corollary 3.3.3 for conditions (i) and (ii), and $\alpha_n = \frac{1}{\gamma^n}$ in Theorem 3.3.1 for condition (iii). ☐

Corollary 3.3.5 *Let* $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ *be a weighted translation operator and* $\beta \in \mathbb{C}$. *Then the following conditions are equivalent.*

(i) $\beta T_{a,w}$ *is topologically mixing.*

(ii) *Both sequences (depending on* a *)*

$$w_{n,\beta} := |\beta|^n \prod_{s=1}^{n} w * \delta_{a^{-1}}^{s} \quad \text{and} \quad \overline{w}_{n,\beta} := \left(|\beta|^n \prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

converge to 0 *pointwise in* G.

Proof. Repeat the same arguments as in the proof of Theorem 3.2.14. ☐

Corollary 3.3.6 *Let* $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ *be a weighted translation operator and let* $|\beta| = 1$. *Then the following conditions are equivalent.*

(i) $T_{a,w}$ *is topologically mixing.*

(ii) $\beta T_{a,w}$ *is topologically mixing.*

Proof. Put $|\beta| = 1$ in Corollary 3.3.5. ☐

Using the above results, we obtain the following theorem.

Theorem 3.3.7 *Let $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ be a weighted translation operator.*

(i) *If $|\beta| = 1$, then $\beta T_{a,w}$ is supercyclic with respect to (α_{n_k}) if, and only if, $T_{a,w}$ is supercyclic with respect to (α_{n_k}).*

(ii) *If $|\beta| = 1$, then $\beta T_{a,w}$ is hypercyclic (topologically mixing) if, and only if, $T_{a,w}$ is hypercyclic (topologically mixing).*

(iii) *If $|\beta| \neq 1$, then $\beta T_{a,w}$ is hypercyclic if, and only if, $T_{a,w}$ is supercyclic with respect to (β^{n_k}).*

(iv) *If $|\beta| \neq 1$, then $\beta T_{a,w}$ is supercyclic with respect to $(\frac{1}{\beta^{n_k}})$ if, and only if, $T_{a,w}$ is hypercyclic.*

(v) *If $T_{a,w}$ is hypercyclic, then $\beta T_{a,w}$ is supercyclic.*

Theorem 3.3.8 *Let $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ be a weighted translation operator. If $\beta_1 T_{a,w}$ and $\beta_2 T_{a,w}$ are topologically mixing for some β_1, β_2 satisfying $|\beta_1| < |\beta_2|$, then $\beta T_{a,w}$ is topologically mixing for every $|\beta| \in [|\beta_1|, |\beta_2|]$.*

Proof. If $\beta_1 T_{a,w}$ and $\beta_2 T_{a,w}$ are topologically mixing with $|\beta_1| < |\beta| < |\beta_2|$, then

$$w_{n,\beta_1} < w_{n,\beta} < w_{n,\beta_2} \qquad \text{and} \qquad \overline{w}_{n,\beta_1} > \overline{w}_{n,\beta} > \overline{w}_{n,\beta_2}$$

on G in Corollary 3.3.5. This implies $\beta T_{a,w}$ is topologically mixing for every $|\beta| \in [|\beta_1|, |\beta_2|]$. □

It has been shown in [2, Theorem 1.6] that if an operator T is such that $t_1 T \oplus t_2 T$ is hypercyclic for some $0 < t_1 < t_2$, then tT is hypercyclic for every $t \in [t_1, t_2]$. For weighted translation operators, we have the following result.

Corollary 3.3.9 *Let* $T_{a,w} : \ell^p(G/H) \to \ell^p(G/H)$ *be a weighted translation operator and let* $0 < \beta_1 < \beta_2$. *The following conditions are equivalent.*

(i) $\beta_1 T_{a,w} \oplus \beta_2 T_{a,w}$ *is hypercyclic.*

(ii) *For* $j = 1, 2$, *both sequences (depending on* a *)*

$$w_{n,\beta_j} := \beta_j^n \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \qquad \textit{and} \qquad \overline{w}_{n,\beta_j} := \left(\beta_j^n \prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit subsequences (w_{n_k,β_j}) *and* $(\overline{w}_{n_k,\beta_j})$ *which converge to* 0 *pointwise in* G.

Proof. Repeat the similar arguments as in the proof of Theorem 3.2.8. ☐

Chapter 4

The discrete Laplacian

In this chapter, we study the Laplacian $\mathcal{L}$ on weighted homogeneous graphs. A weighted homogeneous graph is a homogeneous space of a discrete group G. The Laplacian $\mathcal{L}$ can be viewed as a convolution operator on such a homogeneous space. Therefore Theorem 2.2.7 enables us to give a full description of the spectrum $\mathrm{Spec}(\mathcal{L})$ of $\mathcal{L}$ on a homogeneous graph in terms of irreducible representations of the group G. We compare the eigenvalues of $\mathcal{L}$ with eigenvalues of the Laplacian on a weighted regular tree, and obtain a Dirichlet eigenvalue comparison theorem. For a connected homogeneous graph, we characterise its invariance in terms of group structures and show that all positive $\mathcal{L}$-harmonic functions on an invariant connected homogeneous graph are constant. A Harnack inequality has been proved in [23] for the Laplacian $\mathcal{L}$ on an invariant unweighted homogeneous graph. We extend this Harnack inequality for a Schrödinger operator $\mathcal{L} + \varphi$ on an invariant weighted homogeneous graph.

In Section 4.1, we study a homogeneous graph and describe the spectrum $\mathrm{Spec}(\mathcal{L})$. The Dirichlet eigenvalue comparison theorem will be developed in Section 4.2. We conclude with some properties of an invariant connected homogeneous graph and a version of Harnack inequality in the last section. Results in Section 4.1 and Section 4.3 have been published in [13].

4.1 Spectrum of a homogeneous graph

Applying Theorem 2.2.7, we now describe the spectrum Spec($\mathcal{L}$) of the Laplacian $\mathcal{L}$ on a weighted homogeneous graph under some weight condition.

We denote a graph by (V, E) where V is the set of vertices and E is the set of edges. In a weighted graph (V, E), finite or infinite, let d_v and $w : V \times V \longrightarrow [0, \infty)$ denote respectively the degree of a vertex $v \in V$ and the weight $w(v, u) = w(u, v)$, satisfying $d_v = \sum_{(v,u)\in E} w(v, u) < \infty$. The *Laplacian* $\mathcal{L}$, acting on real or complex functions f on V, is defined by

$$\mathcal{L}f(v) = f(v) - \frac{1}{d_v} \sum_{\substack{u \\ (v,\, u) \,\in\, E}} f(u)w(v, u) \qquad (v \in V). \tag{4.1}$$

This follows from that $\mathcal{L}$ is represented as $\nabla^*\nabla$ [30, 50] where the gradient is given by $\nabla f(v, u) = f(v) - f(u)$ for $(v, u) \in E$. By $\langle \mathcal{L}f, g\rangle_{d_v} = \langle \nabla f, \nabla g\rangle_w$ with a simple computation, we have

$$\sum_{v\in V} \mathcal{L}f(v)\overline{g(v)}\; d_v = \sum_{v\in V} \sum_{(v,u)\in E} (f(v) - f(u))\overline{g(v)}\; w(v, u)$$

which implies (4.1).

An important problem in spectral geometry is the estimation of the spectrum Spec($\mathcal{L}$) of $\mathcal{L}$. Many results concerning Spec($\mathcal{L}$) have appeared in the literature [21, 22, 40, 41, 48, 51]. We refer to [21] for a survey and results for finite graphs. Let (V, E) have n vertices with weight $w \equiv 1$. Then the eigenvalues of $\mathcal{L}$ are arranged as follows [21]:

$$\lambda_0 = 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq 2.$$

If (V, E) is also connected, then $\lambda_1 > 0$ and $\lambda_1 > 1 - \sqrt{1 - h^2}$ where h is the Cheeger constant of the graph [21]. Moreover, we have the so-called Cheeger inequality $\frac{h^2}{2} < \lambda_1 \leq 2h$ [21]. We note that the connection between a finite homogeneous graph Laplacian and group representations has been discussed in [21] and [22]. Our result involves convolution operators and applies to infinite graphs as well.

We call (V, E) a *homogeneous graph* (cf. [21]), if the vertex set V is a homogeneous space of a discrete group G with a graph condition, by which we mean G acts transitively on V by a right action $(v, g) \in V \times G \mapsto vg \in V$ so that V is represented as a right coset space G/H of G by a *finite* subgroup H and the edge set E is described by a finite subset $K = K^{-1} \subset G$ in that $(v, u) \in E$ if, and only if, $u = va$ for some $a \in K$. Henceforth we denote a homogeneous graph by (V, K), with the edge generating set K having finite cardinality $|K|$. We note that (V, K) is a Cayley graph if H reduces to the identity of G, in which case we write (G, K) for the graph.

For a simple example, the cycle $\mathcal{C}_n$ on n vertices can be viewed as a homogeneous graph with vertex set $V = \mathbb{Z}/n\mathbb{Z}$. In fact, $\mathcal{C}_n$ is a Cayley graph since $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ is a group. Although one can consider a more general notion of a homogeneous graph $(G/H, K)$ in which the isotropy subgroup H can be infinite, we only consider this case in the other two sections of this chapter. We refer to [21, 22, 23] for some interesting examples of homogeneous graphs.

The Laplacian for a weighted homogeneous graph (V, K) can be written as

$$\begin{aligned}\mathcal{L}f(v) &= f(v) - \frac{1}{|K|}\sum_{a\in K} f(va)w(v,va) \\ &= \frac{1}{|K|}\sum_{a\in K}(f(v)-f(va))w(v,va) \qquad (v\in V) \end{aligned} \tag{4.2}$$

where $\sum_{a\in K} w(v,va) = |K|$. We describe the spectrum of $\mathcal{L}$ completely in terms of irreducible representations of G when the weight w is given by a measure μ on G which is symmetric (cf. Section 2.1, p.16) and constant on each set xHy $(x, y \in G)$.

Let (V, K) be a homogeneous graph with $V = G/H$ and let μ be a positive symmetric measure on G, supported by K (i.e. $\sum_{a\in K}\mu(a) = |K|$), satisfying

$$\mu(xcy) = \mu(xy) \qquad (x, y \in G, c \in H).$$

We can define a weight w on $V \times V$ by

$$w(Hx, Hy) = \mu(x^{-1}y) = \mu(y^{-1}x).$$

In this case and in the sequel, $w(v, va) = \mu(a)$ and the Laplacian has the form

$$(\mathcal{L}f)(v) = f(v) - \frac{1}{|K|}\sum_{a\in K} f(va)\mu(a) = f * \left(\delta_e - \frac{\mu}{|K|}\right)(v) \tag{4.3}$$

which is a convolution operator $T_\sigma : \ell^2(G/H) \longrightarrow \ell^2(G/H)$ with $\sigma = \delta_e - \mu/|K|$, where $\mu/|K|$ is a probability measure. Since $\mu = \sum_{a\in K}\mu(a)\delta_a$ and $\widehat{\delta}_a(\pi) = \pi(a^{-1})$ for each $\pi \in \widehat{G}$, we have the following description of the spectrum Spec($\mathcal{L}$) by Theorem 2.2.7.

Corollary 4.1.1 *Let* (V, K) *be a homogeneous graph with* $V = G/H$ *and weight* w *given by a measure* μ *as above. The spectrum of the Laplacian in* (4.3) *is given by*

$$\mathrm{Spec}(\mathcal{L}) = 1 - \bigcup\left\{\mathrm{Spec}\left(\sum_{a\in K}\mu(a)|K|^{-1}\pi(a)\right) : \pi \in \widehat{G}_r, \ker\pi \supset \ker\rho_{_H}\right\}.$$

Remark 4.1.2 In [22], a Laplacian acting on vector valued functions $f : G/H \longrightarrow X$ has been considered and the resulting spectrum is called the *vibrational spectrum*. For the vector space X of $n \times n$ matrices, the spectrum of a convolution operator acting on X-valued functions on a group G has been described in [19], which yields the vibrational spectrum of a Cayley graph (G, K) in this case.

Let $\ell^2(V)$ be the complex Hilbert space of square integrable functions with respect to the normalized discrete measure on V, with the inner product:

$$\langle f, g\rangle = \sum_{v\in V} f(v)\overline{g(v)}.$$

We note that $\langle f, g\rangle = \langle g, f\rangle$ if f and g are real-valued, and $\mathcal{L} : \ell^2(V) \longrightarrow \ell^2(V)$ is a self-adjoint operator:

$$\begin{aligned}
& \langle \mathcal{L}h, g\rangle - \langle h, \mathcal{L}g\rangle \\
= \; & \frac{1}{2|K|}\sum_{v\in V}\sum_{a\in K}\Big(h(v)\overline{g(v)} - h(va)\overline{g(v)} + h(va)\overline{g(va)} - h(v)\overline{g(va)}\Big)\,\mu(a) \\
- \; & \frac{1}{2|K|}\sum_{v\in V}\sum_{a\in K}\Big(h(v)\overline{g(v)} - h(v)\overline{g(va)} + h(va)\overline{g(va)} - h(va)\overline{g(v)}\Big)\,\mu(a) \\
= \; & 0.
\end{aligned}$$

In fact, $\mathcal{L}$ is a positive operator since the inner product

$$\langle \mathcal{L}f, f\rangle = \frac{1}{2|K|}\sum_{v\in V}\sum_{a\in K}|f(v) - f(va)|^2\mu(a) \qquad (f \in \ell^2(V)) \tag{4.4}$$

is nonnegative. Hence we always have $\mathrm{Spec}(\mathcal{L}) \subset [0, 2]$ as $\|\mathcal{L}\| \leq \|\delta_e - \frac{\mu}{|K|}\| \leq 2$.

Example 4.1.3 Let $G = \mathbb{Z}$ with $K = \{-1, 1\}$. Consider the Cayley graph $(\mathbb{Z}, \{-1, 1\})$. Let μ be the following measure on $\mathbb{Z}$ supported by $\{-1, 1\}$: $\mu = \delta_1 + \delta_{-1}$. Then

$$\mathcal{L}f(n) = f(n) - \frac{1}{2}f(n-1) - \frac{1}{2}f(n+1)$$

for $n \in \mathbb{Z}$ and $f : \mathbb{Z} \to \mathbb{R}$. Since $\mathbb{Z}$ is abelian and $\widehat{\mathbb{Z}} = \mathbb{T}$, we have

$$\widehat{\delta}_1(\alpha) = \int_{\mathbb{Z}} \alpha(x^{-1}) d\delta_1(x) = \sum_{n\in\mathbb{Z}} \alpha^{-n}\delta_1\{n\} = \alpha^{-1}$$

$$\text{and} \qquad \widehat{\delta}_{-1}(\alpha) = \sum_{n\in\mathbb{Z}} \alpha^{-n}\delta_{-1}\{n\} = \alpha \qquad (\alpha \in \mathbb{T}).$$

Hence

$$\operatorname{Spec}(\mathcal{L}) = \left\{1 - \frac{1}{2}\alpha - \frac{1}{2}\alpha^{-1} : \alpha \in \mathbb{T}\right\} = \{1 - \cos\theta : \theta \in \mathbb{R}\} = [0, 2].$$

If we consider the Cayley graph $(\mathbb{Z}, \{0, \pm 1\})$ where loops are allowed and let $\mu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1} + 2\delta_0$, then

$$\mathcal{L}f(n) = f(n) - \frac{1}{6}f(n-1) - \frac{1}{6}f(n+1) - \frac{2}{3}f(n).$$

Therefore

$$\operatorname{Spec}(\mathcal{L}) = \left\{1 - \frac{1}{6}\alpha - \frac{1}{6}\alpha^{-1} - \frac{2}{3} : \alpha \in \mathbb{T}\right\} = \left\{\frac{1}{3} - \frac{1}{3}\cos\theta : \theta \in \mathbb{R}\right\} = \left[0, \frac{2}{3}\right].$$

Example 4.1.4 Let $V = \mathbb{Z}^2/(n\mathbb{Z} \times m\mathbb{Z})$ with a finite generating set $K = -K \subset \mathbb{Z}^2$. The character group $\widehat{\mathbb{Z}^2}$ is the product $\mathbb{T} \times \mathbb{T}$ of two copies of the circle group $\mathbb{T}$. Each $\pi \in \widehat{\mathbb{Z}^2}$ identifies with $(\pi(1,0), \pi(0,1)) \in \mathbb{T} \times \mathbb{T}$, and $\pi(n\mathbb{Z} \times m\mathbb{Z}) = \{1\}$ if, and only if, $\pi = (e^{2\pi ik/n}, e^{2\pi i\ell/m})$ for $(k, \ell) \in \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$. For such π, we have

$$\pi(a, b) = e^{2\pi i(ka/n + \ell b/m)} \qquad ((a, b) \in K).$$

Hence

$$\operatorname{Spec}(\mathcal{L}) = \left\{1 - \left(\sum_{(a,b)\in K} \frac{\mu(a,b)}{|K|} \cos 2\pi(ka/n + \ell b/m)\right) : (k, \ell) \in \mathbb{Z}_n \times \mathbb{Z}_m\right\}.$$

Example 4.1.5 Let G be the *discrete* Heisenberg group

$$\left\{ \begin{pmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} : m, n, p \in \mathbb{Z} \right\}$$

which is non-abelian. The characters of G are known (cf. [3, 31, 49]). Let $\mathbb{R}/\mathbb{Z}$ be the real numbers mod $\mathbb{Z}$ and denote an element of G by (m, n, p). As in [49], $\widehat{G}_r$ contains, among others, the one-dimensional unitary representations

$$\{\chi_{\alpha,\beta} : \alpha, \beta \in \mathbb{R}/\mathbb{Z}\}$$

where

$$\chi_{\alpha,\beta}(m, n, p) = e^{2\pi i(\alpha m+\beta n)}.$$

Consider the Cayley graph (G, K) with $K = \{(\pm m, 0, 0), (0, \pm n, 0)\}$ and $m, n \neq 0$. Let μ be the following measure on G supported by K:

$$\mu = \frac{1}{2}\delta_{(m,0,0)} + \frac{1}{2}\delta_{(-m,0,0)} + \frac{3}{2}\delta_{(0,n,0)} + \frac{3}{2}\delta_{(0,-n,0)}.$$

We have

$$\begin{aligned} \mathrm{Spec}(\mathcal{L}) &= 1 - \bigcup_{\pi\in\widehat{G}_r} \mathrm{Spec}\left(\frac{1}{4}\sum_{a\in K} \mu(a)\pi(a)\right) \\ &\supset 1 - \bigcup \left\{\frac{1}{4}\sum_{a\in K} \mu(a)\chi_{\alpha,\beta}(a) : \alpha, \beta \in \mathbb{R}/\mathbb{Z}\right\} \\ &= \left\{1 - \left(\frac{1}{4}\cos(2\pi\alpha m) + \frac{3}{4}\cos(2\pi\beta n)\right) : \alpha, \beta \in \mathbb{R}/\mathbb{Z}\right\} = [0, 2]. \end{aligned}$$

It follows that $\mathrm{Spec}(\mathcal{L}) = [0, 2]$.

4.2 Eigenvalue comparison theorems

In [50], Urakawa gave a graph theoretic analogue of Cheng's eigenvalue comparison theorems for the Laplacian of complete Riemannian manifolds [16, 17].

Urakawa compared eigenvalues of the Laplacian on unweighted connected graphs with eigenvalues of the Laplacian on unweighted regular trees. In this section, we extend Urakawa's results to comparison of weighted connected homogeneous graphs with weighted regular trees.

From now on, a graph may be finite or infinite. Let (V, K) be a connected homogeneous graph with weight μ as defined in Section 4.1. In the remaining chapter, the Laplacian $\mathcal{L}$ on (V, K) is defined by

$$\mathcal{L}f(v) = f(v) - \frac{1}{|K|}\sum_{a\in K} f(va)\mu(a) \qquad (v \in V) \tag{4.5}$$

where K is the generating set with finite cardinality $|K|$ and $0 < \mu(a) = \mu(a^{-1})$ satisfying $|K| = \sum_{a\in K}\mu(a) < \infty$.

Lemma 4.2.1 *Suppose that $v \in V$ satisfies $\mathcal{L}f(v) \geq 0$ and $f(va) \geq f(v)$ for all $a \in K$. Then $f(va) = f(v)$ for all $a \in K$.*

Proof. Considering $\mathcal{L}f(v) \geq 0$, we have

$$|K|f(v) \geq \sum_{a\in K} f(va)\mu(a).$$

This implies

$$0 \geq \sum_{a\in K}(f(va) - f(v))\mu(a) \geq 0.$$

Hence $f(v) = f(va)$ for all $a \in K$. $\square$

Let $\mathrm{dist}(u, v)$ be the distance between two vertices u and v, i.e. the number of edges in a shortest path in V connecting u and v. We denote by $B(v_0, R)$ the (open) *ball* centred at $v_0 \in V$, with radius $0 < R < \infty$, where

$$B(v_0, R) = \{v \in V : \mathrm{dist}(v_0, v) < R\}.$$

The *boundary* of $B(v_0, R)$ is denoted by

$$\begin{aligned}\delta B(v_0, R) &= \{v \in V \setminus B(v_0, R) : v \text{ is adjacent to some } u \in B(v_0, R)\} \\ &= \{v \in V : \operatorname{dist}(v_0, v) = R\}.\end{aligned}$$

We consider the space $\ell^2(B(v_0, R))$ and the Dirichlet problem on $B(v_0, R)$:

$$(*) \begin{cases} \mathcal{L}f(v) = \lambda f(v) & \text{on } B(v_0, R) \\ f(v) = 0 & \text{on } \delta B(v_0, R) \end{cases}$$

for functions $f : B(v_0, R) \cup \delta B(v_0, R) \to \mathbb{R}$, where $\lambda \in [0, 2]$. By a slight abuse of notation, we denote

$$D^* = \{g \in \ell^2(B(v_0, R)) \setminus \{0\} : g(v) = 0 \;\; \forall v \in \delta B(v_0, R)\}.$$

We note that the Dirichlet problem has eigenvalues arranged as follows [21, p.128]:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq 2$$

where $n = |B(v_0, R)| = \dim \ell^2(B(v_0, R))$, the cardinality of $B(v_0, R)$. Indeed, 0 is not an eigenvalue. To see this, let $f \in D^*$ satisfy $\mathcal{L}f = 0$. Then, by (4.4),

$$0 = \langle \mathcal{L}f, f \rangle = \frac{1}{2|K|} \sum_{v \in B(v_0,R)} \sum_{a \in K} (f(v) - f(va))^2 \mu(a)$$

which implies that $f(v) = f(va)$ for all $v \in B(v_0, R)$ and $a \in K$. Hence $f(v) = 0$ for all $v \in B(v_0, R) \cup \delta B(v_0, R)$ by connectedness and the boundary condition.

For the first eigenvalue λ_1 and its eigenfunction, called the *first eigenfunction* of $(*)$, we have the following properties (cf. [30, Lemma 1.9]).

Lemma 4.2.2 *Let* λ_1 *be the first eigenvalue of* $(*)$*. Then* λ_1 *is simple, and there is a positive first eigenfunction of* $(*)$*.*

Proof. Let $f \in D^*$ be a first eigenfunction of $(*)$. Then

$$\lambda_1 = \inf_{g \in D^*} \frac{\langle \mathcal{L}g, g\rangle}{\langle g, g\rangle} \leq \frac{\langle \mathcal{L}|f|, |f|\rangle}{\langle |f|, |f|\rangle}$$

since $\mathcal{L} = \mathcal{L}^*$ and $|f| \in D^*$. Suppose f takes both positive and negative values in $B(v_0, R)$. Then

$$\begin{aligned} \langle \mathcal{L}f, f\rangle &= \frac{1}{2|K|} \sum_{v \in B(v_0,R)} \sum_{a \in K} (f(v) - f(va))^2 \mu(a) \\ &> \frac{1}{2|K|} \sum_{v \in B(v_0,R)} \sum_{a \in K} (|f(v)| - |f(va)|)^2 \mu(a) \\ &= \langle \mathcal{L}|f|, |f|\rangle. \end{aligned}$$

With $\langle f, f\rangle = \langle |f|, |f|\rangle$, this implies

$$\lambda_1 = \frac{\langle \mathcal{L}f, f\rangle}{\langle f, f\rangle} > \frac{\langle \mathcal{L}|f|, |f|\rangle}{\langle |f|, |f|\rangle}$$

which is contradiction. By Lemma 4.2.1 and connectedness, $f(v) \neq 0$ for all $v \in B(v_0, R)$. Hence either $f(v) > 0$ for all $v \in B(v_0, R)$ or $f(v) < 0$ for all $v \in B(v_0, R)$. Now suppose λ_1 is not simple. Let f_1, f_2 be two linearly independent positive first eigenfunctions of $(*)$ and choose a vertex $v \in B(v_0, R)$. Then there exists $(c_1, c_2) \neq (0, 0)$ such that $c_1 f_1 + c_2 f_2$ is a first eigenfunction vanishing on v which is impossible. Hence λ_1 is a simple eigenvalue. □

We now prove a weighted version of the discrete Barta theorem in [50, Theorem 2.1].

Theorem 4.2.3 *Let* λ_1 *be the first eigenvalue of* $(*)$*. If* $g > 0$ *on* $B(v_0, R)$ *and* $g = 0$ *on* $\delta B(v_0, R)$*, then*

$$\inf_{v \in B(v_0,R)} \frac{\mathcal{L}g(v)}{g(v)} \leq \lambda_1 \leq \sup_{v \in B(v_0,R)} \frac{\mathcal{L}g(v)}{g(v)}.$$

Proof. Let f be a positive first eigenfunction of $(*)$ and set $h = f - g$. Then

$$\begin{aligned} \lambda_1 &= \frac{\mathcal{L}f}{f} \\ &= \frac{\mathcal{L}(g+h)}{g+h} \\ &= \frac{\mathcal{L}g}{g} + \frac{g(\mathcal{L}h) - h(\mathcal{L}g)}{g(g+h)}. \end{aligned}$$

Since both g and $\mathcal{L}h$ are real-valued, we have

$$\sum_{v \in B(v_0,R)} (g(v)\mathcal{L}h(v) - h(v)\mathcal{L}g(v)) = \langle g, \mathcal{L}h\rangle - \langle h, \mathcal{L}g\rangle = 0$$

which implies either $g(\mathcal{L}h) - h(\mathcal{L}g) = 0$ or $g(\mathcal{L}h) - h(\mathcal{L}g)$ changes sign. The former implies $\lambda_1 = \frac{\mathcal{L}g(v)}{g(v)}$. In the latter case, the sign is negative at some $v \in B(v_0, R)$, so

$$\lambda_1 < \frac{\mathcal{L}g(v)}{g(v)} \leq \sup_{B(v_0,R)} \frac{\mathcal{L}g}{g}.$$

The sign is positive at some $u \in B(v_0, R)$, so

$$\inf_{B(v_0,R)} \frac{\mathcal{L}g}{g} \leq \frac{\mathcal{L}g(u)}{g(u)} < \lambda_1.$$

$\square$

We recall that a connected graph without cycle is called a *tree*, and a *regular tree* with *degree* d is a tree which has d edges at each vertex. Let T_d be a regular tree with degree d and weight w. The Laplacian Δ on T_d is defined by

$$\Delta f(x) = f(x) - \frac{1}{d} \sum_{\substack{y \\ y \sim x}} f(y)w(x,y) \qquad (x \in T_d)$$

where $0 \leq w(x,y) = w(y,x)$ and for all $x \in V$, $\sum_{(x,y)\in E} w(x,y) = d$.

Let $B(x_0, R) = \{x \in T_d : \operatorname{dist}(x_0, x) < R\}$ be the ball centred at $x_0 \in T_d$ with finite radius $R > 0$. We consider the Dirichlet problem on $B(x_0, R)$:

$$(**) \quad \begin{cases} \Delta f(x) = \nu f(x) & \text{on } B(x_0, R) \\ f(x) = 0 & \text{on } \delta B(x_0, R), \end{cases}$$

where $\nu \in [0, 2]$, for functions $f : B(x_0, R) \cup \delta B(x_0, R) \to \mathbb{R}$. Since $B(x_0, R)$ is finite, this problem has eigenvalues arranged as follows:

$$0 < \nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \leq \nu_m$$

where $m = |B(x_0, R)|$, the cardinality of $B(x_0, R)$. Let $x_0 \in T_d$ be fixed. For each $x \in T_d$, we define $r(x) = \operatorname{dist}(x_0, x)$. As in the Dirichlet problem $(*)$ for homogeneous graphs, we can always find a positive first eigenfunction for $(**)$. We have the following property for a first eigenfunction of $(**)$ (cf. [50, Lemma 3.1]).

Lemma 4.2.4 *There is a positive first eigenfunction f of $(**)$ such that $f(x) = f(y)$ whenever $r(x) = r(y)$. Therefore, we may put $f(r) = f(x)$ whenever $r = r(x)$, in which case $f(r)$ is monotone decreasing in r.*

Proof. Let g be a positive first eigenfunction of $(**)$. Define $f : B(x_0, R) \to \mathbb{R}$ by

$$f(x) = \frac{\sum_{z \in B(x_0, r(x)+1) \setminus B(x_0, r(x))} g(z)}{|B(x_0, r(x) + 1) \setminus B(x_0, r(x))|}.$$

Then f is also a positive first eigenfunction of $(**)$ since g satisfies $(**)$ for ν_1 which implies $\Delta f = \nu_1 f$. Moreover, if $r(x) = r(y)$, then $f(x) = f(y)$. Now put $f(r) = f(x)$ whenever $r = r(x)$. Then for $r(x) = 0$,

$$0 \leq \nu_1 f(0) = \Delta f(0) = f(0) - f(1)$$

which implies $f(1) \leq f(0)$. Assume $f(t) \leq f(t-1)$ for $r(x) = t$. We show $f(t+1) \leq f(t)$. Otherwise, we have $f(t+1) > f(t)$. With

$$\nu_1 f(t) = \Delta f(t) = f(t) - \frac{1}{d}\left(\sum_{r(y)=t-1} f(t-1) w(x, y) + \sum_{r(y)=t+1} f(t+1) w(x, y) \right),$$

we have

$$-d\nu_1 f(t) + df(t) = \sum_{r(y)=t-1} f(t-1)w(x,y) + \sum_{r(y)=t+1} f(t+1)w(x,y) > df(t).$$

Then $f(t) < 0$ which gives a contradiction. $\square$

As in [50, Lemma 3.4], we consider the following two conditions for our weighted graphs.

Lemma 4.2.5 *Let (V,K) be a connected homogeneous graph with weight μ and T_d a regular tree with weight w. Fix $v_0 \in V$ and $x_0 \in T_d$. Let $r_1(v) = \mathrm{dist}(v_0, v)$ and $r_2(x) = \mathrm{dist}(x_0, x)$ for $v \in V$ and $x \in T_d$.*

(i) *If*

$$(\textbf{condition A}) \qquad \inf_{\substack{v\,\in\, B(v_0,\,R)\\ r_1(v)\,=\,t}} \frac{\sum_{r_1(va)=t-1}\mu(a)}{|K|} \geq \frac{\sum_{r_2(y)=t-1} w(x,y)}{d}$$

for all $x \in B(x_0,R)$ with $r_2(x) = t$, then

$$\frac{\sum_{r_1(va)=t+1}\mu(a)}{|K|} \leq \frac{\sum_{r_2(y)=t+1} w(x,y)}{d} \qquad (v \in B(v_0,R), x \in B(x_0,R))$$

for $r_1(v) = r_2(x) = t$.

(ii) *If*

$$(\textbf{condition B})\ (V,K) \textit{ is a tree and} \quad \sup_{\substack{v\,\in\, B(v_0,\,R)\\ r_1(v)\,=\,t}} \frac{\sum_{r_1(va)=t-1}\mu(a)}{|K|} \leq \frac{\sum_{r_2(y)=t-1} w(x,y)}{d}$$

for all $x \in B(x_0,R)$ with $r_2(x) = t$, then

$$\frac{\sum_{r_1(va)=t+1}\mu(a)}{|K|} \geq \frac{\sum_{r_2(y)=t+1} w(x,y)}{d} \qquad (v \in B(v_0,R), x \in B(x_0,R))$$

for $r_1(v) = r_2(x) = t$.

Proof. (i) Note that $\sum_{r_2(y)=t} w(x,y) = 0$ as there does not exist $y \in B(x_0, R)$ with $r_2(y) = t$. Since $|K| = \sum_{a\in K} \mu(a)$ and $d = \sum_y w(x,y)$, we have

$$\begin{aligned}
\frac{\sum_{r_1(va)=t+1} \mu(a)}{|K|} &= \frac{|K| - \sum_{r_1(va)=t-1} \mu(a) - \sum_{r_1(va)=t} \mu(a)}{|K|} \\
&\leq 1 - \frac{\sum_{r_1(va)=t-1} \mu(a)}{|K|} \leq 1 - \inf_{\substack{v \in B(v_0,R) \\ r_1(v)=t}} \frac{\sum_{r_1(va)=t-1} \mu(a)}{|K|} \\
&\leq 1 - \frac{\sum_{r_2(y)=t-1} w(x,y)}{d} = \frac{\sum_{r_2(y)=t+1} w(x,y)}{d}.
\end{aligned}$$

(ii) If (V, K) is a tree, then $\sum_{r_1(va)=t} \mu(a) = 0$ as before. Hence

$$\begin{aligned}
\frac{\sum_{r_1(va)=t+1} \mu(a)}{|K|} &= \frac{|K| - \sum_{r_1(va)=t-1} \mu(a) - \sum_{r_1(va)=t} \mu(a)}{|K|} \\
&= 1 - \frac{\sum_{r_1(va)=t-1} \mu(a)}{|K|} \geq 1 - \sup_{\substack{v \in B(v_0,R) \\ r_1(v)=t}} \frac{\sum_{r_1(va)=t-1} \mu(a)}{|K|} \\
&\geq 1 - \frac{\sum_{r_2(y)=t-1} w(x,y)}{d} = \frac{\sum_{r_2(y)=t+1} w(x,y)}{d}.
\end{aligned}$$

□

Remark 4.2.6 For unweighted graphs, we have $\mu = w \equiv 1$, and **condition A** reduces to $\frac{m_-(v)}{|K|} \geq \frac{1}{d}$ for all $v \in B(v_0, R)$ where $m_-(v) = |\{va : r_1(va) = r_1(v) - 1\}|$. Also **condition B** reduces to $\frac{m_-(v)}{|K|} \leq \frac{1}{d}$ for all $v \in B(v_0, R)$, and the above result is identical with Urakawa's result in [50, Lemma 3.4].

Now we are ready to prove a Dirichlet eigenvalue comparison theorem for weighted graphs which extends [50, Theorem 3.3].

Theorem 4.2.7 *Let (V, K) be a connected homogeneous graph with weight μ and T_d a regular tree with weight w. Choose $v_0 \in V$ and $x_0 \in T_d$. Then the first*

Dirichlet eigenvalues of $(*)$ *and* $(**)$ *are related as follows:*
(i) **condition A** *implies*

$$\lambda_1(B(v_0,R)) \le \nu_1(B(x_0,R));$$

(ii) **condition B** *implies*

$$\lambda_1(B(v_0,R)) \ge \nu_1(B(x_0,R)).$$

Proof. Fix $v_0 \in V$ and $x_0 \in T_d$. Write $r_1(v) = \mathrm{dist}(v_0, v)$ for $v \in V$ and $r_2(x) = \mathrm{dist}(x_0, x)$ for $x \in T_d$. Let f be a positive first eigenfunction of $(**)$. Define $f(r) = f(v)$ for $r = r_1(v)$ and $v \in B(v_0,R)$. By Theorem 4.2.3, we have

$$\inf_{B(v_0,R)} \frac{\mathcal{L}f}{f} \le \lambda_1 \le \sup_{B(v_0,R)} \frac{\mathcal{L}f}{f}.$$

For $v \in B(v_0,R)$ with $t = r_1(v) = r_2(x) < R$ for some $x \in B(x_0,R)$, we have, by $(*)$,

$$\begin{aligned}
\mathcal{L}f(v) &= f(v) - \frac{1}{|K|}\sum_{a\in K} f(va)\mu(a) \\
&= f(t) - \frac{1}{|K|}\left(\sum_{r_1(va)=t}\mu(a)f(t) + \sum_{r_1(va)=t-1}\mu(a)f(t-1) + \sum_{r_1(va)=t+1}\mu(a)f(t+1)\right) \\
&= \frac{\sum_{r_1(va)=t-1}\mu(a)}{|K|}(f(t)-f(t-1)) + \frac{\sum_{r_1(va)=t+1}\mu(a)}{|K|}(f(t)-f(t+1)).
\end{aligned}$$

By $(**)$, we have

$$\begin{aligned}
\Delta f(x) &= f(x) - \frac{1}{d}\sum_{\substack{y \\ y\sim x}} f(y)w(x,y) \\
&= f(t) - \frac{1}{d}\left(\sum_{r_2(y)=t-1} w(x,y)f(t-1) + \sum_{r_2(y)=t+1} w(x,y)f(t+1)\right) \\
&= \frac{\sum_{r_2(y)=t-1} w(x,y)}{d}(f(t)-f(t-1)) + \frac{\sum_{r_2(y)=t+1} w(x,y)}{d}(f(t)-f(t+1)).
\end{aligned}$$

This implies

$$\begin{aligned}\mathcal{L}f(v) - \Delta f(x) &= \left(\frac{\sum_{r_1(va)=t-1}\mu(a)}{|K|} - \frac{\sum_{r_2(y)=t-1} w(x,y)}{d}\right)(f(t)-(t-1)) \\ &+ \left(\frac{\sum_{r_1(va)=t+1}\mu(a)}{|K|} - \frac{\sum_{r_2(y)=t+1} w(x,y)}{d}\right)(f(t)-(t+1)).\end{aligned}$$

Moreover we have

$$\mathcal{L}f(v_0) = f(0) - f(1) = \Delta f(x_0)$$

since $r_1(v_0) = r_2(x_0) = 0$, $\sum_{r_1(v_o a)=1}\mu(a) = |K|$ and $\sum_{r_2(y)=1} w(x_0, y) = d$.

(i) If **condition A** holds, we have $\mathcal{L}f(v) \le \Delta f(x)$ by Lemma 4.2.4 and Lemma 4.2.5. This implies

$$\lambda_1 \le \sup_{B(v_0,R)} \frac{\mathcal{L}f}{f} \le \sup_{B(x_0,R)} \frac{\Delta f}{f} = \nu_1.$$

(ii) If **condition B** holds, we have $\mathcal{L}f(v) \ge \Delta f(x)$ by Lemma 4.2.4 and Lemma 4.2.5. This implies

$$\lambda_1 \ge \inf_{B(v_0,R)} \frac{\mathcal{L}f}{f} \ge \inf_{B(x_0,R)} \frac{\Delta f}{f} = \nu_1.$$

□

Using Theorem 4.2.7, we can estimate the bottom of the spectrum of the discrete Laplacian $\mathcal{L}$ for an infinite weighted connected homogeneous graph (V, K) (cf. [50, Corollary 3.11]). Let

$$\lambda_0(V, K) = \inf \operatorname{Spec}(\mathcal{L})$$

be the bottom of the spectrum. It is known in [50] that

$$\lambda_0(V, K) = \lim_{R\to\infty} \lambda_1(B(v_0, R)).$$

Similarly, we write

$$\nu_0(T_d) = \inf \operatorname{Spec}(\Delta)$$

for the bottom of the spectrum of the discrete Laplacian Δ on an infinite weighted regular tree T_d with weight w.

Corollary 4.2.8 *Let* (V, K) *be an infinite connected homogeneous graph with weight* μ *and* T_d *an infinite regular tree with weight* w.
(i) *If* **condition A** *holds for all* $R > 0$, *then*

$$\lambda_0(V, K) \leq \nu_0(T_d).$$

(ii) *If* **condition B** *holds for all* $R > 0$, *then*

$$\lambda_0(V, K) \geq \nu_0(T_d).$$

Proof. By Theorem 4.2.7. □

Remark 4.2.9 If $w \equiv 1$ in Corollary 4.2.8, we have $\nu_0(T_d) = 1 - \frac{2\sqrt{d-1}}{d}$ by [40, p.225], which gives $\lambda_0(V, K) \leq 1 - \frac{2\sqrt{d-1}}{d}$ in (i); but the reverse inequality in (ii).

As another application of Theorem 4.2.7, one can obtain some estimates of the spectrum of $\mathcal{L}$ for an infinite weighted connected homogeneous graph (V, K) (cf. [50, Theorem 5.2]).

Corollary 4.2.10 *Let* (V, K) *be an infinite connected homogeneous graph with weight* μ *and* T_d *an infinite regular tree with weight* w. *If* **condition B** *holds for all* $R > 0$, *then*

$$\mathrm{Spec}(\mathcal{L}) \subset [\nu_0(T_d), 2 - \nu_0(T_d)].$$

In particular, if $w \equiv 1$, *then*

$$\mathrm{Spec}(\mathcal{L}) \subset \left[1 - \frac{2\sqrt{d-1}}{d}, 1 + \frac{2\sqrt{d-1}}{d}\right].$$

Proof. Since a tree is bipartite, we have, by [50],

$$\mathrm{Spec}(\Delta) = [\nu_0(T_d), 2 - \nu_0(T_d)].$$

In **condition B**, (V, K) is also a tree and so by [50] again, we have

$$\mathrm{Spec}(\mathcal{L}) = [\lambda_0(V, K), 2 - \lambda_0(V, K)].$$

Hence

$$\mathrm{Spec}(\mathcal{L}) \subset [\nu_0(T_d), 2 - \nu_0(T_d)]$$

by Corollary 4.2.8 (ii). □

4.3 Harnack inequality

We begin this section by showing the relationship between certain graph invariance and group structures. We then prove a version of the Harnack inequality for an invariant homogeneous graph.

In the sequel, we do not assume that the isotropy group H is finite in a homogeneous graph $(G/H, K)$, instead we assume that G acts as graph automorphisms of G/H, that is, two vertices Hx and Hy are adjacent if, and only, if Hxg and Hyg are adjacent for all $g \in G$. A homogeneous graph (V, K) is called *invariant* in [23] if the edge generating set K satisfies $aK = Ka$ for each $a \in K$. This condition imposes some structure on the group G acting on V. It turns out that a connected Cayley graph (G, K) is invariant (for some edge generating set K) if, and only if, G is an $[\mathrm{IN}_0]$-group as defined in [20]. A locally compact group G is called an $[\mathrm{IN}_0]$-*group* if $G = \bigcup_{n=1}^{\infty} C^n$ for some compact neighbourhood C of the identity satisfying $gC = Cg$ for each $g \in G$.

Proposition 4.3.1 *Let $V = G/H$ be a homogeneous space of a discrete group G. The following conditions are equivalent.*

(i) *(V, K) is a connected invariant homogeneous graph for some finite set $K \subset G$.*

(ii) *$G = \bigcup_{n=0}^{\infty} HK^n$ with $K^0 = \{e\}$ for some finite set $K = K^{-1}$ satisfying $aK = Ka$ and $HgK = HKg$ for $a \in K$ and $g \in G$.*

In particular, (G, K) is a connected invariant Cayley graph for some finite set $K \subset G$ if, and only if, G is an $[\mathrm{IN}_0]$*-group.*

Proof. (i) $\Longrightarrow$ (ii). Denote by $v \sim u$ the adjacency of two points in V. We first show $G = \bigcup_{n=0}^{\infty} HK^n$. Let $g \in G$ and $g \notin H$. Then $Hg \neq H$. Since V is connected, we have $Hg \sim Hg_1 \sim \cdots \sim Hg_n \sim H$ for some $g_1, \ldots, g_n \in G$, and hence $Hg = (Hg_1)a_1 = (Hg_2)a_2a_1 = \cdots = (Hg_n)a_n \cdots a_1 = Ha_{n+1}a_n \cdots a_1$ where $a_1, \ldots, a_{n+1} \in K$. So $g \in HK^{n+1}$. This proves $G = H \cup HK \cup HK^2 \cup \cdots$.

Next, let $a \in K$ and $g \in G$. Then $H \sim Ha$ which implies $Hg \sim Hag$ since G acts on V as automorphisms of V. Hence $Hag = Hga_1$ for some $a_1 \in K$, and we have $HKg \subset HgK$. Similarly, $HgK \subset HKg$ using $Hg \sim Hga$ implies $H \sim Hgag^{-1}$.

(ii) $\Longrightarrow$ (i). Define adjacency $\sim$ in V by K. Given $v \sim u$ in V with $u = va$ for some $a \in K$, we have, for each $g \in G$, that $ug = vag = vga'$ for some $a' \in K$, that is, $ug \sim vg$. Hence (V, K) is a homogeneous graph which is clearly invariant and connected.

Finally, if (G, K) is an invariant connected Cayley graph, then $C = K \cup \{e\}$ is an invariant neighbourhood of the identity by (ii) and $G = \bigcup_{n=1}^{\infty} C^n$ is an $[\mathrm{IN}_0]$-group.

Conversely, if G is an $[\mathrm{IN}_0]$-group with $G = \bigcup_{n=1}^{\infty} C^n$, then (G, K) is a connected invariant graph with $K = C \cup C^{-1}$. □

We note that the product $O(n) \times \mathbb{R}$ of the orthogonal group $O(n)$ and the additive group $\mathbb{R}$ is an [IN$_0$]-group [20]. Evidently, a homogeneous graph $(G/H, K)$ is invariant if G is abelian or K is a subgroup of G. We refer to [21] for more examples of invariant homogeneous graphs.

Let $\mathcal{L}$ be the Laplacian on an invariant weighted homogeneous graph (V, K) as defined in (4.5). We now prove that the positive $\mathcal{L}$-harmonic functions on (V, K), that is, the positive 0-eigenfunctions of $\mathcal{L}$, are constant. Let $V = G/H$ and let $q : G \to G/H$ be the quotient map. Let $C = K \cup \{e\}$ which is an invariant neighbourhood of $e \in G$. The discrete subgroup

$$G_0 = \bigcup_{n=1}^{\infty} C^n \subset G$$

is an [IN$_0$]-group. The measure $\mu/|K|$ in the Laplacian $\mathcal{L}$ has support $K \subset G_0$ and restricts to a probability measure μ_0 on G_0. A real function h on G_0 is called *μ_0-harmonic* if $h = h * \mu_0$. Given an $\mathcal{L}$-harmonic function $f : V \longrightarrow \mathbb{R}$, the equation $\mathcal{L}f = 0$ gives

$$f(Hx) = \left(f * \frac{\mu}{|K|}\right)(Hx) = \int_G f(Hxy^{-1}) \frac{d\mu}{|K|}(y) = \int_{G_0} f(Hxy^{-1}) d\mu_0(y)$$

and hence $f \circ q$ restricts to a μ_0-harmonic function on G_0.

A function $\varphi : G_0 \longrightarrow (0, \infty)$ is called *exponential* if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G_0$ (cf. [20]).

Proposition 4.3.2 *Let* (V, K) *be a connected invariant homogeneous graph with Laplacian* $\mathcal{L}$ *given by* (4.5). *Then all positive* $\mathcal{L}$*-harmonic functions on* V *are constant.*

Proof. Let f be a positive function on $V = G/H$ satisfying $\mathcal{L}f = 0$. By the above remark, the quotient map $q : G \to G/H$ lifts f to a positive μ_0-harmonic function $f \circ q$ on G_0. Since G_0 is an $[\mathrm{IN}_0]$-group and the support of μ_0 generates G_0, it follows from [20, Theorem 9] that $f \circ q|_{G_0}$ is an integral

$$f \circ q(x) = \int_{\mathcal{E}} h(x) dP(h) \qquad (x \in G_0)$$

of (constant multiples of) exponential functions with respect to a probability measure P on $\mathcal{E}$, where $\mathcal{E}$ consists of constant multiples $\alpha\varphi$ of exponential functions φ on G_0 satisfying

$$\int_{G_0} \varphi(x^{-1}) d\mu_0(x) = 1.$$

We show that $\varphi = 1$ for all such φ. Indeed, if $\varphi(a) \neq 1$ for some $a \in K$, then $\varphi(a)+\varphi(a^{-1}) = \varphi(a)+\varphi(a)^{-1} > 2$ and $1 = \int_{G_0} \varphi(x^{-1})d\mu_0(x) = \sum_{b \in K} \varphi(b)\mu(b)/|K|$ implies

$$\begin{aligned} |K| &= \varphi(a)\mu(a) + \varphi(a)^{-1}\mu(a) + \sum_{b \in K\setminus\{a,a^{-1}\}} \varphi(b)\mu(b) \\ &> 2\mu(a) + \sum_{b \in K\setminus\{a,a^{-1}\}} \varphi(b)\mu(b) \\ &\geq \sum_{b \in K} \mu(b) = |K| \end{aligned}$$

which is impossible. Hence $\varphi = 1$ on $C = K \cup \{e\}$ and therefore, on $\bigcup_{n=1}^{\infty} C^n = G_0$.

It follows that $f \circ q$ is constant on G_0. Since $G = \bigcup_{n=1}^{\infty} HC^n$ by connectedness of the graph and Proposition 4.3.1, we have $f(Hx) = f(H)$ for all $x \in G$. $\square$

A Harnack inequality for eigenfunctions of the Laplacian on a finite unweighted invariant homogeneous graph has been shown in [23]. This inequality can be proved similarly for the Laplacian in (4.5) for weighted graphs. We will extend the idea in [23] to deduce a version of Harnack inequality for a Schrödinger operator $\mathcal{L}+\varphi$.

Let (V,K) be a weighted invariant homogeneous graph in which the weight is given by a symmetric measure μ satisfying

$$\mu(a)=\mu(bab^{-1})>0 \qquad (a,b\in K). \tag{4.6}$$

Let $w_a=\mu(a)/|K|$ for $a\in K$ so that the Laplacian in (4.5) is written

$$\mathcal{L}f(v)=\sum_{a\in K}w_a(f(v)-f(va)). \tag{4.7}$$

Chung and Yau [23] have proved a Harnack inequality for eigenfunctions of $\mathcal{L}$ on unweighted (V,K) where $\mu(a)=1$ for all $a\in K$. By Proposition 4.3.2, the positive eigenfunctions of $\mathcal{L}$ corresponding to the eigenvalue $\lambda=0$ are constant. By [18, Corollary 3.14], the ℓ^p-eigenfunctions of $\mathcal{L}$ for $\lambda=0$ and $1\le p<\infty$ are also constant. Extending the idea in [23], we consider below eigenfunctions corresponding to eigenvalues $\lambda>0$ for a Schrödinger operator $\mathcal{L}+\varphi$ which is a positive operator on the Hilbert space $\ell^2(V)$ if $\varphi\ge 0$, but may be unbounded if V is infinite.

We note that if K is a subgroup of G in an invariant homogeneous graph (V,K), then V is a disjoint union of connected components: $V=\bigcup_{v\in V_0}vK$ for some set of vertices V_0. The vertex set S of any union of these components satisfies $SK\subset S$.

Theorem 4.3.3 *Let (V,K) be an invariant homogeneous graph. Let $\varphi\ge 0$ be a function on V and let f be a real function on V satisfying*

$$\mathcal{L}f+\varphi f=\lambda f \qquad (\lambda>0). \tag{4.8}$$

Then on any finite subgraph with vertex set S satisfying $SK\subset S$, we have

$$\sum_{a\in K}w_a[f(v)-f(va)]^2+\alpha\lambda f^2(v)\le\left(\frac{\alpha^2\lambda}{\alpha-2}+\frac{4}{(\alpha-2)\lambda}\sup_S\varphi\right)\sup_S f^2$$

for $v \in S$ and $\alpha > 2$. In particular, the inequality holds for all $v \in V$ if V is finite, with $S = V$.

Proof. We extend the arguments in [23] and include the details for later reference. Define

$$\rho(v) = \sum_{a \in K} w_a [f(v) - f(va)]^2 \qquad (v \in S)$$

and let $\mathcal{L}$ act on the functions ρ and f^2. First consider

$$\begin{aligned} \mathcal{L}\rho(v) &= \sum_{b \in K} w_b \sum_{a \in K} w_a \left([f(v) - f(va)]^2 - [f(vb) - f(vba)]^2 \right) \\ &= -\sum_{b \in K} w_b \sum_{a \in K} w_a [f(v) - f(va) - f(vb) + f(vba)]^2 \\ &+ 2 \sum_{b \in K} w_b \sum_{a \in K} w_a [f(v) - f(va) - f(vb) + f(vba)][f(v) - f(va)]. \end{aligned}$$

Let X denote the second term above. We have

$$\begin{aligned} X &= 2 \sum_{b \in K} w_b \sum_{a \in K} w_a [f(v) - f(va) - f(vb) + f(vba)][f(v) - f(va)] \\ &= 2 \sum_{a \in K} w_a \left(\sum_{b \in K} w_b [f(v) - f(va) - f(vb) + f(vab)] \right) [f(v) - f(va)] \\ &+ 2 \sum_{a \in K} w_a \left(\sum_{b \in K} w_b [f(vba) - f(vab)] \right) [f(v) - f(va)] \\ &= 2\lambda \sum_{a \in K} w_a [f(v) - f(va)]^2 + 2 \sum_{a \in K} w_a [\varphi(va) f(va) - \varphi(v) f(v)][f(v) - f(va)] \end{aligned}$$

where

$$\begin{aligned} \sum_{b \in K} w_b [f(v) - f(vb)] &= \lambda f(v) - \varphi(v) f(v), \\ \sum_{b \in K} w_b [f(va) - f(vab)] &= \lambda f(va) - \varphi(va) f(va) \end{aligned}$$

and $\sum_{b \in K} w_b [f(vba) - f(vab)] = 0$ follows from (4.6), the symmetry of μ and graph invariance. It follows that

$$\begin{aligned}
&\mathcal{L}\rho(v) \le X \\
= \;& 2\lambda \sum_{a\in K} w_a[f(v)-f(va)]^2 + 2\sum_{a\in K} w_a[\varphi(va)f(va)-\varphi(v)f(v)][f(v)-f(va)] \\
\le \;& 2\lambda \sum_{a\in K} w_a[f(v)-f(va)]^2 + 2\sum_{a\in K} w_a[\varphi(va)f(va)f(v)+\varphi(v)f(v)f(va)].
\end{aligned}$$

Next we consider

$$\begin{aligned}
\mathcal{L}f^2(v) &= \sum_{a\in K} w_a[f^2(v)-f^2(va)] \\
&= 2\sum_{a\in K} w_a f(v)[f(v)-f(va)] - \sum_{a\in K} w_a[f(v)-f(va)]^2 \\
&= 2(\lambda-\varphi(v))f^2(v) - \sum_{a\in K} w_a[f(v)-f(va)]^2.
\end{aligned}$$

Putting the last two inequalities above together, we arrive at

$$\begin{aligned}
&\mathcal{L}(\rho(v)+\alpha\lambda f^2(v)) \\
\le \;& 2\alpha\lambda(\lambda-\varphi(v))f^2(v) - (\alpha-2)\lambda\sum_{a\in K} w_a[f(v)-f(va)]^2 \\
& +2f(v)\sum_{a\in K} w_a\varphi(va)f(va) + 2\varphi(v)f(v)\sum_{a\in K} w_a f(va).
\end{aligned}$$

We can find $s \in S$ such that

$$\rho(s)+\alpha\lambda f^2(s) = \sup\{\rho(v)+\alpha\lambda f^2(v) : v \in S\}.$$

Since $SK \subset S$, we have

$$\begin{aligned}
0 \;\le\;& \mathcal{L}(\rho(s)+\alpha\lambda f^2(s)) \\
\le\;& 2\alpha\lambda(\lambda-\varphi(s))f^2(s) - (\alpha-2)\lambda\sum_{a\in K} w_a[f(s)-f(sa)]^2 \\
& +2f(s)\sum_{a\in K} w_a\varphi(sa)f(sa) + 2\varphi(s)f(s)\sum_{a\in K} w_a f(sa).
\end{aligned}$$

This implies

$$\begin{aligned}
&\sum_{a\in K} w_a[f(s)-f(sa)]^2 \\
\leq\ & \frac{1}{(\alpha-2)\lambda}\left(2\alpha\lambda(\lambda-\varphi(s))f^2(s)+2f(s)\sum_{a\in K} w_a\varphi(sa)f(sa)+2\varphi(s)f(s)\sum_{a\in K} w_a f(sa)\right).
\end{aligned}$$

Hence for every $v\in S$, we have

$$\begin{aligned}
&\sum_{a\in K} w_a[f(v)-f(va)]^2+\alpha\lambda f^2(v) \\
\leq\ & \frac{1}{(\alpha-2)\lambda}\left(2\alpha\lambda(\lambda-\varphi(s))f^2(s)+2f(s)\sum_{a\in K} w_a\varphi(sa)f(sa)\right. \\
&\left. +\,2\varphi(s)f(s)\sum_{a\in K} w_a f(sa)+\alpha\lambda(\alpha-2)\lambda f^2(s)\right) \\
\leq\ & \frac{1}{(\alpha-2)\lambda}\left(\alpha^2\lambda^2 f^2(s)+2f(s)\sum_{a\in K} w_a\varphi(sa)f(sa)\right. \\
&\left. +\,2\varphi(s)f(s)\sum_{a\in K} w_a f(sa)\right) \\
\leq\ & \frac{1}{(\alpha-2)\lambda}\left(\alpha^2\lambda^2 f^2(s)+\sum_{a\in K} w_a\varphi(sa)(f^2(s)+f^2(sa))\right. \\
&\left. +\sum_{a\in K} w_a\varphi(s)(f^2(s)+f^2(sa))\right) \\
\leq\ & \frac{\alpha^2\lambda}{\alpha-2}\sup_S f^2+\frac{4}{(\alpha-2)\lambda}\sup_S\varphi\sup_S f^2.
\end{aligned}$$

□

Remark 4.3.4 For $\varphi=0$ and $w_a=\frac{1}{|K|}$ in Theorem 4.3.3, the inequality there is identical with the Harnack inequality for finite V in [23].

As mentioned in Section 4.1, an important problem in spectral geometry is to obtain lower or upper bounds of the first positive eigenvalue λ_1 of the Laplacian

$\mathcal{L}$ on a finite graph (V, E) with $|V| = n$, where $|V|$ is the cardinality of V. The *diameter* D of a graph is defined by

$$D = \sup\{\operatorname{dist}(u, v) : u, v \in V\}.$$

Many results concerning lower or upper bounds of λ_1 can be stated in terms of D and $|V|$. For instance, if (V, E) is d-regular and connected, then we have

$$\lambda_1 \geq \frac{1}{dnD}$$

by [21, Lemma 1.9]. Moreover, for any connected graph (V, E), one has [21]

$$\lambda_1 > \frac{2}{n^4}$$

by Cheeger inequality. Here we obtain a lower bound for λ_1 of $\mathcal{L}$ on a finite weighted graph in terms of its diameter using the above Harnack inequality. This result is similar to Chung and Yau's results in [23, 24] for Dirichlet and Neumann first eigenvalues.

Corollary 4.3.5 *Let (V, K) be a finite invariant homogeneous graph with $|V| = n$. Let $\varphi = 0$ on V and let f be a real function on V corresponding to the first positive eigenvalue λ_1 of* (4.8). *Then*

$$w_a[f(v) - f(va)]^2 \leq 8\lambda_1$$

for all $v \in V$ and $a \in K$. Moreover, we have

$$\lambda_1 \geq \frac{k}{8D^2}$$

where $k = \min\{w_a : a \in K\}$ and D is the diameter of (V, K).

Proof. We have $V = S$ in Theorem 4.3.3. We note that $\sum_{v \in V} f(v) = 0$ since $f \perp 1$ where 1 is an eigenfunction corresponding to the eigenvalue 0. By normalizing, we can choose f such that

$$\sup_{v \in V} |f(v)| = 1 = \sup_{v \in V} f(v).$$

Then for all $v \in V$ and $a \in K$, we have

$$w_a[f(v) - f(va)]^2 \leq 8\lambda_1$$

by letting $\alpha = 4$ in Theorem 4.3.3. Now let $f(u) = 1$ and $f(s) \leq 0$ for some $u, s \in V$. Then there exists a shortest path P in V joining u and s. Suppose P has vertices $(u = v_0, v_1, \cdots, v_t = s)$ where $v_{j+1} = v_j a_j$ with $a_j \in K$ for $0 \leq j \leq t-1$. We consider

$$X = \sum_{j=0}^{t-1} w_{a_j}[f(v_j) - f(v_{j+1})]^2.$$

Then $X \leq 8t\lambda_1 \leq 8D\lambda_1$. Also, we have

$$\begin{aligned} X \cdot \frac{D}{k} &\geq \left(\sum_{j=0}^{t-1} w_{a_j}[f(v_j) - f(v_{j+1})]^2\right)\left(\sum_{j=0}^{t-1} \frac{1}{w_{a_j}}\right) \\ &\geq [f(u) - f(s)]^2 \geq 1. \end{aligned}$$

It follows that

$$\lambda_1 \geq \frac{k}{8D^2}.$$

□

Finally we derive a version of Harnack inequality for Dirichlet eigenfunctions on a finite convex subgraph of an invariant homogeneous graph (V, K), extending the result in [24]. The *boundary* δS of a subgraph of (V, K) with vertex set S is defined by $\delta S = \{v \in V \setminus S : v \sim u \text{ for some } u \in S\}$, where $\sim$ denotes adjacency. A subgraph of (V, K) with vertex set S is called *convex* [24] if, for any subset $Y \subset \delta S$, its neighborhood $N(Y) = \{v \in V : v \sim u \text{ for some } u \in Y\}$ satisfies the boundary expansion property:

$$|N(Y) \setminus (S \cup \delta S)| = |\{v \notin S \cup \delta S : v \sim u \text{ for some } u \in Y\}| \geq |Y|.$$

An eigenfunction f on $S \cup \delta S$ of a Schrödinger operator $\mathcal{L} + \varphi$ is said to satisfy the *Dirichlet boundary condition* if $f(v) = 0$ for $v \in \delta S$. First, we give two useful observations.

Lemma 4.3.6 *Let S be a finite convex subgraph of a homogeneous graph (V, K). Let $\varphi \geq 0$ and let f be a real function on $S \cup \delta S$ satisfying*

$$\mathcal{L}f(v) + \varphi(v)f(v) \left(= \sum_{a \in K} w_a(f(v) - f(va)) + \varphi(v)f(v) \right) = \lambda f(v) \qquad (\lambda > 0) \tag{4.9}$$

for $v \in S$ and $f(v) = 0$ for $v \in \delta S$. Then f can be extended to all vertices of V which are adjacent to some vertex in $S \cup \delta S$ such that

$$\mathcal{L}f(v) + \varphi(v)f(v) = \lambda f(v) \qquad (\lambda > 0)$$

for $v \in \delta S$.

Proof. We note that δS is finite since K is finite. As in the proof of [24, Theorem 1], we consider a system of $|\delta S|$ equations:

$$\sum_{a \in K} w_a(f(v) - f(va)) + \varphi(v)f(v) = \lambda f(v)$$

for each $v \in \delta S$. This implies that for each $v \in \delta S$, we have

$$\sum_{\substack{a \,\in\, K \\ va \,\notin\, S \cup \delta S}} w_a f(va) = - \sum_{\substack{g \,\in\, K \\ vg \,\in\, S}} w_g f(vg).$$

The boundary expansion property enables us to find solutions of the above equations for the value $f(va)$, for each $va \notin S \cup \delta S$ where $a \in K$. Hence f can be extended to a function satisfying (4.9) on δS. $\square$

Lemma 4.3.7 *Let S be a finite convex subgraph of a homogeneous graph (V, K) and let f be a real function on $S \cup \delta S$ satisfying* (4.9). *Let*

$$\phi_a(v) = w_a[f(v) - f(va)]^2 + \alpha\lambda f^2(v) \qquad (v \in S \cup \delta S, a \in K).$$

Then for $\alpha > |K|/\lambda k$ with $k = \min\{w_a : a \in K\}$, there exist some $s \in S$ and $b \in K$ such that

$$\phi_b(s) = \sup\{\phi_a(v) : v \in S \cup \delta S, a \in K\}.$$

Proof. For any $v \in \delta S, a \in K$, there exist some $b \in K$ and $s \in S$ with $s \sim v$ such that $\phi_b(s) \geq \phi_a(v)$. This can be seen in the three cases below.

(i) If $va \in \delta S$, then

$$\phi_a(v) = w_a[0 - 0]^2 + \alpha\lambda 0^2 = 0.$$

(ii) If $va \in S$, then

$$\begin{aligned}
\phi_a(v) &= w_a[0 - f(va)]^2 + \alpha\lambda 0^2 = w_a f^2(va) \\
&\leq w_a f^2(va) + \alpha\lambda f^2(va) \\
&= w_{a^{-1}}[f(va) - f(v)]^2 + \alpha\lambda f^2(va) \\
&= \phi_{a^{-1}}(va).
\end{aligned}$$

(iii) If $va \notin S \cup \delta S$, then let

$$w_a f(va) = - \sum_{\substack{g \in K \\ vg \in S}} w_g f(vg)$$

and $f(vg) = 0$ for all $g \in K \setminus \{a\},\ vg \notin S \cup \delta S$. Let

$$f^2(vh) = \sup\{f^2(vg) : g \in K, vg \in S\}$$

for some $h \in K$. Then

$$\begin{aligned}
\phi_{h^{-1}}(vh) &= w_{h^{-1}} f^2(vh) + \alpha\lambda f^2(vh) \ge \frac{|K|}{k} f^2(vh) \\
&\ge \frac{|K|}{w_a} f^2(vh) \ge \frac{1}{w_a} \sum_{\substack{g \in K \\ vg \in S}} f^2(vg) \\
&\ge \frac{1}{w_a} \left[\sum_{\substack{g \in K \\ vg \in S}} w_g^2 \right] \left[\sum_{\substack{g \in K \\ vg \in S}} f^2(vg) \right] \\
&\ge \frac{1}{w_a} \left[\sum_{\substack{g \in K \\ vg \in S}} w_g f(vg) \right]^2 = w_a \left[\frac{\sum_{\substack{g \in K \\ vg \in S}} w_g f(vg)}{w_a} \right]^2 \\
&= w_a f^2(va) = \phi_a(v).
\end{aligned}$$

□

Theorem 4.3.8 *Let S be a finite convex subgraph of an invariant homogeneous graph (V, K). Let $\varphi \ge 0$ and let f be a real function on $S \cup \delta S$ satisfying*

$$\mathcal{L}f(v) + \varphi(v) f(v) = \lambda f(v) \qquad (\lambda > 0) \tag{4.10}$$

for $v \in S$ and $f(v) = 0$ for $v \in \delta S$. Then we have the inequality

$$w_a[f(v) - f(va)]^2 + \alpha\lambda f^2(v) \le \left(\frac{\alpha^2 \lambda}{\alpha - 2} + \frac{4}{(\alpha - 2)\lambda} \sup_S \varphi \right) \sup_S f^2$$

for $v \in S$, $a \in K$ and $\alpha > \max\{2, |K|/\lambda k\}$ where $k = \min\{w_a : a \in K\}$.

Proof. By Lemma 4.3.6, f can be extended to a function, still denoted by f, on all vertices of V adjacent to $S \cup \delta S$ so that equation (4.10) also holds on δS. As in the proof of Theorem 4.3.3, one can apply similar arguments to the function

$$w_a[f(v) - f(va)]^2 + \alpha\lambda f^2(v) \qquad (v \in S,\ a \in K).$$

Define

$$\rho_a(v) = w_a[f(v) - f(va)]^2 \qquad (v \in S, a \in K)$$

and let $\mathcal{L}$ act on the functions ρ_a and f^2. First consider

$$\begin{aligned}
\mathcal{L}\rho_a(v) &= w_a \sum_{b\in K} w_b \left([f(v) - f(va)]^2 - [f(vb) - f(vba)]^2\right) \\
&= -w_a \sum_{b\in K} w_b[f(v) - f(va) - f(vb) + f(vba)]^2 \\
&+ 2w_a \sum_{b\in K} w_b[f(v) - f(va) - f(vb) + f(vba)][f(v) - f(va)].
\end{aligned}$$

Let X denote the second term above. We have

$$\begin{aligned}
X &= 2w_a \sum_{b\in K} w_b[f(v) - f(va) - f(vb) + f(vba)][f(v) - f(va)] \\
&= 2w_a \left(\sum_{b\in K} w_b[f(v) - f(va) - f(vb) + f(vab)]\right)[f(v) - f(va)] \\
&+ 2w_a \left(\sum_{b\in K} w_b[f(vba) - f(vab)]\right)[f(v) - f(va)] \\
&= 2\lambda w_a[f(v) - f(va)]^2 + 2w_a[\varphi(va)f(va) - \varphi(v)f(v)][f(v) - f(va)]
\end{aligned}$$

where

$$\begin{aligned}
\sum_{b\in K} w_b[f(v) - f(vb)] &= \lambda f(v) - \varphi(v)f(v), \\
\sum_{b\in K} w_b[f(va) - f(vab)] &= \lambda f(va) - \varphi(va)f(va)
\end{aligned}$$

and $\sum_{b\in K} w_b[f(vba) - f(vab)] = 0$ follows from (4.6), the symmetry of μ and graph invariance. It follows that

$$\begin{aligned}
&\mathcal{L}\rho_a(v) \leq X \\
&= 2\lambda w_a[f(v) - f(va)]^2 + 2w_a[\varphi(va)f(va) - \varphi(v)f(v)][f(v) - f(va)] \\
&\leq 2\lambda w_a[f(v) - f(va)]^2 + 2w_a[\varphi(va)f(va)f(v) + \varphi(v)f(v)f(va)].
\end{aligned}$$

Next we consider $\mathcal{L}f^2$. As in the proof of Theorem 4.3.3, we have

$$\begin{aligned} \mathcal{L}f^2(v) &= 2(\lambda - \varphi(v))f^2(v) - \sum_{a\in K} w_a[f(v) - f(va)]^2 \\ &\leq 2(\lambda - \varphi(v))f^2(v) - w_a[f(v) - f(va)]^2. \end{aligned}$$

Putting the last two inequalities above together, we arrive at

$$\begin{aligned} & \mathcal{L}(\rho_a(v) + \alpha\lambda f^2(v)) \\ \leq\ & 2\alpha\lambda(\lambda - \varphi(v))f^2(v) - (\alpha - 2)\lambda w_a[f(v) - f(va)]^2 \\ & +2w_a\left(\varphi(va)f(v)f(va) + \varphi(v)f(v)(va)\right). \end{aligned}$$

Given $\alpha > |K|/\lambda k$, by Lemma 4.3.7, we can find $s \in S$ and $b \in K$ satisfying

$$\rho_b(s) + \alpha\lambda f^2(s) = \sup\{\rho_a(v) + \alpha\lambda f^2(v) : v \in S \cup \delta S, a \in K\}.$$

Hence

$$\begin{aligned} 0 &\leq \mathcal{L}(\rho_b(s) + \alpha\lambda f^2(s)) \\ &\leq 2\alpha\lambda(\lambda - \varphi(s))f^2(s) - (\alpha - 2)\lambda w_b[f(s) - f(sb)]^2 \\ &\quad +2w_b\left(\varphi(sb)f(s)f(sb) + \varphi(s)f(s)f(sb)\right). \end{aligned}$$

This implies

$$\begin{aligned} & w_b[f(s) - f(sb)]^2 \\ \leq\ & \frac{1}{(\alpha - 2)\lambda}\left(2\alpha\lambda(\lambda - \varphi(s))f^2(s) + 2w_b\varphi(sb)f(s)f(sb) + 2w_b\varphi(s)f(s)f(sb)\right). \end{aligned}$$

Hence for every $v \in S$, we have

$$\begin{aligned}
& w_a[f(v) - f(va)]^2 + \alpha\lambda f^2(v) \\
\leq\ & \frac{1}{(\alpha-2)\lambda}\big(2\alpha\lambda(\lambda - \varphi(s))f^2(s) + 2w_b\varphi(sb)f(s)f(sb) \\
& + 2w_b\varphi(s)f(s)f(sb) + \alpha\lambda(\alpha-2)\lambda f^2(s)\big) \\
\leq\ & \frac{1}{(\alpha-2)\lambda}\big(\alpha^2\lambda^2 f^2(s) + 2w_b\varphi(sb)f(s)f(sb) + 2w_b\varphi(s)f(s)f(sb)\big) \\
\leq\ & \frac{1}{(\alpha-2)\lambda}\big(\alpha^2\lambda^2 f^2(s) + w_b\varphi(sb)(f^2(s) + f^2(sb)) \\
& + w_b\varphi(s)(f^2(s) + f^2(sb))\big) \\
\leq\ & \frac{\alpha^2\lambda}{\alpha-2}\sup_S f^2 + \frac{4}{(\alpha-2)\lambda}\sup_S \varphi \sup_S f^2.
\end{aligned}$$

where the last inequality follows from $w_b \leq 1$. □

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