Graded representations
of
Khovanov-Lauda-Rouquier algebras

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Abstract

The Khovanov–Lauda–Rouquier algebras $\mathscr{R}_n$ are a relatively new family of $\mathbb{Z}$-graded algebras. Their cyclotomic quotients $\mathscr{R}_n^\Lambda$ are intimately connected to a smaller family of algebras, the cyclotomic Hecke algebras $\mathscr{H}_n^\Lambda$ of type A, via Brundan and Kleshchev's Graded Isomorphism Theorem. The study of representation theory of $\mathscr{H}_n^\Lambda$ is well developed, partly inspired by the remaining open questions about the modular representations of the symmetric group $\mathfrak{S}_n$.

There is a profound interplay between the representations for $\mathfrak{S}_n$ and combinatorics, whereby each irreducible representation in characteristic zero can be realised as a Specht module whose basis is constructed from combinatorial objects. For $\mathscr{R}_n^\Lambda$, we can similarly construct their representations as analogous Specht modules in a combinatorial fashion. Many results can be lifted through the Graded Isomorphism Theorem from the symmetric group algebras, and more so from $\mathscr{H}_n^\Lambda$, to the cyclotomic Khovanov–Lauda–Rouquier algebras, providing a foundation for the representation theory of $\mathscr{R}_n^\Lambda$.

Following the introduction of $\mathscr{R}_n^\Lambda$, Brundan, Kleshchev and Wang discovered that Specht modules over $\mathscr{R}_n^\Lambda$ have $\mathbb{Z}$-graded bases, giving rise to the study of graded Specht modules. In this thesis we solely study graded Specht modules and their irreducible quotients for $\mathscr{R}_n^\Lambda$. One of the main problems in graded representation theory of $\mathscr{R}_n^\Lambda$, the Graded Decomposition Number Problem, is to determine the graded multiplicities of graded irreducible $\mathscr{R}_n^\Lambda$-modules arising as graded composition factors of graded Specht modules.

We first consider $\mathscr{R}_n^\Lambda$ in level one, which is isomorphic to the Iwahori–Hecke algebra of type A, and research graded Specht modules labelled by hook partitions in this context. In quantum characteristic two, we extend to $\mathscr{R}_n^\Lambda$ a result of Murphy for the symmetric groups, determining graded filtrations of Specht modules labelled by hook partitions, whose factors appear as Specht modules labelled by two-part partitions. In quantum characteristic at least three, we determine an analogous $\mathscr{R}_n^\Lambda$-version of Peel's Theorem for the symmetric groups, providing an alternative approach to Chuang, Miyachi and Tan.

We then study graded Specht modules labelled by hook bipartitions for $\mathscr{R}_n^\Lambda$ in level two, which is isomorphic to the Iwahori–Hecke algebra of type B. In quantum characterisitic at least three, we completely determine the composition factors of Specht modules labelled by hook bipartitions for $\mathscr{R}_n^\Lambda$, together with their graded analogues.

Declaration

I, Louise Sutton, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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The contents in Part III Chapters 5 to 7 will form the paper *Graded Specht modules labelled by hook bipartitions I*, arXiv:1707.01851, 2017.

The contents in Part III Chapters 8 to 13 will form the paper *Graded Specht modules labelled by hook bipartitions II*, currently in preparation.

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Contents

Part I

Introductory materials

The universe is an enormous direct product of representations of symmetry groups.

Hermann Weyl

Introduction

Symmetry permeates the natural world, from the intricacy of spider webs to the vastness of the Milky Way; its beauty reaches far. Its prevalence in our surroundings has inspired many in history to understand the collection of symmetries of a tangible object; it is logical to ask if the concept of symmetry can be formalised abstractly. The story of the symmetric group as we know it today, however, does not begin with this question.

The foundations of permutation groups, and in particular, the symmetric group, date back to Galois' work [G] on solutions to polynomial equations from the nineteenth century. Cauchy [Cau] formally defined the notion of a permutation group (which he called *a system of conjugate substitutions*), and more generally, of a finite group. The study of permutation groups in their own right was continued by Cayley [Cay], who introduced the first abstract definition of a group. One of the most fundamental theorems in the beginnings of group theory is attributed to Cayley: every group is isomorphic to a permutation group. Since every subgroup of a symmetric group arises as a permutation group, Cayley's Theorem emphasises the importance of the symmetric group.

The beginning of the twentieth century gave rise to the representation theory of the symmetric group; one began to question the linear actions of these groups on a vector space. Frobenius [F] inaugurated the ordinary irreducible representations of the symmetric group in 1900, swiftly followed by Young's independent approach [Y1, Y2]. The original construction of the ordinary representations of the symmetric group as a special class of modules, the Specht modules S_λ labelled by partitions, was given by Specht [Sp]. What we now refer to as the classical theory of Specht modules was developed by James [J2], who built on the combinatorics established by Young to afford these modules with a combinatorial basis; the dimensions of Specht modules were discovered by Frame, Robinson and Thrall's [FRT] Hook Length Formula. By taking particular quotients D_μ of Specht modules, James completely classified the modular representations of the symmetric group. The main problems we face in representation theory of the symmetric group are to determine the multiplicities $[S_\lambda : D_\mu]$ of these quotients arising as composition factors of Specht modules, known as *decomposition numbers*, and to completely understand the structure of Specht modules.

Around the same time, Schur introduced his own algebras in his PhD thesis [Sch], now referred to as *Schur algebras*. Schur classified the ordinary irreducible representations of the general linear group by exploiting his newly defined algebras, together with the symmetric group algebras, and thus revealing a deep connection between the representations of the symmetric group and those of the general linear group. Dipper and James developed the q-analogues of Schur's algebras [DJ2], motivated to classify the modular irreducible representations of $GL_n(q)$ in non-defining characteristic.

The *Iwahori–Hecke algebras* are closely related to the q-Schur algebras, which emerged from the study of endomorphism algebras of representations of groups induced by representations of subgroups (see [I] and [Bou]) and were later established in [DJ1]. These algebras are q-deformations of permutation groups of different Coxeter types: q-deformations of the symmetric group algebra in type A [M1], and of the signed symmetric group algebra in type B [DJ3].

Lascoux, Leclerc and Thibon made a momentous advance in the representation theory of the Iwahori–Hecke algebras when they presented a remarkable algorithm [LLT], now known as the LLT algorithm. Using the representation theory of the quantum affine algebra $U_v(\hat{sl}_n)$, for every pair of partitions λ and e-regular μ, Lascoux, Leclerc and Thibon introduced the polynomials $d_{\lambda,\mu}(v)$ with integer coefficients by applying their algorithm. These polynomials, called *v-decomposition numbers*, appear as coefficients of the canonical basis elements of $U_v(\hat{sl}_n)$. They conjectured that the v-decomposition numbers are v-analogues of decomposition numbers for Iwahori–Hecke algebras of type A over a field of characteristic zero.

Ariki [A2] subsequently proved Lascoux, Leclerc and Thibon's conjecture for a larger class of algebras, the *Ariki–Koike algebras*. The Ariki–Koike algebras are a natural generalisation of the Iwahori–Hecke algebras: Ariki and Koike [AK] associated a Hecke algebra to each complex reflection group $(\mathbb{Z}/l\mathbb{Z})\wr\mathfrak{S}_n$ of type $G(l,1,n)$ in the Shephard–Todd classification [ST]. These algebras, defined over the polynomial ring $\mathbb{Z}[Q_1,\dots,Q_l,q,q^{-1}]$, lie in the *cyclotomic Hecke algebras*, a larger family of algebras that were developed by Broué and Malle [BM]. Their construction of the cyclotomic Hecke algebras yielded Hecke algebras for some other complex reflection groups, in particular, for all complex reflection groups of type $G(l,2,n)$ (where 2 divides l) as well as of some exceptional types. This class of algebras was further generalised by Ariki [A1], and in doing so, he introduced Hecke algebras for the remaining complex reflection groups.

Khovanov and Lauda [KL1, KL2] introduced an extraordinary family of diagram algebras, that were independently discovered via Rouquier's algebraic approach [Rou2]. These algebras are know aptly named the *Khovanov–Lauda–Rouquier algebras* $\mathscr{R}_n$, which are naturally non-trivially $\mathbb{Z}$-graded. Astonishingly, Brundan and Kleshchev [BK2] showed that each cyclotomic Hecke algebra of type A is isomorphic to a cyclotomic quotient of the Khovanov-Lauda-Rouquier algebra via their Graded Isomorphism Theorem. Brundan and Kleshchev's isomorphism affirms independent speculations by Rouquier [Rou1, Remark 3.11] and Turner [T] that the cyclotomic Hecke algebras of type A could be attributed non-trivial gradings. This remarkable set of papers motivated the study of *graded* representation theory of Khovanov–Lauda–Rouquier algebras, and, in particular, of the symmetric group algebras.

Shortly after the discovery of the Khovanov–Lauda–Rouquier algebras, Brundan, Kleshchev and Wang [BKW] introduced a non-trivially $\mathbb{Z}$-grading of Specht modules over $\mathscr{R}_n$, which yields a recursive combinatorial formula for their *graded* dimensions. So, we can study *graded* Specht modules and their corresponding *graded decomposition numbers* $[S_\lambda : D_\mu]_v$, which encode grading shifts of their composition factors. Brundan and Kleshchev in [BK3] showed that these graded decomposition numbers are the same as the v-decomposition numbers as mentioned above, and provide a generalised graded analogue of Lascoux, Leclerc and Thibon's conjecture. This extra structure now afforded to cyclotomic Hecke algebras raises new interesting questions about the representations in this graded world: Can we determine their graded decomposition numbers? Can we find a graded analogue to Frame, Robinson and Thrall's Hook Length Formula?

This thesis is split into three distinct parts; the rest of Part I contains Chapter 1, where we provide a review of the necessary background material for our study into these questions. In particular, we introduce the graded family of algebras we will be working over, the Khovanov–Lauda–Rouquier algebras and their cyclotomic quotients, and the Specht modules defined over these algebras. The remaining parts of this thesis tackle these fundamental questions for the cyclotomic Khovanov–Lauda–Rouquier algebras.

In Part II we restrict our study from the general Khovanov–Lauda–Rouquier algebras to the Iwahori–Hecke algebras of type A, and study graded Specht modules in this setting. Our main results, determining part of the graded decomposition matrix for $\mathscr{R}_n^\Lambda$, provide an alternative approach to part of Chuang, Miyachi and Tan's result [CMT, Theorem 1], which predates the theory of Khovanov–Lauda–Rouquier algebras.

We begin with an investigation of graded dimensions of certain Specht modules for the Iwahori–Hecke algebra of type A in Chapter 2. We discover a closed formula for the graded dimension of Specht modules labelled by two-part partitions in quantum characteristic two, illustrating the complicated nature of these gradings on Specht modules and that a graded Hook Length Formula is most unlikely.

In Chapter 3, we study Specht modules $S_{(n-m,1^m)}$ labelled by hook partitions in quantum characteristic two. Inspired by Murphy's work [Mu2] on hook representations and their connection with two-part representations, we find that every $S_{(n-m,1^m)}$ has a Specht filtration whose factors appear as Specht modules labelled by two-part partitions. Moreover, from a result of James [JM2, Theorem 4.15], we know that Specht modules labelled by two-part partitions are irreducible when n is odd in characteristic zero, establishing that the filtrations of $S_{(n-m,1^m)}$ are in fact composition series in this case. This leads us to determine the corresponding graded decomposition matrices for the Iwahori–Hecke algebras of type A, comprising rows corresponding to hook partitions.

With quantum characteristic at least three, we continue the study of Specht modules labelled by hook partitions in Chapter 4. In this context, Peel's Theorem [P, Theorem 2] gives us the ungraded decomposition matrices comprising rows corresponding to hook partitions. We provide results on the graded dimension of $S_{(n-m,1^m)}$, and thus determine the analogous graded decomposition matrices.

In Part III, we study graded Specht modules $S_{((n-m),(1^m))}$ labelled by hook bipartitions for the Iwahori–Hecke algebras of type B, quotients of the general Khovanov–Lauda–Rouquier algebras, in quantum characteristic at least three. The structure of $S_{((n-m),(1^m))}$ depends on parameters κ and n, and thus we examine these Specht modules in four separate cases. For Chapters 5 to 7, we forget the grading on Specht modules.

We begin with Chapter 5: we provide an introduction to the study of $S_{((n-m),(1^m))}$. We present a combinatorial description of the standard basis elements of $S_{((n-m),(1^m))}$ and understand the action of the Khovanov–Lauda–Rouquier basis elements $\{y_1,\dots,y_n\}\cup\{\psi_1,\dots,\psi_{n-1}\}$ on these standard basis elements.

Having established how $\mathscr{R}_n$ acts on standard basis elements of Specht modules, we now introduce certain Specht module homomorphisms in Chapter 6, which will be instrumental in finding the composition factors of $S_{((n-m),(1^m))}$. We show when these Specht module homomorphisms exhibit exact sequences, by ascertaining bases for the kernels and images of these homomorphisms.

In Chapter 7, we discover that the ungraded composition factors of $S_{((n-m),(1^m))}$ arise as certain quotients of the kernels and images of the aforementioned Specht module homomorphisms. To determine irreducibility of these $\mathscr{R}_n^\Lambda$-modules, we begin Chapter 7 by giving an explicit description of how the Khovanov–Lauda–Rouquier generators $\psi_1,\dots,\psi_{n-1}$ act on an arbitrary standard basis element of $S_{((n-m),(1^m))}$. By finding that every non-zero submodule of $S_{((n-m),(1^m))}$ contains a standard basis vector, we apply the action of $\mathscr{R}_n$ to completely determine the ungraded composition series of Specht modules labelled by hook bipartitions.

We draw on Brundan and Kleshchevs's i-restriction and i-induction functors from [BK1, §2.2] in Chapter 8, originating from Robinson's work [Rob], and compose these functors to introduce new induction and restriction functors. We first introduce the sgn-*restriction functor* to determine the bipartition that labels the irreducible head of the sign representation $S_{(\varnothing,(1^n))}$, and then introduce *arm* and *leg functors*, and variations thereof. These functors provide us with an understanding of how to find the irreducible labels of the composition factors of $S_{((n-m),(1^m))}$, which we determine in Chapter 9.

In Chapter 10 we completely determine the ungraded decomposition matrices for the Iwahori–Hecke algebra of type B comprising rows corresponding to hook bipartitions, by determining the distinct irreducible labels of the composition factors of $S_{((n-m),(1^m))}$.

We observe that we can obtain the analogous graded decomposition matrices to those given in Chapter 10, by finding the graded dimensions of $S_{((n-m),(1^m))}$ together with the graded dimensions of its graded composition factors. Chapter 11 is self contained, whereby we determine the graded dimensions of $S_{((n-m),(1^m))}$. In fact, it is necessary to only find the leading and second leading terms in the graded dimensions of the composition factors of $S_{((n-m),(1^m))}$, which we determine in Chapter 12. Collating our results from Chapters 10 to 12 we completely determine the graded decomposition matrices for the Iwahori–Hecke algebra of type B comprising rows corresponding to hook bipartitions in Chapter 13.

Chapter 1

Background

In this chapter we establish our notation and review fundamental background material. Let $\mathbb{F}$ be an arbitrary field throughout, and denote the characteristic of $\mathbb{F}$ by $\mathrm{char}(\mathbb{F})$.

1.1 Hecke algebras

This section serves to introduce the Khovanov–Lauda–Rouquier algebras. Later on we will establish that these algebras are non-trivially graded, which we first prepare for by providing an overview of graded algebra.

1.1.1 Graded algebras and graded modules

An $\mathbb{F}$-algebra A is called *graded*, more precisely, $\mathbb{Z}$-graded, if there exists a direct sum decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$ such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$. There are only finitely many non-zero summands A_i in the direct sum of A whenever A is finite dimensional. An element in the summand A_i is said to be *homogeneous*, and to have *degree* i. For $a_i \in A_i$, we write $\deg(a_i) = i$. We note that every algebra A can be trivially graded by setting $A_0 := A$ in the above decomposition.

Given a graded $\mathbb{F}$-algebra A, we say that the (left) A-module M is $\mathbb{Z}$*-graded* if there exists a direct sum decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ such that $A_iM_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. If M is a graded A-module, then we obtain the module $M\langle k\rangle$ by shifting the grading in M by $k \in \mathbb{Z}$. For an indeterminate v, we set $M\langle k\rangle = v^kM$, so that the grading on M is defined by $(M\langle k\rangle)_i = \left(v^kM\right)_i = M_{i-k}$.

For a graded A-module M, we define its *graded dimension* to be the Laurent polynomial

$$\mathrm{grdim}(M) = \sum_{i\in\mathbb{Z}} \dim(M_i)v^i.$$

Given graded A-modules M and N, an A-module homomorphism $f : M \to N$ satisfying $f(M_i) \subseteq N_{i+k}$ for all $i \in \mathbb{Z}$ is said to be *homogeneous of degree* k. If $k = 0$, then f is said to be a *degree preserving homomorphism.*

Let A be a graded $\mathbb{F}$-algebra and M be a graded A-module. A submodule $N \subset M$ is a *graded submodule* of M if, for every $n = \bigoplus_{i\in\mathbb{Z}} n_i \in N$, then each of its homogeneous components n_i also lies in N.

For a graded A-module M, we say that a *graded filtration* of M is a strictly increasing sequence of graded submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M, \tag{1.1.1}$$

whose factors are M_{i+1}/M_i for $i \in \{0, \ldots, r-1\}$. We refer to M_r/M_{r-1} as the top factor in the filtration of M, and similarly, to M_0 as its bottom factor.

Lemma 1.1. *[NO, Theorem 4.4.6 and Remark 4.4.8] Let M be a graded finitely generated A-module. The radical of M is a graded submodule of M.*

Let A be a graded $\mathbb{F}$-algebra. If a graded A-module M only has graded submodules 0 and itself, then we say that M is *irreducible* (or simple). The following result indicates that the theory of graded algebra is at least as rich as its ungraded counterpart.

Lemma 1.2. *[NO, Theorem 4.4.4] If M is an irreducible graded A-module, then the A-module obtained by forgetting the grading on M is also irreducible.*

We now observe how every irreducible graded module of a finite-dimensional algebra arises.

Lemma 1.3. *[NO, Theorem 9.6.8] Let A be a graded finite-dimensional algebra. If M is an irreducible ungraded A-module, then there is a unique grading on M up to grading shift and isomorphism.*

If the factors M_{i+1}/M_i arising in the graded filtration Equation (1.1.1) of M are irreducible, then we call this filtration a *graded composition series* for M and we say that the quotients M_{i+1}/M_i of M are the *graded composition factors* of M, which are well-defined by the existence of a graded analogue of the Jordan–Hölder theorem. It now makes sense to study *graded decomposition numbers* $[M : L]_v$ of M, where L is a graded irreducible A-module; the graded multiplicity of L as a composition factor of M is defined to be the Laurent polynomial

$$[M : L]_v = \sum_{i\in\mathbb{Z}} [M : L\langle i\rangle] v^i.$$

Note that by setting $v = 1$ in the above definitions, we recover the usual (ungraded) analogues.

1.1.2 Permutation groups

We review two important permutation groups.

1.1.2.1 The symmetric group

Definition 1.4. *The* symmetric group *on n letters, denoted $\mathfrak{S}_n$, is the set of $n!$ permutations π of $\{1,\dots,n\}$.*

In fact, $\mathfrak{S}_n$ is a Coxeter group generated by $s_1,\dots,s_{n-1}$, where s_i is the simple transposition $(i,i+1)$, for $0 \leqslant i \leqslant n-1$.

For $1 \leqslant i \leqslant j \leqslant n-1$, we define

$$s\overset{j}{\underset{i}{\downarrow}} := s_j s_{j-1}\dots s_i, \qquad s\overset{j}{\underset{i}{\uparrow}} := s_i s_{i+1}\dots s_j.$$

We say that a *reduced expression* for a permutation $\pi \in \mathfrak{S}_n$ is a minimal length word $\pi = s_{r_1}\dots s_{r_m}$ for $1 \leqslant r_i < n$ and $0 \leqslant i \leqslant m$. Let $\leqslant$ be the *Bruhat order* on $\mathfrak{S}_n$, defined as follows. For $\pi_1, \pi_2 \in \mathfrak{S}_n$, we write $\pi_1 \leqslant \pi_2$ if there is a reduced expression for π_1 which is a subexpression of a reduced expression for π_2.

1.1.2.2 The signed symmetric group

Definition 1.5. *The* signed symmetric group *of degree n, denoted $(\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n$, or equivalently, the hyperoctahedral group, is the set of $2^n n!$ permutations π of $\{-n,\dots,n\}$ such that $\pi(-i) = -\pi(i)$ for all i.*

For example, the signed symmetric group of degree 8 is the dihedral group $\mathfrak{S}_2 \wr \mathfrak{S}_2$.

In fact, $(\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n$ is a Coxeter group generated by $s_0, s_1,\dots,s_{n-1}$, where s_i is the permutation $(i,i+1)(-1-i,-i)$, for $i > 0$, and $s_0 = (-1,1)$.

1.1.3 Lie-theoretic notation

For $q \in \mathbb{F}^\times$, let e be the smallest positive integer such that $1+q+q^2+\dots+q^{e-1} = 0$, and set $e = \infty$ if no such integer exists. If an algebra A depends on the parameter q, we say that e is the *quantum characteristic* of A.

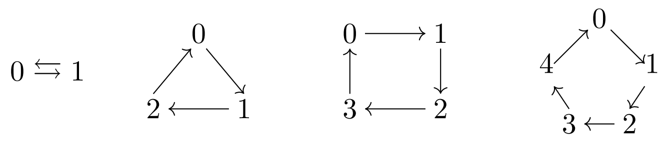
Define $I := \mathbb{Z}/e\mathbb{Z}$. If e is finite, then we identify I with the set $\{0,1,\dots,e-1\}$, whereas, if e is infinite, then we identify I with the set of integers $\mathbb{Z}$.

We let Γ be the quiver with vertex set I and directed edges $i \to i+1$ for each $i \in I$. If no directed edge exists between two vertices $i \neq j$, we write $i \not\to j$. If e is infinite, then Γ is the integral linear quiver of type A_∞ with the following orientation

$$\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots .$$

Otherwise, if e is finite, then Γ is the integral cyclic quiver on e vertices of type $A^{(1)}_{e-1}$.

For example, the cyclic quiver diagrams for $e = 2$, 3, 4 and 5, respectively, are



The associated Cartan matrix $C_\Gamma = (c_{i,j})_{i,j \in I}$ is defined by

$$c_{i,j} := \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } j \not- i, \\ -1 & \text{if } i \to j \text{ or } i \leftarrow j, \\ -2 & \text{if } i \leftrightarrows j. \end{cases}$$

The notation $i \leftrightarrows j$ indicates that $i = j - 1 = j + 1$, which only occurs when $e = 2$.

The generalised Cartan matrix C_Γ corresponds to a Kac–Moody algebra $\mathfrak{g}(C_\Gamma)$, as given in [Kac]. It follows that we have the simple roots $\{\alpha_i \mid i \in I\}$, the fundamental dominant weights $\{\Lambda_i \mid i \in I\}$, and the invariant symmetric bilinear form (,) such that $(\alpha_i, \alpha_j) = c_{i,j}$ and $(\Lambda_i, \alpha_j) = \delta_{ij}$, for all $i, j \in I$. Let $Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geqslant 0} \alpha_i$ be the positive part of the root lattice. A *root* $\alpha \in Q_+$ is a linear combination $\sum_{i \in I} a_i \alpha_i$ of its simple roots where $a_i \in \mathbb{Z}$, and the *height of* α is the sum $\sum_{i \in I} a_i$, denoted by $\mathrm{ht}(\alpha)$.

We now fix a *level* $l \in \mathbb{N}$. The symmetric group $\mathfrak{S}_l$ acts on the left by place permutation on the set I^l of all l-tuples. An *e-multicharge* of l is an ordered l-tuple $\kappa = (\kappa_1, \ldots, \kappa_l) \in I^l$. We define its associated dominant weight Λ of level l to be $\Lambda := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_l}$.

1.1.4 Iwahori–Hecke alebras

The Iwahori–Hecke algebras are q-deformations of the group algebras of Coxeter groups, whereby we recover the group algebra of the Coxeter group in question by setting $q = 1$.

1.1.4.1 Iwahori–Hecke algebras of type A

The Iwahori–Hecke algebras of type A are q-analogues of group algebras of symmetric groups.

Let $q \in \mathbb{F}^\times$. Then the Iwahori–Hecke algebra $\mathscr{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ [M1, §2] of the symmetric group, or equivalently, the Hecke algebra of type A, is the unital associative $\mathbb{F}$-algebra with generators

$$\{T_1, T_2, \ldots, T_{n-1}\}$$

subject to the relations

- $(T_i + 1)(T_i - q) = 0$, for $1 \leqslant i \leqslant n - 1$;

- ◇ $T_iT_j = T_jT_i$, for $0 \leqslant i < j-1 \leqslant n-2$;
- ◇ $T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i$, for $1 \leqslant i \leqslant n-2$.

We recover the symmetric group group algebra $\mathbb{F}\mathfrak{S}_n$ when we set $q = 1$.

1.1.4.2 Iwahori–Hecke algebras of type B

The Iwahori–Hecke algebras of type B are q-analogues of group algebras of signed symmetric groups.

Let $q, Q \in \mathbb{F}^\times$. Then the Iwahori–Hecke algebra $\mathscr{H}_{\mathbb{F},q,Q}((\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n)$ [DJ3, §3] of the signed symmetric group, or equivalently, the Hecke algebra of type B, is the unital associative $\mathbb{F}$-algebra with generators

$$\{T_0, T_1, T_2 \ldots, T_{n-1}\}$$

subject to the relations

- ◇ $(T_0 + 1)(T_0 - Q) = 0$;
- ◇ $T_0T_1T_0T_1 = T_1T_0T_1T_0$;
- ◇ $(T_i + 1)(T_i - q) = 0$, for $1 \leqslant i \leqslant n-1$;
- ◇ $T_iT_j = T_jT_i$, for $0 \leqslant i < j-1 \leqslant n-2$;
- ◇ $T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i$, for $1 \leqslant i \leqslant n-2$.

We recover the signed symmetric group algebra $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$ when we set $q = Q = 1$.

1.1.5 Ariki–Koike algebras

Iwahori–Hecke algebras of types A and B were generalised by Ariki and Koike [AK] to a larger family of algebras, the *Ariki–Koike algebras*, also referred to as the *cyclotomic Hecke algebras*. These algebras are Hecke algebras for each complex reflection group $(\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n$ of type $G(l, 1, n)$ in the Shephard–Todd classification [ST].

Let $q, Q_1, \ldots, Q_l \in \mathbb{F}^\times$ and set $Q = (Q_1, \ldots, Q_l)$. Then the *Ariki–Koike algebra* $\mathscr{H}_{\mathbb{F},q,Q}((\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n)$ is the unital associative $\mathbb{F}$-algebra with generators

$$\{T_0, T_1, \ldots, T_{n-1}\}$$

subject to the relations

- ◇ $(T_0 - Q_1)(T_0 - Q_2) \ldots (T_0 - Q_l) = 0$;
- ◇ $T_0T_1T_0T_1 = T_1T_0T_1T_0$;
- ◇ $(T_i + 1)(T_i - q) = 0$, for $1 \leqslant i \leqslant n-1$;

- $T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i$, for $1 \leqslant i \leqslant n-2$;
- $T_iT_j = T_jT_i$, for $0 \leqslant i < j-1 \leqslant n-2$.

By setting $l = 1$, we see that the Ariki–Koike algebra for $\mathfrak{S}_n$ of type $G(1,1,n)$ is the Iwahori–Hecke algebra of type A, and by setting $l = 2$, the Ariki–Koike algebra for $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$ of type $G(2,1,n)$ is the Iwahori–Hecke algebra of type B.

1.1.6 Khovanov–Lauda–Rouquier algebras

The Khovanov–Lauda–Rouquier algebras were discovered by Khovanov–Lauda [KL1], and independently, Rouquier [Rou2]. Brundan and Kleshchev transformed their work to give the following presentation.

Definition 1.6. *[BK2] Let $\alpha \in Q_+$ such that $\mathrm{ht}(\alpha) = n$, and define the set*

$$I^\alpha = \{\mathbf{i} \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \alpha\}.$$

Then the algebra $\mathscr{R}_\alpha$ is defined to be the unital associative $\mathbb{F}$-algebra generated by the elements

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}, \tag{1.1.2}$$

subject only to the following relations:

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i}); \qquad \textstyle\sum_{\mathbf{i}\in I^\alpha} e(\mathbf{i}) = 1; \tag{1.1.3}$$

$$y_re(\mathbf{i}) = e(\mathbf{i})y_r; \qquad \psi_re(\mathbf{i}) = e(s_r\mathbf{i})\psi_r; \tag{1.1.4}$$

$$y_ry_s = y_sy_r; \tag{1.1.5}$$

$$\psi_ry_s = y_s\psi_r \qquad \textit{if } s \neq r, r+1; \tag{1.1.6}$$

$$\psi_r\psi_s = \psi_s\psi_r \qquad \textit{if } |r-s| > 1; \tag{1.1.7}$$

$$\psi_ry_{r+1}e(\mathbf{i}) = (y_r\psi_r + \delta_{i_r,i_{r+1}})e(\mathbf{i}); \tag{1.1.8}$$

$$y_{r+1}\psi_re(\mathbf{i}) = (\psi_ry_r + \delta_{i_r,i_{r+1}})e(\mathbf{i}); \tag{1.1.9}$$

$$\psi_r^2e(\mathbf{i}) = \begin{cases} 0 & \textit{if } i_r = i_{r+1}, \\ e(\mathbf{i}) & \textit{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r)e(\mathbf{i}) & \textit{if } i_r \rightarrow i_{r+1}, \\ (y_r - y_{r+1})e(\mathbf{i}) & \textit{if } i_r \leftarrow i_{r+1}, \\ (y_{r+1} - y_r)(y_r - y_{r+1})e(\mathbf{i}) & \textit{if } i_r \leftrightarrows i_{r+1}; \end{cases} \tag{1.1.10}$$

$$\psi_r\psi_{r+1}\psi_re(\mathbf{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}) & \textit{if } i_{r+2} = i_r \rightarrow i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}) & \textit{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 2y_{r+1} + y_r + y_{r+2})e(\mathbf{i}) & \textit{if } i_{r+2} = i_r \leftrightarrows i_{r+1}, \\ \psi_{r+1}\psi_r\psi_{r+1}e(\mathbf{i}) & \textit{otherwise}; \end{cases} \tag{1.1.11}$$

for all admissible $\mathbf{i}, \mathbf{j}, r, s$.

We see that the relations of the algebra $\mathscr{R}_\alpha$, given in Equations (1.1.3) to (1.1.11), depends on the quantum characteristic of $\mathscr{R}_\alpha$ without directly involving the parameter q. The above presentation endows $\mathscr{R}_\alpha$ with a canonical $\mathbb{Z}$-grading.

Theorem 1.7. *[BK2, Corollary 1] There is a unique $\mathbb{Z}$-grading on $\mathscr{R}_\alpha$ such that*

$$\deg(e(\mathbf{i})) = 0, \quad \deg(y_r) = 2, \quad \deg(\psi_r e(\mathbf{i})) = -c_{i_r, i_{r+1}},$$

for all admissible r and $\mathbf{i} \in I^\alpha$.

We now define the *affine Khovanov–Lauda–Rouquier algebra* $\mathscr{R}_n$ to be the direct sum

$$\bigoplus_{\substack{\alpha \in Q_+ \\ \mathrm{ht}(\alpha)=n}} \mathscr{R}_\alpha,$$

and thus $\mathscr{R}_n$ exhibits a non-trivial $\mathbb{Z}$-grading too.

If e is finite, then we can obtain the presentation of $\mathscr{R}_n$ by tweaking the presentation of $\mathscr{R}_\alpha$. We do this by replacing α with n, where necessary. Notice that we abuse notation and write y_r and ψ_r for the generators corresponding to $\mathscr{R}_\alpha$, regardless of $\alpha \in Q_+$. Thus the generator y_r (respectively, ψ_r) of $\mathscr{R}_n$ is the sum of the corresponding y_r generators (respectively, ψ_r generators), where each summand corresponds to an $\mathscr{R}_\alpha$ for each $\alpha \in Q_+$ such that $\mathrm{ht}(\alpha) = n$.

1.1.7 Cyclotomic Khovanov–Lauda–Rouquier algebras

For a positive root $\alpha \in Q_+$ and a dominant weight Λ, the cyclotomic algebra $\mathscr{R}_\alpha^\Lambda$ is defined to be the quotient algebra of $\mathscr{R}_\alpha$, subject to the *cyclotomic relations*

$$y_1^{(\Lambda, \alpha_{i_1})} e(\mathbf{i}) = 0, \qquad (1.1.12)$$

for all $\mathbf{i} \in I^\alpha$. These cyclotomic relations are homogeneous, so $\mathscr{R}_\alpha^\Lambda$ inherits a non-trivial $\mathbb{Z}$-grading. We define the *affine cyclotomic Khovanov–Lauda–Rouquier algebra* $\mathscr{R}_n^\Lambda$ to be the direct sum

$$\bigoplus_{\substack{\alpha \in Q_+ \\ \mathrm{ht}(\alpha)=n}} \mathscr{R}_\alpha^\Lambda,$$

and thus $\mathscr{R}_\alpha^\Lambda$ is non-trivially $\mathbb{Z}$-graded too.

Brundan and Kleshchev remarkably discovered a connection, via their Graded Isomorphism Theorem, between the well established representation theory of the cyclotomic Hecke algebras and the newly introduced cyclotomic Khovanov–Lauda–Rouquier algebras.

Theorem 1.8. *[BK2] Let $q \in \mathbb{F}^\times$ and $e \in \{2, 3, \dots, \} \cup \{\infty\}$ be such that* $\operatorname{char}(\mathbb{F}) \nmid e$ *if $e < \infty$ and $q \neq 1$ when $l > 1$. Then there is an isomorphism of algebras*

$$\mathscr{R}_n^\Lambda \cong \mathscr{H}_{\mathbb{F},q,Q}\left((\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n\right),$$

where $Q = (q^{a_1}, \dots, q^{a_l})$ such that $a_i \equiv \kappa_i \pmod{e}$.

In particular, when $e = \operatorname{char}(\mathbb{F})$ and $l = 1$, we have the following isomorphism of algebras: $\mathscr{R}_n^\Lambda \cong \mathbb{F}\mathfrak{S}_n$. Thus, we can consider the cyclotomic Hecke algebras $\mathscr{H}_{F,q,Q}\left((\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n\right)$ as non-trivially $\mathbb{Z}$-graded algebras, in particular the symmetric group algebras, by identifying them with the cyclotomic Khovanov–Lauda–Rouquier algebras $\mathscr{R}_n^\Lambda$.

Corollary 1.9. *[M2, Corollary 3.1.3] Let $q_1, q_2 \in \mathbb{F}^\times$ be distinct primitive eth roots of unity. Then there is an isomorphism of cyclotomic Hecke algebras $\mathbb{F}$-*

$$\mathscr{H}_{\mathbb{F},q_1,Q}\left((\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n\right) \cong \mathscr{H}_{\mathbb{F},q_2,Q}\left((\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n\right),$$

as $\mathbb{F}$-algebras.

It follows that cyclotomic Hecke algebras, and hence cyclotomic Khovanov–Lauda–Rouquier algebras, do not depend on the parameter $q \in \mathbb{F}^\times$, and hence depend only on the quantum characteristic e, dominant weight Λ and the ground field $\mathbb{F}$.

1.1.8 Graded duality

There exists a homogeneous Khovanov–Lauda–Rouquier algebra anti-involution

$$* : \mathscr{R}_n \longrightarrow \mathscr{R}_n,\ e(\mathbf{i}) \mapsto e(\mathbf{i}),\ y_r \mapsto y_r,\ \psi_s \mapsto \psi_s,$$

for $\mathbf{i} \in I^n$, $1 \leqslant r \leqslant n$ and $1 \leqslant s < n$, which factors through to a homogeneous anti-involution for the cyclotomic quotient $\mathscr{R}_n^\Lambda$. We denote the element $x \in \mathscr{R}_n$ mapped under $*$ by x^*.

Given a graded $\mathscr{R}_n$-module M, we define the *graded dual* of M to be the $\mathbb{Z}$-graded $\mathscr{R}_n$-module $M^{\circledast}$ such that

$$\left(M^{\circledast}\right)_k = \operatorname{Hom}_{\mathbb{F}}\left(M\langle k\rangle, \mathbb{F}\right)$$

for all $k \in \mathbb{Z}$, with $\mathscr{R}_n$-action given by $(xf)(m) = f(x^*m)$, for all $x \in \mathscr{R}_n$, $f \in M^{\circledast}$ and $m \in M$.

1.1.9 The sign representation

The sign representation sgn, a one-dimensional $\mathscr{R}_n^\Lambda$-representation, attaches a sign to a generator of $\mathscr{R}_n^\Lambda$ as described below.

For $\mathbf{i} = (i_1, \dots, i_n) \in I^n$, we define

$$-\mathbf{i} := (-i_1, \dots, -i_n).$$

If we write $\alpha = \sum_{i \in I} a_i \alpha_i$, then we let $\alpha' = \sum_{i \in I} a_i \alpha_{-i}$. We define the sign representation to be the unique homogeneous algebra isomorphism

$$\mathrm{sgn} : \mathscr{R}_\alpha \longrightarrow \mathscr{R}_{\alpha'},$$

where

$$\mathrm{sgn}\,(e(\mathbf{i})) = e(-\mathbf{i}),\ \mathrm{sgn}\,(y_r) = -y_r \text{ and } \mathrm{sgn}\,(\psi_s) = -\psi_s,$$

for all $\mathbf{i} \in I^\alpha$, $r \in \{1, \dots, n\}$ and $s \in \{1, \dots, n-1\}$ such that $\mathrm{ht}(\alpha) = n$.

For $\kappa = (\kappa_1, \dots, \kappa_l)$ we define $\kappa' := (-\kappa_l, \dots, -\kappa_1)$, which determines the dominant weight

$$\Lambda' = \Lambda_{-\kappa_l} + \cdots + \Lambda_{-\kappa_1} \in P_+.$$

In fact, the sign representation factors through to an algebra isomorphism on the cyclotomic quotients

$$\mathrm{sgn} : \mathscr{R}^\Lambda_\alpha \longrightarrow \mathscr{R}^{\Lambda'}_{\alpha'}.$$

We define the corresponding sign-twisted algebras

$$\mathscr{R}_{n'} := \bigoplus_{\substack{\alpha \in Q_+ \\ \mathrm{ht}(\alpha) = n}} \mathscr{R}_{\alpha'} \quad \text{and} \quad \mathscr{R}^{\Lambda'}_{n'} := \bigoplus_{\substack{\alpha \in Q_+ \\ \mathrm{ht}(\alpha) = n}} \mathscr{R}^{\Lambda'}_{\alpha'}.$$

Then, for a module $M \in \mathscr{R}^{\Lambda'}_{n'}$, we define the sign-twisted module M^{sgn} to be $M \otimes \mathrm{sgn}$, where multiplication from $\mathscr{R}^\Lambda_n$ is given by

$$a \cdot m = \mathrm{sgn}(a) m,$$

for all $a \in \mathscr{R}^\Lambda_n$ and $m \in M^{\mathrm{sgn}}$.

1.2 Combinatorics

We introduce necessary notation for the purposes of our combinatorial approach.

1.2.1 Young diagrams and partitions

A *composition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of non-negative integers such that $\sum_{i=1}^{\infty} \lambda_i = n$. For $i \geqslant 1$, we refer to the integers λ_i as the *parts* of λ. A *partition* of n is a composition λ for which $\lambda_i \geqslant \lambda_{i+1}$ for all $i \geqslant 1$. We denote the *empty partition* $(0, \dots, 0)$ by $\varnothing$. We define $(1^0) := \varnothing$.

We fix a positive integer l and an e-multicharge $\kappa = (\kappa_1, \dots, \kappa_l)$. An *$l$-multicomposition* of n is an ordered l-tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ of compositions such that $\sum_{i=1}^{l} |\lambda^{(i)}| = n$. We refer to $\lambda^{(i)}$ as the ith *component* of λ. When each component of an l-multicomposition λ is a partition, λ is an *l-multipartition*. We also write $\varnothing$ for the *empty multipartition* $(\varnothing, \dots, \varnothing)$; it is generally clear which level l the empty multipartition $\varnothing$ belongs to. We denote the set of all l-multipartitions of n by $\mathscr{P}_n^l$.

Given l-multicompositions λ and μ of n, we say that λ *dominates* μ, if

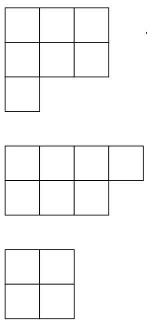
$$\sum_{i=1}^{m-1} |\lambda^{(i)}| + \sum_{j=1}^{k} |\lambda_j^{(m)}| \geqslant \sum_{i=1}^{m-1} |\mu^{(i)}| + \sum_{j=1}^{k} |\mu_j^{(m)}|,$$

for all $1 \leqslant m \leqslant l$ and $k \geqslant 1$. We write $\lambda \trianglerighteq \mu$ to mean that λ dominates μ.

The *Young diagram* of the l-multicomposition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ is defined by

$$[\lambda] := \left\{ (i, j, m) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, l\} \mid 1 \leqslant j \leqslant \lambda_i^{(m)} \right\}.$$

Each element $(i, j, m) \in [\lambda]$ is called a *node* of λ, and in particular, an (i, j)-node of $\lambda^{(m)}$. We pictorially represent the Young diagram of an l-multipartition as a column vector of Young diagrams $[\lambda^{(1)}], \dots, [\lambda^{(l)}]$ where $[\lambda^{(i)}]$ lies above $[\lambda^{(i+1)}]$ for all $i \geqslant 1$. For example, $((3^2, 1), (4, 3), (2^2))$ has the Young diagram



.

For nodes $(i_1, j_1, m), (i_2, j_2, l) \in [\lambda]$, we say that node (i_1, j_1, m) is *higher* than node (i_2, j_2, l) if $i_i < i_2$ and $m \leqslant l$. We define a dominance ordering on nodes of a Young diagram. For $\lambda \in \mathscr{P}_n^l$ and $A, B \in [\lambda]$, we say that A *dominates* B, written $A \leqslant B$, if A lies in a higher row in $[\lambda]$ than B.

The *conjugate* of a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ where

$$\lambda'_i = \# \{ j \geqslant 1 \mid \lambda_j \geqslant i \}.$$

Informally, we obtain λ' from λ by swapping the rows and columns of λ. The *conjugate* of an l-multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ is the l-multipartition

$$\lambda' = (\lambda^{(l)'}, \dots, \lambda^{(1)'}).$$

1.2.2 Residues

We fix an e-multicharge $\kappa = (\kappa_1, \dots, \kappa_l)$. The *$e$-residue* of a node $A = (i, j, m)$ lying in the space $\mathbb{N} \times \mathbb{N} \times \{1, \dots, l\}$ is defined by

$$\operatorname{res} A := \kappa_m + j - i \pmod{e}.$$

We call a node of residue i an *i-node*. For $\lambda \in \mathscr{P}_n^l$, the *residue content of* λ is defined to be

$$\operatorname{cont}(\lambda) := \sum_{A \in \lambda} \alpha_{\operatorname{res} A}.$$

1.2.3 Ladders and e-regular partitions

Let $l = 1$ throughout this subsection. We say that $\lambda \in \mathscr{P}_n^1$ is an *e-regular* partition if there is no $i \geqslant 1$ for which $\lambda = \lambda_{i+e-1} > 0$. In other words, a partition λ is e-regular if $[\lambda]$ has no e consecutive rows of the same length. Otherwise, we say that λ is *e-singular*.

For each $r \geqslant 1$, we define the *rth ladder* to be the set of nodes

$$\mathscr{L}_r = \{(i, j) \in \mathbb{N}^2 \mid r = i + (j-1)(e-1)\}.$$

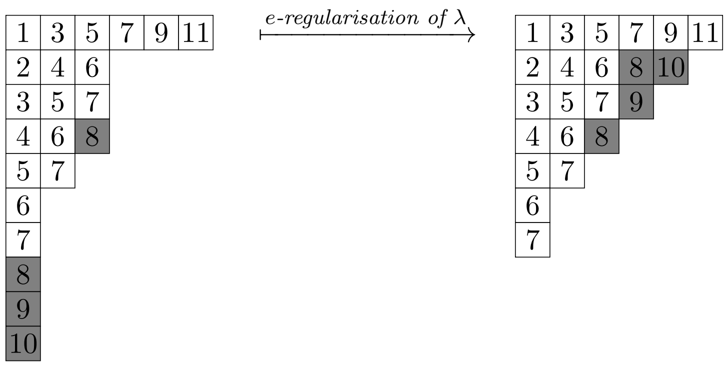
If $(i, j) \in \mathscr{L}_r$, we say that (i, j) has *(column) ladder number* r, denoted ℓ_r, and nodes belonging to the same ladder have the same e-residue. In particular, the *rth ladder of a partition* is the intersection of the Young diagram $[\lambda]$ with the rth ladder $\mathscr{L}_r$. Observe that the ladder numbers lying in nodes in the space $\mathbb{N} \times \mathbb{N}$ are

1	e	$2e-1$	$3e-2$	$4e-3$	$\cdots\cdots$
2	$e+1$	$2e$	$3e-1$	$4e-2$	$\cdots\cdots$
3	$e+2$	$2e+1$	$3e$	$4e-1$	$\cdots\cdots$
4	$e+3$	$2e+2$	$3e+1$	$4e$	$\cdots\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

.

Now we can take any partition λ and map it to an e-regular partition μ by a process called *e-regularisation.* One does this by moving every node with ladder number r in $[\lambda]$ to its highest position in the rth ladder of λ, $\mathscr{L}_r \cap [\lambda]$. We call μ the e-regularisation of λ, denoted by λ^R.

Example 1.10. *Let $e = 3$ and $\lambda = (6, 3^3, 2, 1^5)$. The ladder numbers of λ and the ladder numbers of the 3-regularisation of λ, respectively, are as follows. Nodes which* *have moved up their particular ladder are highlighted.*



We see that $(6, 3^3, 2, 1^5)^R = (6, 5, 4, 3, 2, 1^2)$.

The following result is easily observed, as seen in the previous example, and implies that for any partition λ, the e-regularisation λ^R is indeed e-regular.

Lemma 1.11. *[J1, Statement (1.2)] Let* $\lambda \in \mathscr{P}_n^1$*. Then* λ *is* e*-regular if and only if all the nodes in* $[\lambda]$ *with ladder number* r *lie in the highest possible row in* $\mathscr{L}_r$ *for each* r*.*

In other words, λ is e-regular if and only if $\lambda = \lambda^R$.

1.2.4 Regular multipartitions

Most of the combinatorial definitions in this subsection date back to [K3]; we adopt notation introduced in [Fa5] by Fayers.

For an l-multipartition λ of n, we say that $A \in [\lambda]$ is a *removable node* for λ if $[\lambda]\backslash\{A\}$ is a Young diagram of an l-multipartition. Similarly, we say that $A \notin [\lambda]$ is an *addable node* for λ if $[\lambda] \cup \{A\}$ is a Young diagram of an l-multipartition.

We call a node $A \in [\lambda]$ a *removable* i*-node* of λ if A is a removable node of λ and $\operatorname{res} A = i$. Similarly, a node $A \notin [\lambda]$ is called an *addable* i*-node* of λ if A is an addable node of λ and $\operatorname{res} A = i$. We denote the total number of removable i-nodes of λ by $\operatorname{rem}_i(\lambda)$, and denote the total number of addable i-nodes of λ by $\operatorname{add}_i(\lambda)$. For $\lambda \in \mathscr{P}_n^l$, we write the multipartition obtained by removing all of the removable i-nodes from λ as $\lambda^{\triangledown i}$, and we write the multipartition obtained by adding all of the addable i-nodes to λ as $\lambda^{\vartriangle i}$.

Let $\lambda \in \mathscr{P}_n^l$. We define the *$i$-signature of* λ by reading the Young digram $[\lambda]$ from the top of the first component down to the bottom of the last component, writing a $+$ for each addable i-node and writing a $-$ for each removable i-node, where the leftmost $+$ corresponds to the highest addable i-node of λ. We obtain the *reduced i-signature of* λ by successively deleting all adjacent pairs $+-$ from the i-signature of λ, always of the form $- - \cdots - + + \cdots +$.

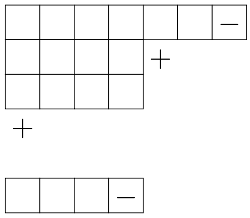
Example 1.12. *Let $e = 3$, $\kappa = (0, 0)$ and $\lambda = ((8, 4^2), (4))$. The 3-residues of λ are*

0	1	2	0	1	2	0
2	0	1	2			
1	2	0	1			

,

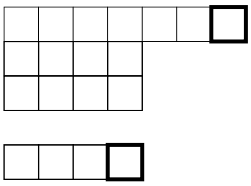
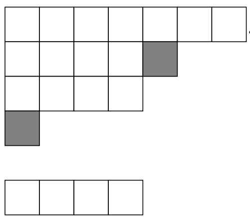
0	1	2	0

and the 0-addable and 0-removable nodes of λ are labelled as follows



. (1.2.1)

Thus, by removing all of the removable 0-nodes from λ (corresponding to the outlined nodes), and respectively, adding all of the removable 0-nodes from λ (corresponding to the shaded nodes) we have the Young diagrams of multipartitions

$[\lambda^{\triangledown_0}] =$  *and* $[\lambda^{\vartriangle_0}] =$  .

Referring to Diagram (1.2.1), the 0-signature of λ is $- + + -$ (corresponding to the $-$ and $+$ labels from top to bottom in the diagram), and the reduced 0-signature is $-+$ (corresponding to the nodes $(1, 7, 1)$ and $(2, 5, 1)$).

The removable i-nodes corresponding to the $-$ signs in the reduced i-signature of λ are called the *normal* i-nodes of λ, and similarly, we call the addable i-nodes corresponding to the $+$ signs in the reduced i-signature of λ the *conormal* i-nodes of λ. We denote the total number of normal i-nodes of λ by $\mathrm{nor}_i(\lambda)$, and we denote the total number of conormal i-nodes of λ by $\mathrm{conor}_i(\lambda)$. The lowest normal i-node of $[\lambda]$, if there is one, is called the *good* i-node of λ, which corresponds to the last $-$ sign in the i-signature of λ. Dually, the highest conormal i-node of $[\lambda]$, if there is one, is called the *cogood* i-node of λ, which corresponds to the first $+$ sign in the i-signature of λ.

For $0 \leqslant r \leqslant \mathrm{nor}_i(\lambda)$, we denote the multipartition obtained from λ by removing the r lowest normal i-nodes of λ by $\lambda \downarrow_i^r$, and for $0 \leqslant r \leqslant \mathrm{conor}_i(\lambda)$, we denote the multipartition obtained from λ by adding the r highest conormal i-nodes of λ by $\lambda \uparrow_i^r$. We set $\uparrow_i := \uparrow_i^1$ for adding the cogood i-node of λ and $\downarrow_i := \downarrow_i^1$ for removing the good i-node of λ. For $\lambda \in \mathscr{P}_n^l$, it is easy to see that A is a cogood node for λ if and only if A

is a good node for $\lambda \cup \{A\}$. The operators $\uparrow_i^r$ and $\downarrow_i^r$ act inversely on a multipartition $\lambda \in \mathscr{P}_n^l$ in the following sense

$$\lambda \downarrow_i^r \uparrow_i^r = \lambda \qquad \text{and} \qquad \lambda \uparrow_i^s \downarrow_i^s = \lambda, \tag{1.2.2}$$

for $0 \leqslant r \leqslant \mathrm{nor}_i(\lambda)$ and $0 \leqslant s \leqslant \mathrm{conor}_i(\lambda)$.

We define the set of all *regular l-multipartitions of n* to be the set

$$\mathscr{R}\mathscr{P}_n^l = \{\varnothing \uparrow_{i_1} \ldots \uparrow_{i_n} \mid i_1, \ldots, i_n \in I\}.$$

If a multipartition λ lies in $\mathscr{R}\mathscr{P}_n^l$, then λ is called *regular*. Hence $\lambda \in \mathscr{P}_n^l$ is *regular* if and only if $[\lambda]$ is obtained by successively adding cogood nodes to $[\varnothing]$. That is, we have a sequence $\varnothing = \lambda(0), \lambda(1), \ldots, \lambda(n) = \lambda$ such that $[\lambda(i)] \cup \{A\} = [\lambda(i+1)]$, where A is a cogood node of $\lambda(i)$.

We can alternatively write the set of all regular l-multipartitions of n as

$$\mathscr{R}\mathscr{P}_n^l = \left\{\lambda \in \mathscr{P}_n^l \mid \lambda \downarrow_{i_1} \ldots \downarrow_{i_n} = \varnothing, \text{for some } i_1, \ldots, i_n \in I\right\}.$$

Example 1.13. *Suppose that $l = 1$. If $e \in \{2, 3, \ldots\}$ is finite, then $\mathscr{R}\mathscr{P}_n^1$ coincides with the set of all e-regular partitions, whereas $\mathscr{R}\mathscr{P}_n^1 = \mathscr{P}_n^1$ if $e = \infty$.*

1.2.5 Tableaux

Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)}) \in \mathscr{P}_n^l$. A λ-tableau T is a bijection $T : [\lambda] \to \{1, \ldots, n\}$. Usually, we depict a λ-tableau T by inserting entries $1, \ldots, n$ into the Young diagram $[\lambda]$; we say that the entry lying in the node $(i, j, m) \in [\lambda]$ is the (i, j, m)-entry of T, denoted $T(i, j, m)$. We refer to the $\lambda^{(i)}$-tableau $T^{(i)}$ as the *ith component* of T for all $i \in \{1, \ldots, l\}$. We say that T is *row standard* if the entries in each row increase from left to right along rows of each component of T, and similarly, we say T is *column standard* if the entries in T increase down each column along the columns of each component of T. We denote the set of all column-standard λ-tableaux by $\mathrm{ColStd}(\lambda)$. If T is row-standard and column-standard, then T is called *standard*. We denote the set of all standard λ-tableaux by $\mathrm{Std}(\lambda)$.

For $\lambda \in \mathscr{P}_n^l$, let T be a standard λ-tableau. Then we define $T_{\leqslant r}$ to be the μ-tableau obtained from T by removing all of the nodes occupied with entries greater than r, where $1 \leqslant r \leqslant n$ and μ is a multipartition of r. It follows that $T_{\leqslant r}$ has the *shape* of the multipartition $\mu \in \mathscr{P}_r^l$.

The *column-initial tableau* T_λ is the λ-tableau where the entries $1, \ldots, n$ appear in order down consecutive columns, working from left to right in components $l, l-1, \ldots, 1$, in turn. Similarly, the *row-initial tableau* T^λ is the λ-tableau where the entries $1, \ldots, n$ appear in order along successive rows, working from top to bottom in components $1, 2, \ldots, l$, in turn. For example, the column- and row-initial $((5, 3), (2^2, 1))$-tableaux are

$$T_{((5,3),(2^2,1))} = \left(\begin{array}{|c|c|c|c|c|}\hline 6 & 8 & 10 & 12 & 13 \\ \hline 7 & 9 & 11 \\ \cline{1-3}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}\right) \text{ and } T^{((5,3),(2^2,1))} = \left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \cline{1-3}\end{array}\ ,\ \begin{array}{|c|c|}\hline 9 & 10 \\ \hline 11 & 12 \\ \hline 13 \\ \cline{1-1}\end{array}\right).$$

The symmetric group $\mathfrak{S}_n$ acts naturally on the left on the set of λ-tableaux. We define the permutations $w_T, w^T \in \mathfrak{S}_n$ from

$$w_T T_\lambda = T = w^T T^\lambda,$$

where T is a λ-tableau for some $\lambda \in \mathscr{P}_n^l$. For example, if

$$S = \left(\begin{array}{|c|c|c|c|c|}\hline 4 & 8 & 10 & 11 & 12 \\ \hline 7 & 9 & 13 \\ \cline{1-3}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 \\ \cline{1-1}\end{array}\right),$$

then $(4\,5\,6)(11\,13\,12)T_{((5,3),(2^2,1))} = S = (1\,4\,11\,2\,8\,13\,3\,10\,5\,12\,6\,7\,9)T^{((5,3),(2^2,1))}$.

We let $w \in \mathfrak{S}_n$ have a fixed reduced expression $w = s_{r_1}, \ldots, s_{r_m}$ throughout, and refer to it as the *preferred reduced expression of* w. We define the associated element of $\mathscr{R}_n^\Lambda$

$$\psi_w := \psi_{r_1} \ldots \psi_{r_m},$$

which, in general, depends on the choice of a preferred reduced expression of w.

We say that $w \in \mathfrak{S}_n$ is *fully commutative* if we can go from any reduced expression of w to any other using only the commuting braid relations $s_i s_j = s_j s_i$ for $\mid i - j \mid > 1$. The elements ψ_w do not depend on the choice of preferred reduced expression if w is a fully commutative element of $\mathfrak{S}_n$. We now define important elements which will aid us in constructing bases for particular $\mathscr{R}_n^\Lambda$-modules

$$\psi^T := \psi_{w^T} \quad \text{and} \quad \psi_T := \psi_{w_T}.$$

Let T be a λ-tableau and write $\operatorname{res}_T(r) = \operatorname{res}(i,j,m)$, where $r = T(i,j,m)$. The *residue sequence* of T is defined to be

$$\mathbf{i}_T = \mathbf{i}_T^\kappa = (\operatorname{res}_T(1), \ldots, \operatorname{res}_T(n)).$$

We set $\mathbf{i}_\lambda = \mathbf{i}_{T_\lambda}$. For example, when $e = 3$ and $\kappa = (0,1)$, the 3-residues of the nodes in

$((5,3),(2^2,1))$ are given by

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&0&1\\\hline 2&0&1\\\cline{1-3}\end{array}\,,$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 0&1\\\hline 2\\\cline{1-1}\end{array}$$

so that $\mathbf{i}_{((5,3),(2^2,1))} = (1,0,2,2,1,0,2,1,0,2,1,0,1)$. We now define the idempotent generator of $\mathscr{R}_n^\Lambda$ to be $e_T := e(\mathbf{i}_T)$ with respect to T.

Let $\lambda \in \mathscr{P}_n^l$ and T be a λ-tableau. Suppose that $T(i_1,j_1,m) = r$ and $T(i_2,j_2,m) = s$ such that $1 \leqslant r \neq s \leqslant n$ (so r and s lie in the same component of T). We write

- $r \nearrow_T s$ if $i_1 > i_2$ and $j_1 < j_2$ (s lies strictly above and to the right of r in the same component of T);
- $r \searrow_T s$ if $i_1 < i_2$ and $j_1 < j_2$ (s lies strictly below and to the right of r in the same component of T);
- $r \rightarrow_T s$ if $i_1 = i_2$ and $j_1 < j_2$ (r and s lie in the same row of the same component of T, where s is strictly to the right of r);
- $r \downarrow_T s$ if $i_1 < i_2$ and $j_1 = j_2$ (r and s lie in the same column of the same component of T, where s is strictly below r).

Lemma 1.14. *[BK2, Lemma 3.3] Let $\lambda \in \mathscr{P}_n^l$ and $T \in \mathrm{Std}(\lambda)$. Then s_rT is also standard if and only if neither $r \rightarrow_T r+1$ nor $r \downarrow_T r+1$.*

For example, given S as above, s_rS is standard for $r = 3,4,6,7,9,12$.

For $\lambda \in \mathscr{P}_n^l$, we define a *Bruhat order on λ-tableaux*. For λ-tableaux S and T, we let $w_S, w_T \in \mathfrak{S}_n$ be permutations such that $S = w_ST_\lambda$ and $T = w_TT_\lambda$. Then we say that T *dominates* S if and only if $w_S \leqslant w_T$, and write $T \trianglerighteq S$.

Let $\lambda \in \mathscr{P}_n^l$ and A be an addable i-node of λ and B be a removable i-node of λ. We set the degree of A to be

$$\begin{aligned} d^A(\lambda) :&= \#\{\text{addable } i\text{-nodes of } \lambda \text{ strictly above } A\} \\ &\quad - \#\{\text{removable } i\text{-nodes of } \lambda \text{ strictly above } A\} \end{aligned}$$

and the degree of B to be

$$\begin{aligned} d_B(\lambda) :&= \#\{\text{addable } i\text{-nodes of } \lambda \text{ strictly below } B\} \\ &\quad - \#\{\text{removable } i\text{-nodes of } \lambda \text{ strictly below } B\}. \end{aligned}$$

For a positive root $\alpha \in Q_+$, we define the *defect* of α to be

$$\mathrm{def}(\alpha) = (\Lambda,\alpha) - \tfrac{1}{2}(\alpha,\alpha). \tag{1.2.3}$$

Let $T \in \mathrm{Std}(\lambda)$ where n lies in node A of λ, and set $\mu = \lambda \backslash \{A\}$. We set $\deg(\varnothing) := 0$ and define the *degree* of T recursively via

$$\deg(T) := d^A(\mu) + \deg(T_{\leqslant n-1}).$$

Similarly, the dual notion of the *codegree* of T is defined to be

$$\mathrm{codeg}(T) := d_A(\mu) + \mathrm{codeg}(T_{\leqslant n-1}).$$

We note that these definitions are the other way around to those given in [BKW]. The degree and codegree of a standard λ-tableau are dual notions of each other via the following result.

Lemma 1.15. *[BKW, Lemma 3.12] Let $\lambda \in \mathscr{P}^l_n$, $\mathrm{cont}(\lambda) = \alpha$ and $T \in \mathrm{Std}(\lambda)$. Then*

$$\deg(T) + \mathrm{codeg}(T) = \mathrm{def}(\alpha).$$

1.3 Graded Specht modules

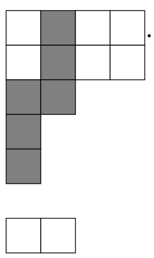
We introduce our main objects of study, graded Specht modules, following the theory of Brundan, Kleshchev and Wang in [BKW]. We remark that we work with the *dual* Specht module throughout, however, we will refer to this module as the Specht module itself for brevity, consistent with James' classical construction of the Specht module over the symmetric group algebras.

1.3.1 Garnir tableaux and Garnir elements

Let $\lambda \in \mathscr{P}^l_n$. We say that a node $A = (i, j, m) \in [\lambda]$ is a *(column) Garnir node* of λ, if $(i, j+1, m)$ is a node in $[\lambda]$. The *(column) Garnir belt* $\mathbf{B}_A$ of A is defined to be the set of nodes

$$\mathbf{B}_A = \{(k, j, m) \in [\lambda] \mid k \geqslant i\} \cup \{(k, j+1, m) \in [\lambda] \mid 1 \leqslant k \leqslant i\}.$$

For example, the Garnir belt of $(3, 1, 1)$ in $((4^2, 2, 1^2), (2))$ is shaded in the following Young diagram



Let $r = T_\lambda(i, j, m)$ and $s = T_\lambda(i, j+1, m)$. We place the entries $r, r+1, \ldots, s$ in $\mathbf{B}_A$ in order from top right to bottom left. The resulting λ-tableau, G_A, is called the *(column)*

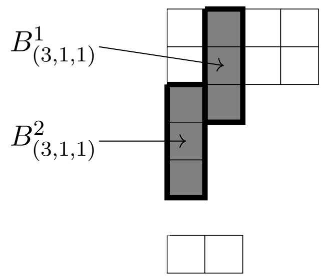
Garnir tableau of A whose entries outside the Garnir belt of A coincide with the entries in T_λ. The Garnir $((4^2,2,1^2),(2))$-tableau of $(3,1,1)$ is

$$G_{(3,1,1)} = \begin{array}{|c|c|c|c|}\hline 3 & 5 & 11 & 13 \\ \hline 4 & 6 & 12 & 14 \\ \hline 8 & 7 \\ \cline{1-2} 9 \\ \cline{1-1} 10 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} = \overset{9}{\underset{7}{s\!\uparrow}}\ \overset{8}{\underset{6}{s\!\uparrow}}\ \overset{7}{\underset{5}{s\!\uparrow}}\ T_{((4^2,2,1^2),(2))}.$$

A *(column) brick* of A is a set of e consecutive nodes

$$\{(i,j,m),(i+1,j,m),\ldots,(i+e-1,j,m)\} \subset \mathbf{B}_A$$

such that $\operatorname{res}(i,j,m) = \operatorname{res} A$. Suppose that there are k bricks lying in the Garnir belt $\mathbf{B}_A$. If $k > 0$, then we label the bricks $B_A^1, B_A^2, \ldots, B_A^k$ in $\mathbf{B}_A$ from top to bottom, firstly down column $j+1$, then down column j. For $e = 3$, the Garnir belt of $(3,1,1)$ in our running example has two bricks, labelled in the following Young diagram



Let n_A be the smallest number in the Garnir tableau G_A in $\mathbf{B}_A$, which also lies in a brick. We define *brick permutations* of $\mathfrak{S}_n$, which are simple transpositions of bricks, by

$$w_A^r := \prod_{a=n_A+e(r-1)}^{n_A+re-1} (a\ a+e) \in \mathfrak{S}_n$$

for each $r \in \{1,\ldots,k-1\}$. Informally, the brick permutation w_A^r swaps the rth and $(r+1)$th bricks in $\mathbf{B}_A$. Let the *(column) brick permutation group* be

$$\mathfrak{S}_A = \left\langle w_A^1, w_A^2, \ldots, w_A^{k-1} \right\rangle \cong \mathfrak{S}_k \subseteq \mathfrak{S}_n.$$

We let T_A be the λ-tableau obtained by placing the bricks $B_A^1, B_A^2, \ldots, B_A^k$ successively down column j and then down column $j+1$ in $[\lambda]$. We consider the following set of λ-tableaux

$$\operatorname{Gar}_A = \{T \in \operatorname{ColStd}(\lambda) \mid T = wT_A,\ \text{for a brick permutation } w \in \mathfrak{S}_A\},$$

called the set of *Garnir λ-tableaux* of A. In our running example, notice that $T_{(3,1,1)} = T_{((4^2,2,1^2),(2))}$. Now, there is only one brick permutation $w^1_{(3,1,1)} = s\overset{7}{\underset{5}{\downarrow}}\, s\overset{8}{\underset{6}{\downarrow}}\, s\overset{9}{\underset{7}{\downarrow}}$, so that $\mathfrak{S}_{(3,1,1)}$ is generated by $w^1_{(3,1,1)}$. We have $w^1_{(3,1,1)}T_{((4^2,2,1^2),(2))} = G_{(3,1,1)}$, hence

$$\mathrm{Gar}_{(3,1,1)} = \left\{G_{(3,1,1)}, T_{((4^2,2,1^2),(2))}\right\}.$$

Each tableau in Gar_A is standard, except for G_A, which is the maximal element of Gar_A under the strong Bruhat order of tableaux, that is, $T \trianglelefteq G_A$ for all $T \in \mathrm{Gar}_A$. The unique minimal element of Gar_A under the Bruhat order is T_A. If $T \in \mathrm{Gar}_A$, then T has the same residue sequence as G_A, that is, $\boldsymbol{i}_T = \boldsymbol{i}_A$ where $\boldsymbol{i}_A := \boldsymbol{i}_{G_A}$, and moreover, $\deg(T) = \deg(G_A)$ for all tableaux T in Gar_A.

In general, we can write Gar_A in terms of T_A as follows. Let f be the number of bricks in column j of $\boldsymbol{B}_A$ and let $\mathscr{D}_A$ be the minimal length left coset representatives of the group of brick permutations that permute bricks within columns. Then

$$\mathrm{Gar}_A = \{wT_A \mid w \in \mathscr{D}_A\}.$$

For any $S \in \mathrm{Gar}_A$, we have $S = u_S T_A$ for some $u_S \in \mathscr{D}_A$. So we can write $w_S = u_S w_{T_A}$ with $l(w_S) = l(u_S) + l(w_{T_A})$, where $w_S, w_{T_A} \in \mathfrak{S}_n$ such that $w_S T_\lambda = S$ and $w_{T_A} T_\lambda = T_A$. By [KMR, Lemma 3.17], the elements $w_S, u_S, w_{T_A} \in \mathscr{D}_A$ are fully commutative. Hence we have elements ψ_S, ψ_{U_S} and ψ_{T_A} that are independent of the choice of preferred reduced expression where $\psi_S = \psi_{U_S}\psi_{T_A}$. We now define elements lying in $\mathscr{R}_n^\Lambda$

$$\sigma_A^r := (-1)^e \psi_{w_A^r} e(\boldsymbol{i}_A) \quad \text{and} \quad \tau_A^r := (\sigma_A^r + 1)e(\boldsymbol{i}_A).$$

We let $w_A^{r_1} \dots w_A^{r_m}$ be a reduced expression for $u_S \in \mathfrak{S}_A$, and write $\tau_A^{u_S} := \tau_A^{r_1} \dots \tau_A^{r_m}$. We know u_S is fully commutative, so τ_A^λ is not dependent on the choice of reduced expression.

Definition 1.16. *Let A be a Garnir node of λ. The* (column) Garnir element *of A is*

$$g_A := \sum_{u_S \in \mathscr{D}_A} \tau_A^{u_S} \psi_{T_A} \ \in \mathscr{R}_n.$$

Since $\tau_A^{u_S}$ is a homogeneous element of $\mathscr{R}_n$ by definition, g_A is also homogeneous. For our running example, the Garnir element $g_{(3,1,1)}$ of $(3,1,1)$ is

$$\begin{aligned}\sum_{u_S \in \mathscr{R}_{(3,1,1)}} \tau_{(3,1,1)}^{u_S} \psi_{T_{(3,1,1)}} &= (1 + \tau^1_{(3,1,1)})e(\mathbf{i}_{(3,1,1)}) \\ &= (2 + \sigma^1_{(3,1,1)})e(\mathbf{i}_{(3,1,1)}) \\ &= 2e(\mathbf{i}_{(3,1,1)}) - \psi_7\psi_6\psi_5\psi_8\psi_7\psi_6\psi_9\psi_8\psi_7 e(\mathbf{i}_{(3,1,1)}).\end{aligned}$$

There exists a dual notion of a *row* Garnir element of a Garnir node – see [KMR, Section 5] for an exposition.

In general, the Garnir elements of Garnir nodes in a λ-tableau are very complicated to compute, however, those of interest to us, the Garnir elements of Garnir nodes lying in partitions $(n-m, m)$ and $(n-m, 1^m)$, and bipartitions $((n-m), (1^m))$ and $((n-m, 1^m), \varnothing)$, happen to be particularly easy to find.

Lemma 1.17. *Let $\lambda \in \mathscr{P}_n^l$ and $A = (a, b, m)$ be a Garnir node of λ. If all the bricks are in the same column of the Garnir belt $\mathbf{B}_A$, then the Garnir element g_A is $\psi_{G_A} e(\mathbf{i}_A)$.*

Proof. Suppose that there are $k \geqslant 0$ bricks lying in the $(b-1)$th column (respectively, bth column) of $\mathbf{B}_A$ and no bricks lying in its bth column (respectively, $(b-1)$th column). We thus observe that $\mathscr{D}_A = \{1\}$ and hence the Garnir element of A is

$$g_A = \tau^{\mathrm{id}} \psi_{G_A} e(\mathbf{i}_A) = \psi_{G_A} e(\mathbf{i}_A).$$

□

1.3.1.1 Garnir elements of $(n-m, m)$

Let $\lambda = (n-m, m)$, $A_i^1 = (1, i)$ and $A_i^2 = (2, i)$ for $i \in \mathbb{N}$. Then the complete set of Garnir nodes of λ is

$$\{A_i^1 \mid 1 \leqslant i \leqslant n-m-1\} \cup \{A_i^2 \mid 1 \leqslant i \leqslant m-1\}.$$

We first find the Garnir element of node A_i^1, for $m+1 \leqslant i \leqslant n-m-1$. The Garnir belt of A_i^1 is $\mathbf{B}_{A_i^1} = \{(1, i)\} \cup \{(1, i+1)\}$, depicted by the shaded area in the following Young diagram of λ

		$\cdots$		$\cdots$		A_i^1		$\cdots$	
		$\cdots$							

.

The Garnir tableau $G_{A_i^1}$ of A_i^1 is

1	$\cdots\cdots$	$2m{-}1$	$2m{+}1$	$\cdots\cdots$	$j{-}1$	$j{+}1$	j	$j{+}2$	$\cdots\cdots$	n
2	$\cdots\cdots$	$2m$								

,

where $j = i+m$, so $G_{A_i^1} = \psi_{m+i} T_\lambda$. It follows from Lemma 1.17 that the Garnir element $g_{A_i^1}$ of A_i^1 is $\psi_{m+i} e(\mathbf{i}_\lambda)$, for $m+1 \leqslant i \leqslant n-m-1$.

We now find the Garnir element of node A_i^1, for $1 \leqslant i \leqslant m$. The Garnir belt of A_i^1 is $\mathbf{B}_{A_i^1} = \{(1, i), (1, i+1)\} \cup \{(2, i)\}$, depicted by the shaded area in the following Young diagram of λ

	$\cdots$		A_i^1			$\cdots$		$\cdots$	
	$\cdots$					$\cdots$			

.

The Garnir tableau $G_{A_i^1}$ of A_i^1 is

1	⋯⋯	$2i-3$	$2i$	$2i-1$	$2i+3$	⋯⋯	$2m-1$	$2m+1$	⋯⋯	n	,
2	⋯⋯	$2i-2$	$2i+1$	$2i+2$	$2i+4$	⋯⋯	$2m$				

where $G_{A_i^1} = s_{2i-1}s_{2i}T_\lambda$. It follows from Lemma 1.17 that the Garnir element $g_{A_i^1}$ of A_i^1 is $\psi_{2i-1}\psi_{2i}e(\mathbf{i}_\lambda)$, for $1 \leqslant i \leqslant m$.

Finally, we find the Garnir element of node A_i^2, for $1 \leqslant i \leqslant m-1$. The Garnir belt of A_i^2 is $\mathbf{B}_{A_i^2} = \{(1, i+1)\} \cup \{(2, i), (2, i+1)\}$, depicted by the shaded area in the following Young diagram of λ

	⋯					⋯		⋯		.
	⋯		A_i^2			⋯				

The Garnir tableau $G_{A_i^2}$ of A_i^2 is

1	⋯⋯	$2i-3$	$2i-1$	$2i$	$2i+3$	⋯⋯	$2m-1$	$2m+1$	⋯⋯	n	,
2	⋯⋯	$2i-2$	$2i+2$	$2i+1$	$2i+4$	⋯⋯	$2m$				

where $G_{A_i^2} = s_{2i+1}s_{2i}T_\lambda$. It follows from Lemma 1.17 that the Garnir element $g_{A_i^2}$ of A_i^2 is $\psi_{2i+1}\psi_{2i}e(\mathbf{i}_\lambda)$, for $1 \leqslant i \leqslant m-1$.

1.3.1.2 Garnir elements of $(n-m, 1^m)$

Let $\lambda = (n-m, 1^m)$ and $A_i := (1, i)$ for $i \in \mathbb{N}$. Then the complete set of Garnir nodes of λ is

$$\{A_i \mid 1 \leqslant i \leqslant n-m-1\}.$$

We first find the Garnir element of node $A_1 = (1,1)$. The Garnir belt of $(1,1)$ is $\mathbf{B}_{A_1} = \{(j,1) \mid 1 \leqslant j \leqslant m+1\} \cup \{(1,2)\}$, depicted by the shaded area in the following Young diagram of λ

A_1			⋯		.
⋮					

The Garnir tableau of A_1 is

$G_{A_1} =$

2	1	$m+3$	$m+4$	⋯⋯	n	,
3						
⋮						
$m+2$						

where $G_{A_1} = s_1 s_2 \dots s_{m+1}T_\lambda$. It follows from Lemma 1.17 that the Garnir element g_{A_1} of A_1 is $\psi_1\psi_2 \dots \psi_{m+1}e(\mathbf{i}_\lambda)$.

We now find the Garnir element of node A_i, for $i > 1$. The Garnir belt $\mathbf{B}_{A_i}$ of A_i

consists only of the two consecutive nodes $(1, i)$ and $(1, i+1)$ in the arm of λ, as shown in the following shaded Young diagram

$$\begin{array}{|c|c|c|c|c|c|}\hline & \cdots & A_i & \blacksquare & \cdots & \\ \hline \\ \cline{1-1} \vdots \\ \cline{1-1} \\ \cline{1-1}\end{array}.$$

Thus, the Garnir λ-tableau of A_i is

$$G_{A_i} = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & m{+}2 & \cdots & j{-}1 & j{+}1 & j & j{+}2 & \cdots & n \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \vdots \\ \cline{1-1} m{+}1 \\ \cline{1-1}\end{array},$$

where $i + m = j$, so $T_{A_i} = G_{A_i}$. It follows from Lemma 1.17 that the Garnir element g_{A_i} of A_i is $\psi_{m+i}e(\mathbf{i}_\lambda)$, for $1 < i \leqslant n - m - 1$.

Remark 1.18. *The Garnir nodes of* $((n - m, 1^m), \varnothing)$ *are* $A_i = (1, i, 1)$, *for* $1 \leqslant i \leqslant n - m - 1$. *We can deduce from the Garnir elements of* $(n - m, 1^m)$ *that the Garnir elements of* $((n - m, 1^m), \varnothing)$ *are*

- ◇ $g_{A_1} = \psi_1\psi_2 \dots \psi_{m+1}e(\mathbf{i}_{((n-m,1^m),\varnothing)})$;
- ◇ $g_{A_i} = \psi_{m+i}e(\mathbf{i}_{((n-m,1^m),\varnothing)})$, *for* $2 \leqslant i \leqslant n - m - 1$.

1.3.1.3 Garnir elements of $((n - m), (1^m))$

Let $\lambda = ((n - m), (1^m))$ and $A_i = (1, i, 1)$ for $i \in \mathbb{N}$. Then the complete set of Garnir nodes of λ is

$$\{A_i \mid 1 \leqslant i \leqslant n - m - 1\}.$$

The Garnir belt $\mathbf{B}_{A_i}$ of A_i consists only of the two consecutive nodes $(1, i, 1)$ and $(1, i + 1, 1)$ in the arm of λ. Thus, the Garnir λ-tableau of A_i is

$$G_{A_i} = \left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline m{+}1 & \cdots & j{-}1 & j{+}1 & j & j{+}2 & \cdots & n \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline m \\ \hline\end{array} \right),$$

where $i + m = j$, so $G_{A_i} = s_{m+i}T_\lambda$. It follows from Lemma 1.17 that the Garnir element of A_i is $\psi_{m+i}e(\mathbf{i}_\lambda)$, for all $i \in \{1, \dots, n - m - 1\}$.

1.3.2 Homogeneous presentation of Specht modules

Kleshchev, Mathas and Ram provide the following presentation of Specht modules for $\mathscr{R}_n^\Lambda$.

Definition 1.19. *[KMR, Definition 7.11] Let $\alpha \in Q_+$ such that $\mathrm{ht}(\alpha) = n$ and $\lambda \in \mathscr{P}_n^l$. The (column)* Specht module *S_λ is defined to be the $\mathscr{R}_\alpha$-module generated by z_λ of degree*

$$\deg(z_\lambda) := \deg(T_\lambda)$$

subject only to the defining relations:

- $e(\mathbf{i})z_\lambda = \delta_{\mathbf{i},\mathbf{i}_\lambda} z_\lambda$;
- $y_r z_\lambda = 0$ *for all* $r \in \{1, \ldots, n\}$;
- $\psi_r z_\lambda = 0$ *for all* $i \in \{1, \ldots, n-1\}$ *such that r and $r+1$ lie in the same column of the same component of T_λ;*
- $g_A z_\lambda = 0$ *for all Garnir nodes A in $[\lambda]$.*

We notice that this Specht module presentation of S_λ is homogeneous, so that S_λ is a non-trivially $\mathbb{Z}$-graded $\mathscr{R}_\alpha$-module. In the light of Equation (1.1.12), one can show that $y_1^{(\Lambda,\alpha_{i_1})} e(\mathbf{i})$ kills S_λ for all $\mathbf{i} \in I^\alpha$, bringing us to the following result.

Theorem 1.20. *[KMR, Corollary 7.21] For $\lambda \in \mathscr{P}_n^l$, S_λ factors through the surjective $\mathbb{F}$-algebra homomorphism $\mathscr{R}_\alpha \to \mathscr{R}_\alpha^\Lambda$. Moreover, S_λ is a $\mathbb{Z}$-graded $\mathscr{R}_\alpha^\Lambda$-module.*

1.3.3 A standard homogeneous basis of Specht modules

Let $\lambda \in \mathscr{P}_n^l$ and T be a λ-tableau. Recall that we define $w_T \in \mathfrak{S}_n$ to be the permutation that satisfies $T = w_T T_\lambda$. Now define an element of S_λ to be

$$v_T := \psi_{w_T} z_\lambda,$$

where ψ_{w_T} is determined by the preferred reduced expression of w_T. In particular, we have $v_{T_\lambda} = z_\lambda$.

Lemma 1.21. *Let $\lambda \in \mathscr{P}_n^l$, T be a λ-tableau and suppose that $v_T = \psi_{w_T} z_\lambda$ be a basis vector of S_λ. Then $e(\mathbf{i})v_T = v_T$ if $\mathbf{i} = w_T \mathbf{i}_\lambda$, otherwise $e(\mathbf{i})v_T = 0$.*

Proof. We have

$$e(\mathbf{i})v_T = e(\mathbf{i})\psi_{w_T} z_\lambda = \psi_{w_T} e(w_T^{-1}\mathbf{i}) z_\lambda = \begin{cases} \psi_{w_T} e(\mathbf{i}_\lambda) z_\lambda & \text{if } \mathbf{i} = w_T \mathbf{i}_\lambda; \\ \psi_{w_T} e(\mathbf{i}') z_\lambda \text{ with } \mathbf{i}' \neq \mathbf{i} & \text{if } \mathbf{i} \neq w_T \mathbf{i}_\lambda, \end{cases}$$

and the result clearly follows from Definition 1.19. □

Whilst the vectors v_T of S_λ also depend on the choice of a preferred reduced expression, in general, the following result does not.

Theorem 1.22. *[BKW, Corollary 4.6] and [KMR, Proposition 7.14] Let $\lambda \in \mathscr{P}_n^l$. The set of vectors*

$$\{v_T \mid T \in \operatorname{Std}(\lambda)\}$$

is a homogeneous $\mathbb{F}$-basis of S_λ of degrees

$$\deg(v_T) = \deg(T).$$

Moreover, for any λ-tableau S, not necessarily standard, v_S can be written as a linear combination of $\mathbb{F}$-basis elements v_T such that $S \trianglerighteq T$.

We call this basis the *standard homogeneous basis of S_λ*. We consider the action of the Khovanov–Lauda–Rouquier generators on basis elements of S_λ.

Lemma 1.23. *[BKW, Lemmas 4.8 and 4.9] Let $\lambda \in \mathscr{P}_n^l$ and $T \in \operatorname{Std}(\lambda)$.*

1. *$y_r v_T$ can be written as a linear combination of basis elements v_S such that $T \triangleright S$, for $1 \leqslant r \leqslant n$;*
2. *If $r \downarrow_T r+1$ or $r \rightarrow_T r+1$, then $\psi_r v_T$ can be written as a linear combination of basis elements v_S such that $T \triangleright S$ and $\mathbf{i}_S = \mathbf{i}_{s_r T}$, for $1 \leqslant r \leqslant n-1$.*

Theorem 1.24. *[BKW, Theorem 4.10(i)] Let $\lambda \in \mathscr{P}_n^l$ and $\lambda \in \operatorname{Std}(\lambda)$. Given reduced expressions $s_{r_1} \dots s_{r_m}$ and $s_{t_1} \dots s_{t_m}$ for w_T,*

$$\psi_{r_1} \dots \psi_{r_m} z_\lambda - \psi_{t_1} \dots \psi_{t_m} z_\lambda$$

can be written as a linear combination of homogeneous basis elements v_S where $S \triangleleft T$.

We now examine the homogeneous elements of Specht modules, since we know from Definition 1.19 that Specht modules are graded over $\mathscr{R}_n^\Lambda$.

Theorem 1.25. *[BKW, Theorem 4.10(ii)] Let $\lambda \in \mathscr{P}_n^l$ and $\lambda \in \operatorname{Std}(\lambda)$. For each r, the vectors $y_r v_T$ and $\psi_r v_T$ are homogeneous, and we have that*

$$\begin{aligned} e(\mathbf{i})v_T &= \delta_{\mathbf{i},\mathbf{i}_T}(v_T) && (\mathbf{i} \in I^n), \\ \deg(y_r v_T) &= \deg(y_r) + \deg(v_T) && (r \in \{1,\dots,n\}), \\ \deg(\psi_r v_T) &= \deg(\psi_r e(\mathbf{i})) + \deg(v_T) && (r \in \{1,\dots,n-1\}). \end{aligned}$$

1.3.4 Graded dimensions of Specht modules

We first review the ungraded dimensions of James' classical Specht modules for the symmetric group algebras. We know from [J3, Theorem 8.4] that the basis elements of

Specht modules S_λ defined over the symmetric group algebra are labelled by standard λ-tableaux, so that the dimension of S_λ equals the total number of distinct standard λ-tableaux. The size of this basis of a Specht module for $\mathbb{F}\mathfrak{S}_n$ is far from trivial to determine.

For a partition $\lambda \in \mathscr{P}_n^1$, we say that a *hook* in λ consists of a node $(i,j) \in [\lambda]$ together with nodes lying directly below it and directly to its right. That is, the hook of (i,j) is the set of nodes

$$\{(i,j) \in [\lambda] \mid j \leqslant l \leqslant \lambda_i\} \cup \{(k,j) \in [\lambda] \mid i \leqslant k \leqslant m\},$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$ has m non-zero parts. The *hook length of* (i,j), denoted $h_{(i,j)}$, is the number of nodes in its hook.

In 1954, Frame, Robinson and Thrall [FRT] introduced the following beautiful combinatorial formula using hook lengths to determine the dimensions of Specht modules for $\mathbb{F}\mathfrak{S}_n$.

Theorem 1.26 (Hook Length Formula)**.** *Let $\lambda \in \mathscr{P}_n^1$. Then*

$$\dim(S_\lambda) = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{(i,j)}}$$

Remark 1.27. *It is clear that the Hook Length Formula is independent of the choice of the ground field $\mathbb{F}$. By generalising the definition of a hook to a multipartition, we can easily extend this result to ungraded Specht modules labelled by multipartitions $\lambda \in \mathscr{P}_n^l$ for the cyclotomic Khovanov–Lauda–Rouquier algebra.*

Example 1.28. *Let $\lambda = ((5,3),(2^2,1))$. The hook lengths of each node in $[\lambda]$ are as follows*

6	5	4	2	1
3	2	1		

.

4	2
3	1
1	

Thus $\dim(S_\lambda) = \frac{13!}{6\cdot5\cdot4^2\cdot3^2\cdot2^3\cdot1^4} = 180180$*, so there are* 180180 *standard* $((5,3),(2^2,1))$*-tableaux!*

We now consider the *graded dimensions* of Specht modules for $\mathscr{R}_n^\Lambda$ by taking into account their $\mathbb{Z}$-gradings.

For a multipartition $\lambda \in \mathscr{P}_n^l$, recall that the *degree* of standard λ-tableau T is defined recursively via

$$\deg(T) := d^A(\mu) + \deg(T_{\leqslant n-1}),$$

where $T_{\leqslant n-1}$ is the standard μ-tableau obtained by removing node A from T, which contains entry n. By Theorem 1.22, we know that the degree of a standard basis vector

v_T is the degree of the standard λ-tableau T, that is, $\deg(v_T) = \deg(T)$. Thus we obtain the graded dimension of S_λ by summing over each degree of every possible standard λ-tableau as follows.

Proposition 1.29. *Let $\lambda \in \mathscr{P}_n^l$. Then the graded dimension of S_λ is*

$$\operatorname{grdim}(S_\lambda) := \sum_{T \in \operatorname{Std}(\lambda)} v^{\deg(T)},$$

where v is an arbitrary indeterminate.

Naturally, we recover the ungraded dimension of S_λ by setting $v = 1$.

Example 1.30. *Let $e = 3$ and $\kappa = (0,0)$. $S_{((1),(1^4))}$ is spanned by basis vectors labelled by tableaux*

$$T_1 = \begin{array}{|c|}\hline 1\\\hline\end{array},\ T_2 = \begin{array}{|c|}\hline 2\\\hline\end{array},\ T_3 = \begin{array}{|c|}\hline 3\\\hline\end{array},\ T_4 = \begin{array}{|c|}\hline 4\\\hline\end{array},\ T_5 = \begin{array}{|c|}\hline 5\\\hline\end{array}.$$

$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$$

We find the degree of T_1. We note that the degree of any node in the first component is 0, so $d^{(1,1,1)} = 0$. The e-residues of $((1),(1))$ are

$$\begin{array}{|c|}\hline 0\\\hline\end{array}.$$

$$\begin{array}{|c|}\hline 0\\\hline\end{array}$$

Thus $(1,1,2)$ has removable 0-node $(1,1,1)$, and hence $d^{(1,1,2)} = -1$. Adding this node, we observe the e-residues of $((1),(1^2))$

$$\begin{array}{|c|}\hline 0\\\hline \square\\\hline\end{array}.$$

$$\begin{array}{|c|}\hline 0\\\hline 2\\\hline\end{array}$$

Thus $(2,1,2)$ has addable 2-node $(2,1,1)$, and hence $d^{(2,1,2)} = 1$. Adding this node, we observe the e-residues of $((1),(1^3))$

$$\begin{array}{|c|c|}\hline 0 & \square\\\hline\end{array}.$$

$$\begin{array}{|c|c|}\hline 0 & \square\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array}$$

Thus $(3,1,2)$ has addable 1-nodes $(1,2,1)$ and $(1,2,2)$, and hence $d^{(3,1,2)} = 2$. Finally, *adding this node, we observe the e-residues of* $((1),(1^4))$

$$\boxed{0}\,.$$

$$\begin{array}{|c|}\hline 0\\ \hline 2\\ \hline 1\\ \hline 0\\ \hline\end{array}$$

Thus $(4,1,2)$ *has removable* 0*-node* $(1,1,1)$*, and hence* $d^{(4,1,2)}=-1$*. Thus*

$$\deg(T_1)=d^{(1,1,1)}+d^{(1,1,2)}+d^{(2,1,2)}+d^{(3,1,2)}+d^{(4,1,2)}=1.$$

Similarly, one can find that $\deg(T_2)=\deg(T_5)=3$*,* $\deg(T_3)=2$ *and* $\deg(T_4)=1$*, and hence* $\operatorname{grdim}\left(S_{((1),(1^4))}\right)=2v^3+v^2+2v$*.*

Unlike the Hook Length Formula given in Theorem 1.26, there is no known closed formula for the graded dimension of a Specht module for $\mathscr{R}_n^\Lambda$. This raises the problem of determining if a graded analogue to the Hook Length Formula exists.

1.3.5 Weights of multipartitions

We define the *block decomposition* of a finite-dimensional $\mathbb{F}$-algebra A to be

$$A=B_0\oplus B_1\oplus\cdots\oplus B_m,$$

where $B_0,\dots,B_m$ are indecomposable, two-sided ideals. We call these B_i, for all $i\in\{0,\dots,m\}$, the *blocks of* A. We now let $e_0,\dots,e_m$ be the primitive central idempotents of A, namely the *block idempotents* of A. If $Ae_i=B_i$, for $i\in\{0,\dots,m\}$, then $e_iB_i=B_i$, otherwise $e_jB_i=0$ for $j\neq i$.

Let M be an A-module. We have $\sum_{i=1}^m e_i=1_A$, so that M decomposes as

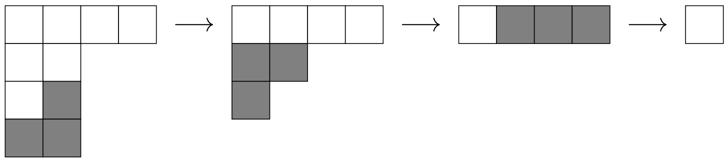
$$M=e_0M\oplus\cdots\oplus e_mM.$$

We say that M *belongs to the block* B_i if $e_iM=M$.

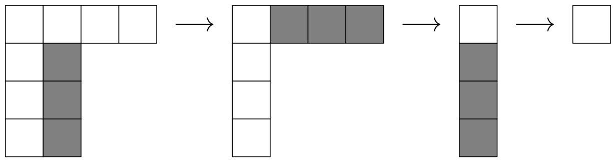
The blocks of the Khovanov–Lauda–Rouquier algebras, in particular the *weights* of these blocks, provide us with a lot of information about the representations of these algebras. For example, each block contains a Specht module, and moreover, each Specht module belongs to a single block. For $\lambda\in\mathscr{P}_n^l$, we know from [LM] that the weight of a block containing the Specht module S_λ corresponds to the combinatorial definition of the *e-weight* of the multipartition λ, which we now introduce.

We first restrict our attention to level one and let $\lambda\in\mathscr{P}_n^1$. We define a *rim e-hook* of λ to be a connected chain of e nodes of $[\lambda]$ such that we obtain a valid Young diagram when these nodes are removed from $[\lambda]$. The partition obtained by removing all possible rim e-hooks from λ is called the *e-core of λ*, denoted $\mathrm{cor}_e(\lambda)$. We say that the *e-weight of λ* is the number of rim e-hooks we remove to obtain the e-core of λ, denoted by $\mathrm{wt}_e(\lambda)$. We set $\mathrm{wt}_e(\lambda) = 0$ if $e = \infty$.

Example 1.31. *Let $e = 3$ and $\lambda = (5, 2^3)$. Observe that we can successively remove rim 3-hooks (shaded below) from $[\lambda]$ to obtain its 3-core as follows*



We see that $\mathrm{cor}_3(\lambda) = (1)$ and $\mathrm{wt}_3(\lambda) = 3$. In fact, notice that we obtain the same 3-core, and hence 3-weight, of λ by successively removing different rim 3-hooks from $[\lambda]$ as follows



We see that the e-core, and hence the e-weight, of a partition are well defined notions.

Theorem 1.32. *[JK, Theorem 2.7.16] The e-core of $\lambda \in \mathscr{P}_n^1$ is uniquely determined.*

Fayers [Fa3, §2.1] generalised the definition of the e-weight of a partition to higher levels of the cyclotomic Hecke algebras, and hence the cyclotomic Khovanov–Lauda–Rouquier algebras, by introducing the e-weight of a multipartition.

In fact, the e-weights of multipartitions are block invariants, so that the e-weights of two multipartitions $\lambda, \mu \in \mathscr{P}_n^l$ are the same whenever the Specht modules S_λ and S_μ lie in the same block. Moreover, we know from [Fa3], for $\alpha \in Q_+$ and $\lambda \in \mathscr{P}_n^l$ such that $\alpha = \mathrm{cont}(\lambda)$, that the defect of α from Equation (1.2.3) is equivalent to the e-weight of λ.

1.3.6 Graded duality of Specht modules

For $\lambda \in \mathscr{P}_n^l$, one can instead study the representations of $\mathscr{R}_n^\Lambda$ as *(row) Specht modules*, denoted by S^λ. Row Specht modules have a similar presentation to column Specht modules, as can be seen in [KMR]. In fact, row and column Specht modules are dual to each other, up to a grading shift.

Theorem 1.33. *[KMR, Theorem 7.25] Let $\lambda \in \mathscr{P}_n^l$. Then we have the following isomorphism of $\mathscr{R}_n^\Lambda$-algebras*

$$S^\lambda \cong (S_\lambda)^\circledast \langle \mathrm{wt}_e(\lambda) \rangle.$$

1.4 Graded irreducible $\mathscr{R}_n^\Lambda$-modules

We determine a classification of the graded irreducible $\mathscr{R}_n^\Lambda$-modules. We can equip the graded $\mathscr{R}_n^\Lambda$-module S_λ with a homogeneous symmetric bilinear form $\langle\ ,\ \rangle$ of degree zero (see [HM1, §2]). We define the radical of S_λ to be

$$\operatorname{rad} S_\lambda = \{v_T \in S_\lambda \mid \langle v_T, v_S \rangle = 0, \forall v_S \in S_\lambda\}.$$

Since $\langle v_T, v_S \rangle = 0$ whenever $\deg(v_T) + \deg(v_S) \neq 0$ (that is, $\deg(T) + \deg(S) \neq 0$), $\operatorname{rad} S_\lambda$ is a graded $\mathscr{R}_n^\Lambda$-submodule of S_λ. We now define the graded quotient $\mathscr{R}_n^\Lambda$-module

$$D_\lambda := S_\lambda / \operatorname{rad} S_\lambda$$

for each $\lambda \in \mathscr{P}_n^l$. We know from [HM1, Lemma 2.9] that D_λ is absolutely irreducible or zero. Moreover, since we also know from [HM1, Lemma 2.9] that $\operatorname{rad} D_\lambda$ is the graded Jacobson radical of S_λ, D_λ is a well-defined graded quotient of S_λ by Lemma 1.1.

Specht modules also exhibit a graded cellular basis, and so, S_λ can be analogously studied as a graded cell module (see [HM1] for an exposition). The next result shows that the irreducible heads of S_λ only arise from regular multipartitions, first conjectured by Ariki and Mathas in [AM].

Theorem 1.34. *[HM1, Corollary 5.11] D_λ is an absolutely irreducible $\mathscr{R}_n^\Lambda$-module if and only if $\lambda \in \mathscr{RP}_n^l$.*

Thus, these non-zero quotient $\mathscr{R}_n^\Lambda$-modules labelled by regular multipartitions give a complete classification of the irreducible $\mathscr{R}_n^\Lambda$-modules.

Theorem 1.35. *[BK3, Theorem 4.11] and [HM1, Proposition 2.18]*

1. *$\left\{D_\lambda\langle i\rangle \mid \lambda \in \mathscr{RP}_n^l, i \in \mathbb{Z}\right\}$ is a complete set of pairwise non-isomorphic irreducible graded $\mathscr{R}_n^\Lambda$-modules.*
2. *For all $\lambda \in \mathscr{RP}_n^l$, $D_\lambda \cong D_\lambda^\circledast$ as graded $\mathscr{R}_n^\Lambda$-modules.*

By the second part of this theorem, we know that irreducible $\mathscr{R}_n^\Lambda$-modules are self-dual. As D_λ and its dual are isomorphic as graded $\mathscr{R}_n^\Lambda$-modules, no grading shifts occur which leads us on to the following result.

Proposition 1.36. *For all $\lambda \in \mathscr{RP}_n^l$, the graded dimension of D_λ is symmetric in v and v^{-1}.*

Together with Proposition 1.29, the following result is an immediate consequence.

Corollary 1.37. *Let $\lambda \in \mathscr{RP}_n^l$ and $\mathscr{T} = \operatorname{Std}(\lambda)$. Then*

$$\operatorname{grdim}(D_\lambda) = v^i \sum_{T \in \mathscr{T}} v^{\deg(T)},$$

where $2i = -\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})$. *Moreover, the highest degree in the graded dimension of* D_λ *is* $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T}))$.

Setting $S = \{v_T \mid T \in \mathscr{T}\}$, we have $\operatorname{grdim}(D_\lambda) = v^i \operatorname{grdim}(\operatorname{span} S)$, and in other words $D_\lambda = \operatorname{span} S\langle i\rangle$.

1.5 Graded decomposition numbers for $\mathscr{R}_n^\Lambda$

Decomposition numbers record information about the structure of Specht modules over $\mathscr{R}_n^\Lambda$. For $\lambda \in \mathscr{P}_n^l$ and $\mu \in \mathscr{RP}_n^l$, we denote the *ungraded decomposition number* by $d_{\lambda,\mu} = [S_\lambda : D_\mu]$, which is the multiplicity of D_μ appearing as a composition factor in a composition series of S_λ. We know that we can afford Specht modules with a grading and study the graded composition factors that arise in their composition series, since a graded version of the Jordan–Hölder theorem exists.

We denote the ungraded decomposition matrix for $\mathscr{R}_n^\Lambda$ by $(d_{\lambda,\mu})$; we write $(d_{\lambda,\mu}^{\mathbb{F}})$ when we want to emphasise the ground field $\mathbb{F}$. We can compute the ungraded decomposition matrices for $\mathscr{R}_n^\Lambda$ over a field of characteristic zero via the generalised Lascoux–Leclerc–Thibon algorithm given by Fayers in [Fa4], whereas, the ungraded decomposition matrices for $\mathscr{R}_n^\Lambda$ over a field of positive characteristic are far more elusive. For $\nu, \mu \in \mathscr{RP}_n^l$, we know from [BK3] that there exists an *adjustment matrix* $(a_{\nu,\mu}^{\mathbb{F}})$ such that the product $(d_{\lambda,\nu}^{\mathbb{C}})(a_{\nu,\mu}^{\mathbb{F}})$ gives us the ungraded decomposition matrices for $\mathscr{R}_n^\Lambda$ over an arbitrary field, and moreover, $(d_{\lambda,\nu}^{\mathbb{C}})(a_{\nu,\mu}^{\mathbb{C}}) = (d_{\lambda,\nu}^{\mathbb{C}})$. However, there exists no algorithm for finding the entries in the adjustment matrix over a field of positive characteristic.

For $\lambda \in \mathscr{P}_n^l$ and $\mu \in \mathscr{RP}_n^l$, we define the *graded decomposition number* (or the graded composition multiplicity) to be

$$d_{\lambda,\mu}(v) = [S_\lambda : D_\mu]_v := \sum_{i \in \mathbb{Z}} a_i v^i \in \mathbb{N}[v, v^{-1}],$$

where a_i is the composition multiplicity of $D_\mu\langle i\rangle$ appearing as a composition factor of S_λ. Note that we recover the ungraded decomposition number by setting $v = 1$.

We record these graded multiplicities in a *graded decomposition matrix*, denoted by $(d_{\lambda\mu}(v))$, where its rows correspond to Specht modules labelled by multipartitions and its columns correspond to irreducible quotients of Specht modules labelled by regular multipartitions. By [BK2, Corrollary 6.3] we know that the graded decomposition matrices for cyclotomic Hecke algebras, and hence for the corresponding cyclotomic Khovanov–Lauda–Rouquier algebras, only depend on the quantum characterisitic e and the characteristic of the ground field $\mathbb{F}$, and not on $\mathbb{F}$ itself, affirming a conjecture by Mathas.

The following result for $\mathscr{R}_n^\Lambda$ is a generalised graded version of [J3, Corollary 12.2] for $\mathbb{F}\mathfrak{S}_n$.

Theorem 1.38. *[BK3, Corollary 5.15] Let* $\lambda \in \mathscr{P}_n^l$ *and* $\mu \in \mathscr{RP}_n^l$. *Then*

1. $d_{\mu,\mu}(v) = 1$;

2. $d_{\lambda,\mu}(v) \neq 0$ *only if* $\mu \trianglerighteq \lambda$.

Theorem 1.39. *[BK3, Theorem 5.17] Let $\mathbb{F}$ be an arbitrary field, $\lambda \in \mathscr{P}_n^l$ and $\mu \in \mathscr{R}\mathscr{P}_n^l$. Then*

$$d_{\lambda,\mu}^{\mathbb{F}}(v) = \sum_{\nu \in \mathscr{R}\mathscr{P}_n^l} d_{\lambda,\nu}^{\mathbb{C}}(v) a_{\nu,\mu}^{\mathbb{F}}(v),$$

for some $a_{\nu,\mu}^{\mathbb{F}}(v) \in \mathbb{N}[v, v^{-1}]$ with $a_{\nu,\mu}^{\mathbb{F}}(v) = a_{\nu,\mu}^{\mathbb{F}}(v^{-1})$.

We say that $a_{\nu,\mu}^{\mathbb{F}}(v)$ is a *graded adjustment number* of $\mathscr{R}_n^\Lambda$ over $\mathbb{F}$; the graded adjustment matrix $(a_{\nu,\mu}^{\mathbb{F}}(v))$ is a square unitriangular matrix whose entries are symmetric in v and v^{-1} and whose rows and columns correspond to regular multipartitions, whereby we recover the ungraded adjustment matrix when we set $v = 1$. It follows that we can obtain the graded decomposition matrix for $\mathscr{R}_n^\Lambda$ over a field of positive characteristic by post-multiplying the graded decomposition matrix $\mathscr{R}_n^\Lambda$ over $\mathbb{C}$ by the graded adjustment matrix, that is,

$$(d_{\lambda,\mu}^{\mathbb{F}}(v)) = (d_{\lambda,\nu}^{\mathbb{C}}(v))(a_{\nu,\mu}^{\mathbb{F}}(v)),$$

for $\lambda \in \mathscr{P}_n^l$, $\mu, \nu \in \mathscr{R}\mathscr{P}_n^l$.

1.6 Induction and restriction of $\mathscr{R}_n^\Lambda$-modules

The *decomposition number problem* of understanding the multiplicities $[S_\lambda : D_\mu]$, for all $\lambda, \mu \in \mathscr{P}_n^l$, is equivalent to the *branching problem* of understanding the multiplicities

$$[\operatorname{res}_{\mathscr{R}_{n-1}^\Lambda}^{\mathscr{R}_n^\Lambda} D_\lambda : D_\mu],$$

for all $\lambda, \mu \in \mathscr{P}_n^l$, which provides the motivation for studying the restriction of an irreducible $\mathscr{R}_n^\Lambda$-module to an $\mathscr{R}_{n-1}^\Lambda$-module. The restriction of the ordinary representations of the symmetric group and their composition factors are well understood via the *Classical Branching Rule for* $\mathbb{F}\mathfrak{S}_n$ (for example, see [J3, Theorem 9.2]). This result was extended to the Ariki–Koike algebras or the cyclotomic Hecke algebras by Ariki–Koike [AK, Corollary 3.12], and hence we introduce an analogous result for the cyclotomic Khovanov–Lauda–Rouquier algebras, recently given by Mathas [M3]. We simultaneously recall the 'dual' results in [HM1] of how Specht modules of the cyclotomic Khovanov–Lauda–Rouquier algebra behave under induction. Induction allows us to understand representations of the the cyclotomic Khovanov–Lauda–Rouquier algebra $\mathscr{R}_{n+1}^\Lambda$ from representations of the subalgebra $\mathscr{R}_n^\Lambda$ that are known to us. We write *res* to denote the functor restricting a $\mathscr{R}_n^\Lambda$-module to a $\mathscr{R}_{n-1}^\Lambda$-module, and write *ind* to denote the functor inducing a $\mathscr{R}_n^\Lambda$-module to a $\mathscr{R}_{n+1}^\Lambda$-module.

We first introduce Brundan and Kleshchev's i-restriction and i-induction operators e_i and f_i acting on $\mathbb{F}\mathfrak{S}_n$-modules, as given in Section 2.2 of [BK1]. These functors originate from Robinson [Rob]; we extend these exact operators to act on $\mathscr{R}_n^\Lambda$-modules.

We let M be a $\mathscr{R}_n^\Lambda$-module. There are *i-restriction functors* $e_i : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{n-1}^\Lambda\text{-mod}$, for $i \in \mathbb{Z}/e\mathbb{Z}$, such that

$$\operatorname{res}_{\mathscr{R}_{n-1}^\Lambda}^{\mathscr{R}_n^\Lambda} M \cong \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} e_i M,$$

by [BK1, Lemma 2.5], which is an analogous result to [AK, Corollary 3.12].

The *graded* classical branching rule for Specht modules is given as follows, whereby the ungraded version is recovered by setting v to be 1.

Theorem 1.40. *[M3, Corollary 5.8] Let $\lambda \in \mathscr{P}_n^l$ and $i \in \mathbb{Z}/e\mathbb{Z}$. Let $A_1, A_2, \dots, A_m$ be the removable i-nodes of λ and $\lambda^{(1)} \lhd \lambda^{(2)} \lhd \dots \lhd \lambda^{(m)}$ be the respective l-multipartitions of $n-1$ such that $\lambda^{(j)} = \lambda \backslash \{A_j\}$ for $1 \leqslant j \leqslant m$. Then $e_i S_\lambda$ has a filtration of $\mathscr{R}_{n-1}^\Lambda$-modules*

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_m = e_i S_\lambda,$$

with $M_j/M_{j-1} \cong v^{d^{A_j}(\lambda)} S_{\lambda^{(j)}}$, for $1 \leqslant j \leqslant m$.

Example 1.41. *Let $e = 3$, $\kappa = (0,2)$ and $\lambda = ((6,5^2,2),(4,3,2))$. We observe that the 3-residues of λ and its addable 2-nodes, shading the removable 2-nodes of λ, are*

0	1	2	0	1	2 (shaded)
2	0	1	2	0	
1	2	0	1	2 (shaded)	
0	1	2			
2					

.

2	0	1	2 (shaded)
1	2	0	
0	1	2	
2			

We label the removable 2-nodes as $A_1 = (1,6,1)$, $A_2 = (3,5,1)$ and $A_3 = (1,4,2)$. It follows that $e_2 S_\lambda$ has a filtration of $\mathscr{R}_{26}^\Lambda$-modules

$$0 \subset M_0 \subset M_1 \subset M_2 \subset M_3 = e_2 S_\lambda,$$

where

$$\begin{aligned} M_1/M_0 &\cong v^{d^{A_1}(\lambda)} S_{\lambda^{(1)}} \cong S_{((5^3,2),(4,3,2))}, \\ M_2/M_1 &\cong v^{d^{A_2}(\lambda)} S_{\lambda^{(2)}} \cong v^{-1} S_{((6,5,4,2),(4,3,2))}, \\ M_3/M_2 &\cong v^{d^{A_3}(\lambda)} S_{\lambda^{(3)}} \cong S_{((6,5^2,2),(3^2,2))}. \end{aligned}$$

Similarly, there are *i-induction functors* $f_i : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{n+1}^\Lambda\text{-mod}$, for $i \in \mathbb{Z}/e\mathbb{Z}$, such that

$$\operatorname{ind}_{\mathscr{R}_n^\Lambda}^{\mathscr{R}_{n+1}^\Lambda} M \cong \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} f_i M,$$

by [BK1, Lemma 2.5].

The 'dual' graded branching rule for Specht modules is given as follows.

Theorem 1.42. *[HM2, Main Theorem] Let $\lambda \in \mathscr{P}_n^l$ and $i \in \mathbb{Z}/e\mathbb{Z}$. Let $A_1, A_2, \dots, A_m$ be the addable i-nodes of λ and $\lambda^{(1)} \triangleright \lambda^{(2)} \triangleright \cdots \triangleright \lambda^{(m)}$ be the respective l-multipartitions of $n+1$ such that $\lambda^{(j)} = \lambda \cup \{A_j\}$ for $1 \leqslant j \leqslant m$. Then $f_i S_\lambda$ has a filtration of $\mathscr{R}_{n+1}^\Lambda$-modules*

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_m = f_i S_\lambda$$

with $M_j/M_{j-1} \cong v^{d^{A_j}} S_{\lambda^{(j)}}$ for $1 \leqslant j \leqslant m$.

Example 1.43. *Let $e = 3$, $\kappa = (0,2)$ and $\lambda = ((6,5^2,2),(4,3,2))$ as in Example 1.41. We label the addable 2-nodes of λ as $B_1 = (4,3,1)$, $B_2 = (5,1,1)$, $B_3 = (3,3,2)$ and $B_4 = (4,1,2)$. Then $f_2 S_\lambda$ has a filtration of $\mathscr{R}_{28}^\Lambda$-modules*

$$0 \subset M_0 \subset M_1 \subset M_2 \subset M_3 \subset M_4 = f_2 S_\lambda,$$

where

$$\begin{aligned}
M_1/M_0 &\cong v^{d^{B_1(\lambda)}} S_{\lambda^{(1)}} \cong v^{-2} S_{((6,5^2,3),(4,3,2))},\\
M_2/M_1 &\cong v^{d^{B_2(\lambda)}} S_{\lambda^{(2)}} \cong v^{-1} S_{((6,5^2,2,1),(4,3,2))},\\
M_3/M_2 &\cong v^{d^{B_3(\lambda)}} S_{\lambda^{(3)}} \cong v^{-1} S_{((6,5^2,2),(4,3^3))},\\
M_4/M_3 &\cong v^{d^{B_4(\lambda)}} S_{\lambda^{(4)}} \cong S_{((6,5^2,2),(4,3,2,1))}.
\end{aligned}$$

The operators e_i and f_i are both left and right adjoint to each other by [K4, Lemma 8.2.2], and so are exact functors.

There are generalisations of the i-restriction and i-induction operators to "divided powers" $e_i^{(r)}$ and $f_i^{(r)}$. For $i \in \mathbb{Z}/e\mathbb{Z}$ and $r \geqslant 0$, there are *divided power i-restriction functors* $e_i^{(r)} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{n-r}^\Lambda\text{-mod}$, which satisfy [BK1, Lemma 2.6]

$$e_i^r M \cong \bigoplus_{k=1}^{r!} e_i^{(r)} M.$$

Similarly, for $i \in \mathbb{Z}/e\mathbb{Z}$, there are *divided power induction i-functors* $f_i^{(r)} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{n+r}^\Lambda\text{-mod}$, which satisfy [BK1, Lemma 2.6]

$$f_i^r M \cong \bigoplus_{k=1}^{r!} f_i^{(r)} M.$$

Notice that $e_i^{(1)} = e_i$ and $f_i^{(1)} = f_i$. The divided powers $e_i^{(r)}$ and $f_i^{(r)}$ are also both left and right adjoint to each other (see [K4, Theorem 8.3.2]), and so are exact functors.

For a non-zero $\mathscr{R}_n^\Lambda$-module M, we define

$$\epsilon_i(M) = \max\{r \geqslant 0 \mid e_i^{(r)} M \neq 0\} \quad \text{and} \quad \varphi_i(M) = \max\{r \geqslant 0 \mid f_i^{(r)} M \neq 0\}, \tag{1.6.1}$$

and now define

$$e_i^{(\max)} M = e_i^{(\epsilon_i M)} M \qquad \text{and} \qquad f_i^{(\max)} M = f_i^{(\varphi_i M)} M.$$

Corollary 1.44. *Let $\lambda \in \mathscr{P}_n^l$ and $i \in \mathbb{Z}/e\mathbb{Z}$.*

1. *Then $\epsilon_i(S_\lambda) = \mathrm{rem}_i(\lambda)$ and $e_i^{(\max)} S_\lambda \cong S_{\lambda^{\triangledown i}}$.*
2. *Then $\varphi_i(S_\lambda) = \mathrm{add}_i(\lambda)$ and $f_i^{(\max)} S_\lambda \cong S_{\lambda^{\vartriangle i}}$.*

1.7 Modular branching rules for $\mathscr{R}_n^\Lambda$-modules

Kleshchev developed the analogous theory for restricting the modular representations of the symmetric group [K1, K2, K3], which Brundan extended to Hecke algebras of type A [B]. These modular branching rules were generalised for cyclotomic Hecke algebras, proven by Ariki in the proof of [A3, Theorem 6.1]. Thus, modular branching rules for the cyclotomic Khovanov–Lauda–Rouquier algebras make sense, which we note here. Recall that D_λ is the irreducible quotient of the Specht module S_λ, where $\lambda \in \mathscr{R}\mathscr{P}_n^l$.

Theorem 1.45. *[BK1, §2.6] Let $\lambda \in \mathscr{P}_n^l$. Then*

$$\epsilon_i(D_\lambda) = \mathrm{nor}_i(\lambda) \text{ and } \varphi_i(D_\lambda) = \mathrm{conor}_i(\lambda).$$

Moreover,

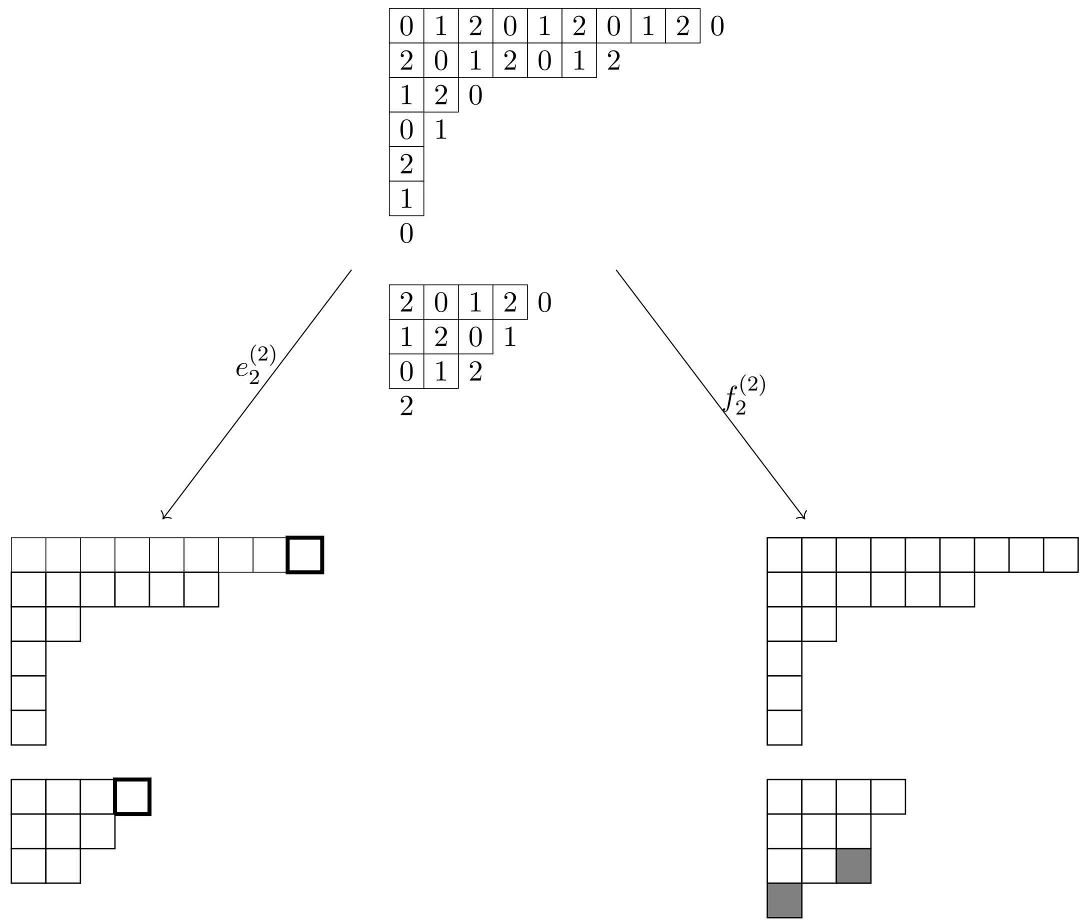
1. *if $A_1, \ldots, A_{\mathrm{nor}_i(\lambda)}$ are the normal i-nodes of λ, then*

$$e_i^{(\max)} D_\lambda \cong D_{\lambda \setminus \{A_1, \ldots, A_{\mathrm{nor}_i(\lambda)}\}},$$

2. *and if $A_1, \ldots, A_{\mathrm{conor}_i(\lambda)}$ are the conormal i-nodes of λ, then*

$$f_i^{(\max)} D_\lambda \cong D_{\lambda \cup \{A_1, \ldots, A_{\mathrm{conor}_i(\lambda)}\}}.$$

Example 1.46. *Let $e = 3$, $\kappa = (0, 2)$ and $\lambda = ((9, 6, 2, 1^3), (4, 3, 2))$. We know that S_λ is irreducible since we can obtain λ from $(\varnothing, \varnothing)$ by adding certain conormal nodes, that is, $\lambda = (\varnothing, \varnothing) \uparrow_2 \uparrow_1 \uparrow_0 \uparrow_2 \uparrow_0 \uparrow_1^2 \uparrow_2^2 \uparrow_0^4 \uparrow_1^4 \uparrow_2 \uparrow_0 \uparrow_2^4 \uparrow_0^2 \uparrow_1^3 \uparrow_2$. We draw the 3-residues of λ and its* *addable nodes as follows.*



Observe that λ *has* 2*-signature* $-+--++$*, and hence the reduced* 2*-signature of* λ *is* $--++$*, so that* λ *has two normal* 2*-nodes and two conormal* 2*-nodes. Note that we have also drawn the bipartition obtained by removing the normal* 2*-nodes from* λ *(that are outlined), as well as the bipartition obtained by adding the conormal* 2*-nodes of* λ *(that are shaded). Then we have*

$$e_2^{(2)} D_\lambda \cong D_{((8,6,2,1^3),(3^2,2))}$$
$$f_2^{(2)} D_\lambda \cong D_{((9,6,2,1^3),(4,3^2,1))}.$$

For each $i \in \mathbb{Z}/e\mathbb{Z}$, there is at most one good i-node of λ, and hence at most e good nodes of λ. It follows from [K2, Theorem 0.5] that the socle of the restriction of an irreducible $\mathscr{R}^{\Lambda}_{n}$-module D_λ to an $\mathscr{R}^{\Lambda}_{n-1}$-module is a direct sum of at most e indecomposable $\mathscr{R}^{\Lambda}_{n}$-summands. Moreover, we also know from [K2] that we can verify that the residue sequence of $\lambda\backslash\{A\}$ is distinct for each good node of λ, so that each summand $D_{\lambda\backslash\{A\}}$ belongs to a distinct block of $\mathscr{R}^{\Lambda}_{n}$. We generalise this result to "divided powers" as follows.

Corollary 1.47. *Let* $\lambda \in \mathscr{P}^{l}_{n}$ *and* $i \in \mathbb{Z}/e\mathbb{Z}$*.*

1. *If $r \leqslant \mathrm{nor}_i(\lambda)$, then*
$$\mathrm{soc}\left(e_i^{(r)} D_\lambda\right) \cong D_{\lambda\downarrow_i^r}.$$

2. *If $r \leqslant \mathrm{conor}_i(\lambda)$, then*
$$\mathrm{soc}\left(f_i^{(r)} D_\lambda\right) \cong D_{\lambda\uparrow_i^r}.$$

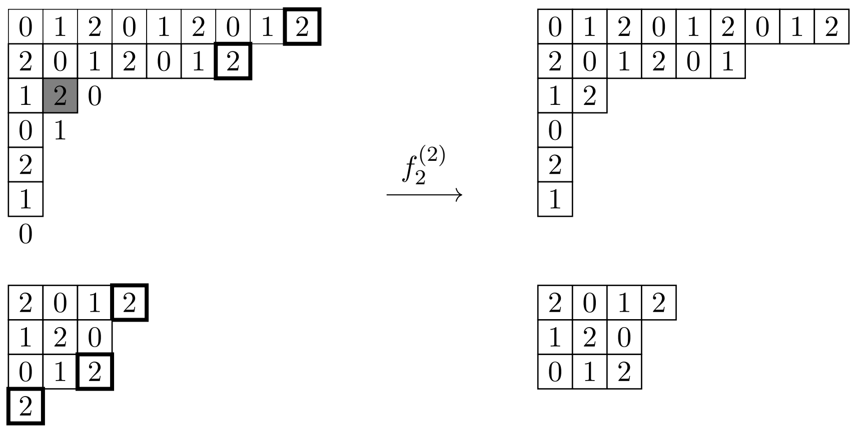
It follows that the modular branching rules for Specht modules of the Khovanov–Lauda–Rouquier algebras $\mathscr{R}_n^\Lambda$, together with the operators $\uparrow_i^r$ and $\downarrow_i^r$, provide a combinatorial algorithm for determining labels of $\mathscr{R}_n^\Lambda$-modules that we know are irreducible.

Proposition 1.48. *Let $\lambda \in \mathscr{P}_n^l$. If D is an irreducible $\mathscr{R}_n^\Lambda$-module with $e_i^{(r)} D \cong D_\lambda$, then $D = D_{\lambda\uparrow_i^r}$.*

Proof. Suppose that $D = D_\mu$ where $\mu \in \mathscr{RP}_n^l$, so that $e_i^{(r)} D = e_i^{(r)} D_\mu \cong D_\lambda$. We know that $r \leqslant \mathrm{nor}_i(\mu)$ since $e_i^{(r)} D \neq 0$, then from the first part of Corollary 1.47 we have $\mathrm{soc}\left(e_i^{(r)} D_\mu\right) \cong D_\nu$ where $\nu = \mu\downarrow_i^r$. Since $e_i^{(r)} D_\mu \cong D_\lambda$, we have $\nu = \lambda$. Then, by Equation (1.2.2), $\lambda = \mu\downarrow_i^r\uparrow_i^r = \mu$, as required. □

Let $0 \leqslant r \leqslant \mathrm{nor}_i(\lambda)$ with $e_i^{(r)} D_\mu \cong D_\lambda$, where $\lambda, \mu \in \mathscr{RP}_n^l$. Then the normal i-nodes of μ and the conormal i-nodes of λ coincide, and hence
$$\mathrm{soc}\left(f_i^{(r)}\left(e_i^{(r)} D_\mu\right)\right) \cong D_{\mu\uparrow_i^r\downarrow_i^r} = D_\mu.$$

Example 1.49. *Let $e = 3$, $\kappa = (0, 2)$ and $\lambda = ((8, 6, 2, 1^3), (3^2, 2))$. By Example 1.46, we have $e_2^{(2)} D_\lambda \cong D_\lambda$. The 3-residues of $((8, 6, 2, 1^3), (3^2, 2))$ are*

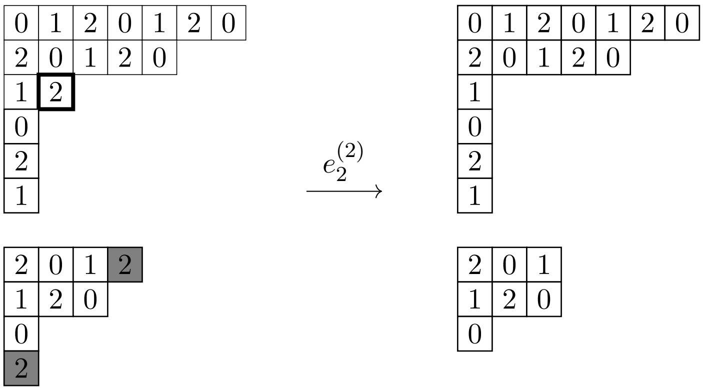


where the removable 2-node of λ is shaded and the addable 2-nodes of λ are outlined. Hence $((8, 6, 2, 1^3), (3^2, 2))$ has 2-signature $++-+++$, and thus reduced 2-signature $++++$. Adding the highest two conormal 2-nodes, it follows that $\mathrm{soc}\left(f_2^{(2)} D_{((8,6,2,1^3),(3^2,2))}\right) \cong D_{((9,6,2,1^3),(4,3,2))}$, as expected.

For non-irreducible $\mathscr{R}_n^\Lambda$-modules, we can determine the labels of their composition factors by applying the same combinatorial algorithm using the following result.

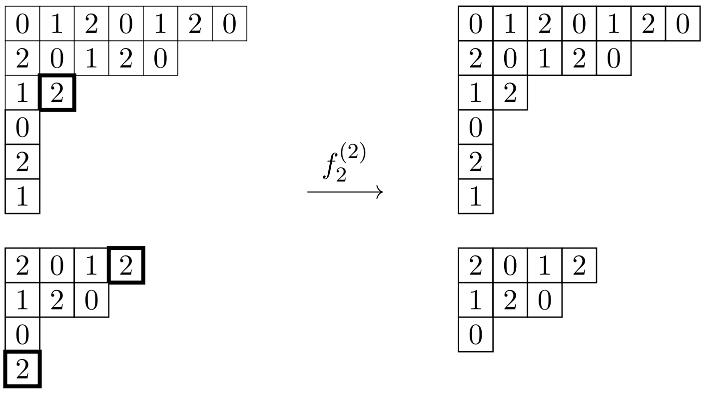
Proposition 1.50. *For $r \geqslant 0$ and an $\mathscr{R}_n^\Lambda$-module M, suppose that $e_i^{(r)}M \cong D_\mu$, where $\mu \in \mathscr{P}_{n-r}^l$. Then one of the composition factors of M is $D_{\mu\uparrow_i^r}$, while all the other composition factors of M are killed by $e_i^{(r)}$.*

Example 1.51. *Let $e = 3$, $\kappa = (0, 2)$ and $\lambda = ((7, 5, 1^4), (3^2, 1))$. The 3-residues of λ*



where the addable 2-node of λ is outlined and the removable 2-nodes of λ are shaded. Hence $e_2^{(2)} S_{((7,5,1^4),(4,3,1^2))} \cong S_{((7,5,1^4),(3^2,1))}$, which we know is irreducible since $S_{((8,6,2,1^3),(3^2,2))}$ is irreducible from Example 1.46 and $((8, 6, 2, 1^3), (3^2, 2)) \downarrow_1^3\downarrow_2 = ((7, 5, 1^4), (3^2, 1))$.

We now observe that the 3-residues of $((7, 5, 1^4), (3^2, 1))$ and its addable 2-nodes are



Observe that $+++$ is the 2-signature of $((7, 5, 1^4), (3^2, 1))$, and thus $\operatorname{soc}\left(f_2^{(2)} D_{((7,5,1^4),(3^2,1))}\right) \cong D_{((7,5,2,1^3),(4,3,1))}$ is a composition factor of S_λ.

Part II

Specht modules labelled by hook partitions

Chapter 2

Gradings on $S_{(n-m,m)}$ and $S_{(n-m,1^m)}$

Our main objects of study in Part II of this thesis are Specht modules $S_{(n-m,1^m)}$ labelled by hook partitions in finite quantum characteristic. In the following chapter, we will see a deep connection between these Specht modules and Specht modules labelled by two-part partitions in quantum characteristic two, inspired by the work [Mu2] of Murphy on hook representations. In preparation, we first determine the ungraded dimension of $S_{(n-m,m)}$ in quantum characteristic two, together with its graded analogue, and subsequently begin to understand the $\mathbb{Z}$-grading on $S_{(n-m,1^m)}$ in finite quantum characteristic.

2.1 The graded dimension of $S_{(n-m,m)}$ when $e = 2$

We first determine the ungraded dimension of Specht modules labelled by two-part partitions, which is independent of the ground field $\mathbb{F}$.

Proposition 2.1. *The (ungraded) dimension of $S_{(n-m,m)}$ is*

$$\frac{n+1-2m}{n+1}\binom{n+1}{m}.$$

Proof. Observe that the hook lengths of each node in $[(n-m,m)]$ are

$n{-}m{+}1$	$n{-}m$	$\cdots\cdots\cdots$	$n{-}2m{+}2$	$n{-}2m$	$n{-}2m{-}1$	$\cdots\cdots\cdots$	1
m	$m{-}1$	$\cdots\cdots\cdots$	1				

written as

- ◇ $h_{(1,m-i)} = n-2m+2+i$ for $0 \leqslant i \leqslant m-1$;
- ◇ $h_{(1,n-m-i)} = i+1$ for $0 \leqslant i \leqslant n-2m-1$;
- ◇ $h_{(2,m-i)} = i+1$ for $0 \leqslant i \leqslant m-1$.

Hence, by applying the Hook Length Formula given in Theorem 1.26, we have

$$\begin{aligned}\dim\left(S_{(n-m,m)}\right) &= \frac{n!}{m!(n-2m)!(n-2m+2)(n-2m+3)\dots(n-m+1)}\\ &= \frac{(n-2m+1)(n+1)!}{(n+1)m!(n-m+1)!}\\ &= \frac{n-2m+1}{n+1}\binom{n+1}{m}.\end{aligned}$$

□

We now let $e = 2$ and observe that the 2-residues for the nodes in the space $\mathbb{N}\times\mathbb{N}$ are

$$\begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 0 & 1 & 0 & \cdots \\ \hline 1 & 0 & 1 & 0 & 1 & \cdots \\ \hline 0 & 1 & 0 & 1 & 0 & \cdots \\ \hline 1 & 0 & 1 & 0 & 1 & \cdots \\ \hline 0 & 1 & 0 & 1 & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \cline{1-5} \end{array},$$

and thus, due to the chequerboard arrangement of these 2-residues, we expect that the graded dimensions of Specht modules in quantum characteristic two are easier to determine than those in quantum characteristic at least three.

Lemma 2.2. *Let $e = 2$ and $0 \leqslant m \leqslant \lfloor \frac{n}{2} \rfloor$. Then the graded dimension of $S_{(n-m,m)}$ is*

1. $\operatorname{grdim}\left(S_{(n-m-1,m)}\right) + v \cdot \operatorname{grdim}\left(S_{(n-m,m-1)}\right)$ *if n is even with $n > 2m$;*
2. $v \cdot \operatorname{grdim}\left(S_{(m,m-1)}\right)$ *if $n = 2m$;*
3. $\operatorname{grdim}\left(S_{(n-m-1,m)}\right) + v^{-1} \cdot \operatorname{grdim}\left(S_{(n-m,m-1)}\right)$ *if n is odd.*

Proof. Let $\lambda = (n-m, m)$. If $n > 2m$, then there are two nodes in $[\lambda]$ which n can lie in, namely $(1, n-m)$ and $(2, m)$. Clearly, there are no addable or removable $(n-m-1)$-nodes strictly above $(1, n-m)$ for λ, and hence $d^{(1,n-m)}(\lambda) = 0$.

1. Suppose that $n > 2m$ is even. Now $\operatorname{res}(2, m)$ $=\operatorname{res}(1, n-m+1)$, so $(1, n-m+1)$ is an addable $(m-2)$-node above $(2, m)$ and thus $d^{(2,m)}(\lambda) = 1$. Hence we have

$$\begin{aligned}&\operatorname{grdim}(S_\lambda)\\ &= \sum_{T\in Std(\lambda)} v^{\deg(T)}\\ &= \sum_{\substack{T\in Std(\lambda)\\ s.t.\ T(1,n-m)=n}} \left(v^{\deg(T)}\right) + \sum_{\substack{T\in Std(\lambda)\\ s.t.\ T(2,m)=n}} \left(v^{\deg(T)}\right)\\ &= \sum_{T\in Std((n-m-1,m))} \left(v^{d^{(1,n-m)}(\lambda)+\deg(T)}\right) + \sum_{T\in Std((n-m,m-1))} \left(v^{d^{(2,m)}(\lambda)+\deg(T)}\right)\end{aligned}$$

$$= \sum_{T\in Std((n-m-1,m))} \left(v^{\deg(T)}\right) + v\left(\sum_{T\in Std((n-m,m-1))} v^{\deg(T)}\right)$$
$$= \operatorname{grdim}\left(S_{(n-m-1,m)}\right) + v\cdot\operatorname{grdim}\left(S_{(n-m,m-1)}\right).$$

2. Suppose $n=2m$, so n always lies in node $(2,m)$ in $(m^2)\in\mathscr{P}_n^1$. Since $\operatorname{res}(2,m)=\operatorname{res}(1,m+1)$, $(1,m+1)$ is an addable $(m-2)$-node strictly above $(2,m)$ for (m^2) and thus $d^{(2,m)}((m^2))=1$. Hence, by the previous part, $\operatorname{grdim}\left(S_{(n-m,m)}\right)=v\cdot\operatorname{grdim}\left(S_{(m,m-1)}\right)$, as required.
3. Suppose n is odd. We have $\operatorname{res}(2,m)=\operatorname{res}(1,n-m)$, and hence $(1,n-m)$ is a removable $(m-2)$-node strictly above $(2,m)$ for λ, that is, $d^{(2,m)}(\lambda)=-1$. Thus, the results follows from the first part.

□

Proposition 2.3. *Let $e=2$ and $n\geqslant 2m$. Then*

$$\operatorname{grdim}\left(S_{(n-m,m)}\right)=\begin{cases}\displaystyle\sum_{i=0}^{m}\left(\frac{n-2m+2-\frac{4i}{n}}{n-2m+2+2i}\binom{\frac{n}{2}}{i}\binom{\frac{n}{2}}{m-i}v^{m-2i}\right) & \text{if } n \text{ is even,}\\ \displaystyle\frac{n+1-2m}{n+1}\sum_{i=0}^{m}\left(\binom{\frac{n+1}{2}}{i}\binom{\frac{n+1}{2}}{m-i}v^{m-2i}\right) & \text{if } n \text{ is odd.}\end{cases}$$

Proof. We proceed by simultaneous induction on n, for n odd and for n even. Firstly, let $n=2a$ for some $a\in\mathbb{N}$, and suppose that

$$\operatorname{grdim}\left(S_{(2a-m-1,m)}\right)=\frac{a-m}{a}\sum_{i=0}^{m}\left(\binom{a}{i}\binom{a}{m-i}v^{m-2i}\right).$$

If $n>2m$, then by Lemma 2.2 we have

$$\begin{aligned}
&\operatorname{grdim}\left(S_{(n-m,m)}\right)\\
&=\operatorname{grdim}\left(S_{(n-1-m,m)}\right)+v\cdot\operatorname{grdim}\left(S_{(n-m,m-1)}\right)\\
&=\operatorname{grdim}\left(S_{(2a-1-m,m)}\right)+v\cdot\operatorname{grdim}\left(S_{(2a-m,m-1)}\right)\\
&=\frac{a-m}{a}\sum_{i=0}^{m}\left(\binom{a}{i}\binom{a}{m-i}v^{m-2i}\right)+v\cdot\frac{a-m+1}{a}\sum_{i=0}^{m-1}\left(\binom{a}{i}\binom{a}{m-i-1}v^{m-2i-1}\right)\\
&=\frac{a-m}{a}\sum_{i=0}^{m}\left(\binom{a}{i}\binom{a}{m-i}v^{m-2i}\right)\\
&\qquad+\frac{a-m+1}{a}\sum_{i=0}^{m}\left(\frac{m-i}{a-m+i+1}\binom{a}{i}\binom{a}{m-i}v^{m-2i}\right)\\
&=\frac{1}{a}\sum_{i=0}^{m}\left(\frac{a(a-m+1)-i}{a-m+i+1}\binom{a}{i}\binom{a}{m-i}v^{m-2i}\right)
\end{aligned}$$

$$= \sum_{i=0}^{m} \left(\frac{n - 2m + 2 - \frac{4i}{n}}{n - 2m + 2 + 2i} \binom{\frac{n}{2}}{i} \binom{\frac{n}{2}}{m-i} v^{m-2i} \right).$$

Similarly, if $n = 2m$, then by Lemma 2.2 we have

$$\begin{aligned} \operatorname{grdim}\left(S_{(m^2)}\right) = v \cdot \operatorname{grdim}\left(S_{(m,m-1)}\right) &= v \cdot \frac{1}{m} \sum_{i=0}^{m-1} \left(\binom{m}{i} \binom{m}{m-i-1} v^{m-2i-1} \right) \\ &= \frac{1}{m} \sum_{i=0}^{m} \left(\frac{m-i}{i+1} \binom{m}{i} \binom{m}{m-i} v^{m-2i} \right). \end{aligned}$$

We now let $n = 2a + 1$ for some $a \in \mathbb{N}$, and suppose that

$$\begin{aligned} \operatorname{grdim}\left(S_{(2a-m,m)}\right) &= \sum_{i=0}^{m} \left(\frac{a^2 - am + a - i}{a(a-m+i+1)} \binom{a}{i} \binom{a}{m-i} v^{m-2i} \right) \\ &= \sum_{i=0}^{m} \left(\frac{a^2 - am + a - m + i}{a(a-i+1)} \binom{a}{i} \binom{a}{m-i} v^{2i-m} \right). \end{aligned}$$

Then, by Lemma 2.2, we have

$$\begin{aligned} &\operatorname{grdim}\left(S_{(n-m,m)}\right) \\ &= \operatorname{grdim}\left(S_{(n-1-m,m)}\right) + v^{-1} \cdot \operatorname{grdim}\left(S_{(n-m,m-1)}\right) \\ &= \operatorname{grdim}\left(S_{(2a-m,m)}\right) + v^{-1} \cdot \operatorname{grdim}\left(S_{(2a+1-m,m-1)}\right) \\ &= \sum_{i=0}^{m} \left(\frac{a(a-m+1) - m + i}{a(a-i+1)} \binom{a}{i} \binom{a}{m-i} v^{2i-m} \right) \\ &\quad + v^{-1} \cdot \sum_{i=0}^{m-1} \left(\frac{a(a-m+2) - m + i + 1}{a(a-i+1)} \binom{a}{i} \binom{a}{m-i-1} v^{2i-m} \right) \\ &= \sum_{i=0}^{m} \left(\frac{a(a-m+1) - m + i}{a(a-i+1)} \binom{a}{i} \binom{a}{m-i} v^{2i-m} \right) \\ &\quad + \sum_{i=0}^{m} \left(\frac{(a(a-m+2) - m + i + 1)(m-i)}{a(a-i+1)(a+1-m+i)} \binom{a}{i} \binom{a}{m-i} v^{2i-m} \right) \\ &= (a-m+1)(a+1) \sum_{i=0}^{m} \left(\frac{1}{(a+1-i)(a+1-m+i)} \binom{a}{i} \binom{a}{m-i} v^{2i-m} \right) \\ &= \frac{a-m+1}{a+1} \sum_{i=0}^{m} \left(\binom{a+1}{i} \binom{a+1}{m-i} v^{2i-m} \right). \\ &= \frac{n+1-2m}{n+1} \sum_{i=0}^{m} \left(\binom{\frac{n+1}{2}}{i} \binom{\frac{n+1}{2}}{m-i} v^{2i-m} \right). \end{aligned}$$

□

It is obvious that we recover the ungraded dimension of $S_{(n-m,m)}$ from Proposition 2.1 when n is odd by setting $v=1$, that is,

$$\frac{n+1-2m}{n+1}\sum_{i=0}^{m}\left(\binom{\frac{n+1}{2}}{i}\binom{\frac{n+1}{2}}{m-i}\right)=\frac{n+1-2m}{n+1}\binom{n+1}{m},$$

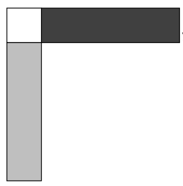
however, the ungraded dimension of $S_{(n-m,m)}$ cannot be directly recovered when n is even.

We observe that the graded dimension of $S_{(n-m,m)}$ with $e=2$, a particularly straightforward graded dimension to determine, is remarkably complicated. Moreover, we note that it seems unlikely to obtain a graded Hook Length Formula by this observation together with others throughout this thesis, and that if one exists, it will most certainly not be as beautifully succinct as Frame, Robinson and Thrall's ungraded version.

2.2 Degree of a standard $(n-m,1^m)$-tableau

In this section we set up preliminary results in order to later determine the graded dimensions of Specht modules labelled by hook partitions and of their composition factors in finite quantum characteristic.

We denote a *hook partition* of n by $(n-m,1^m)$, for $0\leqslant m\leqslant n-1$. We refer to the set of nodes $\{(i,1)\mid 2\leqslant i\leqslant m+1\}$ as the *leg* of $(n-m,1^m)$, and we refer to the set of nodes $\{(1,i)\mid 2\leqslant i\leqslant n-m\}$ as the *arm* of $(n-m,1^m)$. The Young diagram of a hook partition with its leg and arm shaded is

,

where its leg has been shaded lighter than its arm.

For $A\in[(n-m,1^m)]$ such that $\operatorname{res} A=i$, we now determine when there is an addable or removable i-node of $(n-m,1^m)$ lying in the first row of $[\lambda]$, strictly above A.

Lemma 2.4. *Let e be finite and $T\in\operatorname{Std}(n-m,1^m)$ with $T(i,1)=k$, for some i,k, with $2\leqslant i\leqslant m+1$ and $2\leqslant k\leqslant n$. Then $(k-i+1,1^{i-1})$ has neither addable $(1-i)$-node $(1,k-i+2)$ nor removable $(1-i)$-node $(1,k-i+1)$, except in the following cases.*

1. *If $k\equiv 0 \pmod e$, then $(1,k-i+2)$ is an addable $(1-i)$-node of $(k-i+1,1^{i-1})$ strictly above $(i,1)$.*

2. *If $k\equiv 1 \pmod e$ and $T(i,1)>T(1,2)$, then $(1,k-i+1)$ is a removable $(1-i)$-node of $(k-i+1,1^{i-1})$ strictly above $(i,1)$.*

Proof. 1. Suppose that $T(i,1) = \alpha e$ for some $\alpha \geqslant 0$. Then the entries $2, \ldots, \alpha e - 1$ in T lie in the set of nodes $\{(2,1), \ldots, (i-1,1)\} \cup \{(1,2), \ldots, (1, \alpha e - i + 1)\}$. Observe that $\mathrm{res}(i,1) = 1 - i = \alpha e - i + 1 = \mathrm{res}(1, \alpha e - i + 2)$, and since $T(i,1) > T(1, \alpha e - i + 1)$, then $(\alpha e - i + 1, 1^{i-1})$ has addable $(1-i)$-node $(1, \alpha e - i + 2)$.

2. Suppose that $T(i,1) = \alpha e + 1$ for some $\alpha > 0$. Then the entries $2, \ldots, \alpha e$ in T lie in the set of nodes $\{(2,1), \ldots, (i-1,2)\} \cup \{(1,2), \ldots, (1, \alpha e - i + 2)\}$. Observe that $\mathrm{res}(i,1) = 1 - i = \alpha e - i + 1 = \mathrm{res}(1, \alpha e - i + 2)$. Hence, if $T(i,1) > T(1,2)$, then $(1, \alpha e - i + 2)$ is a removable $(1-i)$-node of $(i,1)$. Clearly, if $T(i,1) < T(1,2)$, then the arm of T is empty, and hence $(\alpha e - i + 2, 1^{i-1})$ has no removable $(1-i)$-nodes in the arm of T.

3. Suppose that $T(i,1) = \alpha e + j$, for $j \in \{2, \ldots, e-1\}$, $\alpha \geqslant 0$. Then the entries $2, \ldots, \alpha e + j - 1$ in T lie in the set of nodes $\{(2,1), \ldots, (i-1,2)\} \cup \{(1,2), \ldots, (1, \alpha e + j - i)\}$. We observe that $\mathrm{res}(i,1) = 1 - i$, whereas $\mathrm{res}(1, \alpha e + j - i + 1) = \alpha e + j - i = \mathrm{res}(1, \alpha e + j - i)$. So $\mathrm{res}(i,1) \neq \mathrm{res}(1, \alpha e + j - i), \mathrm{res}(1, \alpha e + j - i + 1)$, and thus $(\alpha e - i + j + 1, 1^{i-1})$ has neither addable $(1-i)$-node $(1, k - i + 2)$ nor removable $(1-i)$-node $(1, k - i + 1)$.

□

We now provide the degree of an arbitrary standard $(n-m, 1^m)$-tableau.

Lemma 2.5. *Let e be finite, $T \in \mathrm{Std}(n-m, 1^m)$ and $2 \leqslant i \leqslant m+1$. Then*

$$\deg(T) = \lfloor \tfrac{m}{e} \rfloor + \#\{i \mid T(i,1) \equiv 0 \pmod{e}\} - \#\{i \mid T(i,1) \equiv 1 \pmod{e}\}.$$

Proof. Let $T(i,1) = k$, for $i \leqslant k \leqslant n$. We note that there are $\lfloor \frac{m}{e} \rfloor$ nodes in the leg of T with residue 0 modulo e and $(k - i + 1, 1^{i-1})$ has addable 0-node if $i \equiv 1 \pmod{e}$ and $k > i$. Then, by Lemma 2.4, we have

$$\begin{aligned}
\deg(T) = \#\{i \mid\ & (k-i+1, 1^{i-1}) \text{ has addable 0-node } (2,2)\} \\
&+ \#\{i \mid (k-i+1, 1^{i-1}) \text{ has addable } (1-i)\text{-node } (1, k-i+2)\} \\
&- \#\{i \mid (k-i+1, 1^{i-1}) \text{ has removable } (1-i)\text{-node } (1, k-i+1)\} \\
=\ & \#\{i \mid i \equiv 1 \pmod{e}\} - \#\{i \mid i \equiv 1 \pmod{e}, k = i\} \\
&+ \#\{i \mid k \equiv 0 \pmod{e}\} \\
&- \#\{i \mid k \equiv 1 \pmod{e}\} + \#\{i \mid k \equiv 1 \pmod{e}, k = i\} \\
=\ & \#\{i \mid i \equiv 1 \pmod{e}\} + \#\{i \mid k \equiv 0 \pmod{e}\} - \#\{i \mid k \equiv 1 \pmod{e}\},
\end{aligned}$$

where $\#\{i \mid i \equiv 1 \pmod{e}\} = \lfloor \frac{m}{e} \rfloor$. □

For any $T \in \mathrm{Std}(n-m, 1^m)$, we define

$$a_T := \#\{i \mid T(i,1) \equiv 0 \pmod{e}\} - \#\{i \mid T(i,1) \equiv 1 \pmod{e}\}.$$

Then, for any non-empty subset $\mathscr{T}$ of $\mathrm{Std}(n-m,1^m)$, we define the set

$$A_{\mathscr{T}} := \{a_T \mid T \in \mathscr{T}\}. \tag{2.2.1}$$

We define the *maximum degree* of $\mathscr{T}$ to be the largest degree of all tableaux in $\mathscr{T}$, written

$$\mathrm{maxdeg}(\mathscr{T}) := \max\{\deg(T) \mid T \in \mathscr{T}\}.$$

Similarly, we define the *minimum degree* of $\mathscr{T}$ to be the smallest degree of all tableaux in $\mathscr{T}$, written

$$\mathrm{mindeg}(\mathscr{T}) := \min\{\deg(T) \mid T \in \mathscr{T}\}.$$

By Lemma 2.5, it follows that

$$\mathrm{maxdeg}(\mathscr{T}) = \lfloor \tfrac{m}{e} \rfloor + \max(A_{\mathscr{T}}), \quad \mathrm{mindeg}(\mathscr{T}) = \lfloor \tfrac{m}{e} \rfloor + \min(A_{\mathscr{T}}). \tag{2.2.2}$$

Chapter 3

Hook representations in quantum characteristic 2

Historically, the *hook representations* of the symmetric group have been some of the easier representations to study, yet they possess rich structure. Recall that a hook representation is a Specht module labelled by a hook partition. Peel [P] studied hook representations for the symmetric group, and subsequently, Murphy [Mu1, Mu2].

We know the decomposable Specht modules for the symmetric group algebra $\mathbb{F}\mathfrak{S}_n$ only occur over a field of characteristic two. In this case, we know that the Specht module S_λ is indecomposable if λ is a 2-regular partition, by [J3, Corollary 13.18]. Murphy studied the decomposable Specht modules labelled by hook partitions for the symmetric group in [Mu1]. Murphy determined that most hook representations are indecomposable, where the only decomposable hook representations occur when n is odd. A more precise result was given in [Mu2, Theorem 3.3] to determine these particular decomposable Specht modules; the cases when $S_{(n-m,1^m)}$ is semisimple occur when $m < 6$. $S_{(n)}$ is clearly semisimple, and for $0 < m < 6$, the sum decompositions of $S_{(n-m,1^m)}$ are as follows:

- ⋄ if n is odd, then $S_{(n-1,1)}$ is semisimple;
- ⋄ if $n \equiv 3 \pmod 4$, then $S_{(n-2,1^2)} \cong S_{(n-2,2)} \oplus S_{(n)}$;
- ⋄ if $n \equiv 1 \pmod 4$, then $S_{(n-3,1^3)} \cong S_{(n-3,3)} \oplus S_{(n-1,1)}$;
- ⋄ if $n \equiv 7 \pmod 8$, then $S_{(n-4,1^4)} \cong S_{(n-4,4)} \oplus S_{(n-2,2)} \oplus S_{(n)}$;
- ⋄ if $n \equiv 1 \pmod 8$, then $S_{(n-5,1^5)} \cong S_{(n-5,5)} \oplus S_{(n-3,3)} \oplus S_{(n-1,1)}$.

Thus, for the most part, Specht modules labelled by hook partitions have no such sum decomposition. However, it can be extracted from [Mu2, §2] that Specht modules labelled by hook partitions over the symmetric group algebra have Specht filtrations whose factors are Specht modules labelled by two part partitions

$$S_{(n-m,m)}, S_{(n-m+2,m-2)}, S_{(n-m+4,m-4)}, \dots,$$

from bottom to top. For $n > 2m$, the partitions $(n-m,m), (n-m+2,m-2), (n-m+4,m-4), \dots$ are 2-regular partitions, and hence the Specht modules labelled by these partitions are indecomposable. Thus, from Murphy's work, $S_{(n-m,1^m)}$ is the first known Specht module filtered by indecomposable Specht modules that confirms Dodge and Fayers' postulation [DF, §8] that every Specht module for the symmetric group has a filtration of indecomposable Specht modules.

The modular representations of the symmetric group over a field of characteristic two are known to closely emulate the representations of the Iwahori–Hecke algebra in quantum characteristic two over the ground field $\mathbb{C}$. Thus, it is natural to ask if we can extend the above filtration arising from G. Murphy's work from the symmetric group to the cyclotomic Khovanov–Lauda–Rouquier algebra; we introduce analagous filtrations of Specht modules labelled by hook partitions for $\mathscr{R}_n^\Lambda$. We set $l = 1$ and $e = 2$ (that is, $q = -1$) throughout this chapter.

3.1 Homogeneous basis elements of $S_{(n-m,1^m)}$

We now define a homogeneous basis of $S_{(n-m,1^m)}$. Given a standard $(n-m,1^m)$-tableau T, we write $a_j := T(j,1)$ for $2 \leqslant j \leqslant m+1$. Then T is completely determined by $a_2, \dots, a_{m+1}$. We can write

$$T = w_T T_{(n-m,1^m)},$$

where

$$w_T = s\overset{a_2-1}{\underset{2}{\downarrow}}\; s\overset{a_3-1}{\underset{3}{\downarrow}} \dots s\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}$$

is a reduced expression. If $a_i = i$ for all $i \in \{2, \dots, m+1\}$, then we set $T = T_{(n-m,1^m)}$. For $j \geqslant i$, we define

$$\Psi\overset{j}{\underset{i}{\downarrow}} = \psi_j \psi_{j-1} \dots \psi_i.$$

Now we can write

$$v_T = \psi_{w_T} z_{(n-m,1^m)}, \tag{3.1.1}$$

where

$$\psi_{w_T} = \Psi\overset{a_2-1}{\underset{2}{\downarrow}}\; \Psi\overset{a_3-1}{\underset{3}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}},$$

and by Theorem 1.22, the v_T form a basis for $S_{(n-m,1^m)}$ as T runs over all standard $(n-m,1^m)$-tableaux. For brevity, we write $v_T = v(a_2, \dots, a_{m+1})$. Thus, if $a_i = i$ for all

$i \in \{2, \ldots, m+1\}$, then $v_T = z_{(n-m,1^m)}$. For $j \in \mathbb{N}$, we write

$$\prod_{i=1}^{j} x_i := x_1 x_2 \ldots x_j,$$

where $x_1, \ldots, x_j$ are elements of $\mathscr{R}_n$ that do not necessarily commute.

3.2 A Specht filtration of $S_{(n-2k,1^{2k})}$

We concern ourselves with the hook representations $S_{(n-2k,1^{2k})}$; we let $\lambda = (n-2k, 1^{2k})$ until the end of Chapter 3.

Definition 3.1. *For $1 \leqslant j \leqslant k+1$, we define the standard λ-tableau*

$$A_j := \prod_{i=3}^{2k-2j+3} \left(s \overset{2i-3}{\underset{i}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(s \overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}} \right) T_\lambda.$$

We note that the entries in A_j that correspond to the left hand product in the definition of A_j are even, whereas the entries corresponding to the right hand product are consecutive entries. A_j is the standard λ-tableau with even entries $2, 4, 6, \ldots, 4k-4j+6$ lying in the first $2k-2j+2$ nodes in its leg, and with consecutive entries $4k-4j+7, 4k-4j+8, \ldots, 4k-2j+3$ lying in the remaining $2j-2$ nodes in its leg. If we set $l = 4k-4j$, then A_j is the λ-tableau

1	3	5	········	l+5	l+2j+4	········	n
2							
4							
⋮							
l+4							
l+6							
l+7							
⋮							
l+2j+3							

,

where we have shaded the consecutive entries lying in A_j. In particular, we see that A_{k+1} is the column-initial λ-tableau T_λ.

We now find the basis vector v_{A_j} of S_λ corresponding to A_j. For $1 \leqslant j \leqslant k+1$, we write

$$v_{A_j} = \prod_{i=3}^{2k-2j+3} \left(\Psi \overset{2i-3}{\underset{i}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(\Psi \overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}} \right) z_\lambda, \tag{3.2.1}$$

by setting $T = A_j$ in Equation (3.1.1).

We know that v_{A_j} is completely determined by the entries lying in the leg of A_j. Thus, by observing the standard λ-tableau A_j above, we can write the basis vector v_{A_j}

as

$$v(2,4,\ldots,4k-4j+4,4k-4j+6,4k-4j+7,\ldots,4k-2j+2,4k-2j+3).$$

In particular, $v_{A_{k+1}} = v(2,3,\ldots,2k+1)$, which is the standard generator z_λ of S_λ. We let M_j be the $\mathscr{R}_n^\Lambda$-module generated by v_{A_j}. We aim to show that S_λ has an ascending filtration $0 \subset M_1 \subset \cdots \subset M_{k+1} = S_\lambda$.

To prove the following lemmas, we require the presentation of Specht modules labelled by hook partitions from Definition 1.19.

Remark 3.2. *S_λ is the $\mathscr{R}_n^\Lambda$-module generated by z_λ, subject only to the relations*

- $\diamond$ $e(\mathbf{i})z_\lambda = \delta_{\mathbf{i},\mathbf{i}_\lambda} z_\lambda$,
- $\diamond$ $y_i z_\lambda = 0$ *for* $i \in \{1,\ldots,n-1\}$,
- $\diamond$ $\psi_i z_\lambda = 0$ *for* $i \in \{1,\ldots,2k\}$,

together with the Garnir relations (by Section 1.3.1.2)

- $\diamond$ $\psi_1\psi_2\ldots\psi_{2k+1}z_\lambda = 0$,
- $\diamond$ $\psi_i z_\lambda = 0$ *for* $i \in \{2k+2,2k+3,\ldots,n-1\}$.

The following lemmas describe how the generators $y_1,\ldots,y_n,\psi_1,\ldots,\psi_{n-1}$ of the Khovanov–Lauda–Rouquier algebras act on a basis vector $v(a_2,\ldots,a_{2k+1})$ of S_λ.

Lemma 3.3. *Let $e = 2$ and suppose i,j are such that $a_{j-1} \geqslant j$ and $3 \leqslant j \leqslant i-1$.*

1. *If both i and $i+1$ lie in $\{a_2,\ldots,a_{m+1}\}$, then $\psi_i v(a_2,\ldots,a_{m+1}) = 0$.*
2. *If i lies in $\{a_2,\ldots,a_{m+1}\}$, then $y_i v(a_2,\ldots,a_{m+1}) = 0$.*
3. *If $i+1$ lies in $\{a_2,\ldots,a_{m+1}\}$, but i does not, then $y_i v(a_2,\ldots,a_{m+1}) = 0$.*

Proof. We prove the three statements simultaneously, by induction on i. We name statement 1, 2 and 3, respectively, A_i, B_i and C_i.

1. We prove that A_i holds by using the inductive statements A_{i-1}, B_i, C_{i-1} and C_{i-2}. Let $a_j = i$ and $a_{j+1} = i+1$. If $i = j$ then all the terms $\Psi\underset{2}{\overset{a_2-1}{\downarrow}} \ldots \Psi\underset{j}{\overset{i-1}{\downarrow}}\ \Psi\underset{j+1}{\overset{i}{\downarrow}}$ are trivial, so

$$\psi_i v(a_2,\ldots,a_{m+1}) = \psi_i \Psi\underset{j+2}{\overset{a_{j+2}-1}{\downarrow}} \ldots \Psi\underset{m+1}{\overset{a_{m+1}-1}{\downarrow}} z_\lambda = \Psi\underset{j+2}{\overset{a_{j+2}-1}{\downarrow}} \ldots \Psi\underset{m+1}{\overset{a_{m+1}-1}{\downarrow}} \psi_i z_\lambda = 0.$$

We suppose that $i \geqslant j+1$.

(i) Let's first consider when i is odd. We have

$$\begin{aligned}
&\psi_i v(a_2, a_3, \dots, a_{j-1}, i, i+1, a_{j+2}, a_{j+3}, \dots, a_{m+1}) \\
&= \psi_i \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi \overset{i-1}{\underset{j}{\downarrow}} \Psi \overset{i}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda \\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} (\psi_i \psi_{i-1} \psi_i) \Psi \overset{i-2}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda.
\end{aligned}$$

We see that

$$s \overset{i-2}{\underset{j}{\downarrow}} s \overset{i-1}{\underset{j+1}{\downarrow}} s \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots s \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} T_\lambda,$$

which we shall call T, is the corresponding λ-tableau to the element

$$\Psi \overset{i-2}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda \in \mathscr{R}_n^\Lambda.$$

One observes that $T(j,1) = i-1$, $T(j+1,1) = i$ and $T(1, i-j+1) = i+1$. If j is odd (resp., even), then $\operatorname{res}(j,1) = 1-j$ is even (resp., odd), $\operatorname{res}(j+1,1) = -j$ is odd (resp., even), and $\operatorname{res}(1, i-j+1) = i-j$ is even (resp., odd) since i is odd. We let e_T be the idempotent with respect of T, where, if j is odd (resp., even), its $(i-1)$th and $(i+1)$th entries are odd (resp., even) and its ith entry is even (resp., odd). We thus have

$$\begin{aligned}
&\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} (\psi_i \psi_{i-1} \psi_i) \Psi \overset{i-2}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} e(\mathbf{i}_\lambda) z_\lambda \\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} (\psi_i \psi_{i-1} \psi_i) e_T \Psi \overset{i-2}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda \\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} (\psi_{i-1} \psi_i \psi_{i-1} + 2y_{i+1} - y_i - y_{i+2}) e_T \Psi \overset{i-2}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \\
&\qquad \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda,
\end{aligned}$$

by Definition 1.6 Equation (1.1.11).

The first term of this expression becomes

$$\begin{aligned}
&\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \psi_{i-1} \psi_i \psi_{i-1} \Psi \overset{i-2}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda \\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \psi_{i-1} \psi_i \psi_{i-1} v(2, \dots, j-1, i-1, i, a_{j+2}, \dots, a_{m+1})
\end{aligned}$$

Now $\psi_{i-1} v(2, \dots, j-1, i-1, i, a_{j+2}, \dots, a_{m+1})$ is zero by A_{i-1}.

The second term of the expression becomes

$$2\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} y_{i+1} \Psi \overset{i-2}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \Psi \overset{a_{j+3}-1}{\underset{j+3}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda$$

$$= 2\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\Psi\overset{i-2}{\underset{j}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{i+2}{\downarrow}}\,y_{i+1}\Psi\overset{i-1}{\underset{j+1}{\downarrow}}\Psi\overset{i+1}{\underset{j+2}{\downarrow}}\Psi\overset{a_{j+3}-1}{\underset{j+3}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda$$
$$= 2\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\Psi\overset{i-2}{\underset{j}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{i+2}{\downarrow}}\,y_{i+1}v(2,\dots,j,i,i+2,a_{j+3},\dots,a_{m+1})$$

where $y_{i+1}v(2,\dots,j,i,i+2,a_{j+3},\dots,a_{m+1})$ equals zero by C_{i-1}.

The third term of the expression becomes

$$-\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\Psi\overset{i-2}{\underset{j}{\downarrow}}\,y_i\Psi\overset{i-1}{\underset{j+1}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda$$
$$= -\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\Psi\overset{i-2}{\underset{j}{\downarrow}}\,y_i v(2,\dots,j,i,a_{j+2},\dots,a_{m+1})$$

where $y_i v(2,\dots,j,i,a_{j+2},\dots,a_{m+1})$ equals zero by B_i.

Finally, the fourth term of the expression becomes

$$-\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\,y_{i+2}\Psi\overset{i-2}{\underset{j}{\downarrow}}\Psi\overset{i-1}{\underset{j+1}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda$$
$$= -\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\Psi\overset{i-2}{\underset{j}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{i+3}{\downarrow}}\,y_{i+2}\Psi\overset{i-1}{\underset{j+1}{\downarrow}}\Psi\overset{i+2}{\underset{j+2}{\downarrow}}\Psi\overset{a_{j+3}-1}{\underset{j+3}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda$$
$$= -\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\Psi\overset{i-2}{\underset{j}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{i+3}{\downarrow}}\,y_{i+2}v(2,\dots,j,i,i+3,a_{j+3},\dots,a_{m+1})$$

where $y_{i+2}v(2,\dots,j,i,i+3,a_{j+3},\dots,a_{m+1})$ equals zero by C_{i-2}.

(ii) Now suppose that i is even. We have

$$\psi_i v(a_2,a_3,\dots,a_{j-1},i,i+1,a_{j+2},a_{j+3},\dots,a_{m+1})$$
$$= \psi_i\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\Psi\overset{i-1}{\underset{j}{\downarrow}}\Psi\overset{i}{\underset{j+1}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda$$
$$= \Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\,\psi_i\psi_{i-1}\psi_i\Psi\overset{i-2}{\underset{j}{\downarrow}}\Psi\overset{i-1}{\underset{j+1}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda$$
$$= \Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\,\psi_{i-1}\psi_i\psi_{i-1}\Psi\overset{i-2}{\underset{j}{\downarrow}}\Psi\overset{i-1}{\underset{j+1}{\downarrow}}\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda$$
$$= \Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}}\,\psi_{i-1}\psi_i\psi_{i-1}v(2,\dots,j-1,i-1,i,a_{j+2},\dots,a_{m+1})$$

where $\psi_{i-1}v(2,\dots,j-1,i-1,i,a_{j+2},\dots,a_{m+1})$ equals zero by A_{i-1}.

2. We prove that B_i holds by using the inductive statements A_{i-2} and B_{i-1}. Let $a_j = i$. If $i = j$ then all the terms $\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{i-1}{\underset{j}{\downarrow}}$ are trivial, so

$$y_i v(a_2,\dots,a_{m+1}) = y_i\Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda = \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}y_i z_\lambda = 0.$$

We suppose that $i \geqslant j+1$.

(i) Suppose i is odd. We have

$$\begin{aligned}
&y_i v(a_2,\ldots,a_{j-1},i,a_{j+1},\ldots,a_{m+1})\\
&= y_i \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; \Psi\overset{i-1}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; y_i\psi_{i-1}\Psi\overset{i-2}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; (\psi_{i-1}y_{i-1}+1)\Psi\overset{i-2}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda.
\end{aligned}$$

The first term of this expression becomes

$$\begin{aligned}
&\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; \psi_{i-1}y_{i-1}\Psi\overset{i-2}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; \psi_{i-1}y_{i-1}v(2,\ldots,j-1,i-1,a_{j+1},\ldots,a_{m+1})
\end{aligned}$$

where $y_{i-1}v(2,\ldots,j-1,i-1,a_{j+1},\ldots,a_{m+1})$ equals zero by B_{i-1}.

Now the second term of the expression becomes

$$\begin{aligned}
&\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \; \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; \Psi\overset{i-2}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \; \Psi\overset{a_{j-1}-1}{\underset{i-1}{\downarrow}} \; \psi_{i-2}\Psi\overset{i-3}{\underset{j-1}{\downarrow}} \; \Psi\overset{i-2}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \; \Psi\overset{a_{j-1}-1}{\underset{i-1}{\downarrow}} \; \psi_{i-2}v(2,\ldots,j-2,i-2,i-1,a_{j+1},\ldots,a_{m+1})
\end{aligned}$$

where $\psi_{i-2}v(2,\ldots,j-2,i-2,i-1,a_{j+1},\ldots,a_{m+1})$ equals zero by A_{i-2}.

(ii) Suppose i is even.

$$\begin{aligned}
&y_i v(a_2,\ldots,a_{j-1},i,a_{j+1},\ldots,a_{m+1})\\
&= y_i \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; \Psi\overset{i-1}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; y_i\psi_{i-1}\Psi\overset{i-2}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; \psi_{i-1}y_{i-1}\Psi\overset{i-2}{\underset{j}{\downarrow}} \; \Psi\overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \ldots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \; z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \; \psi_{i-1}y_{i-1}v(2,\ldots,j-1,i-1,a_{j+1},\ldots,a_{m+1})
\end{aligned}$$

where $y_{i-1}v(2,\ldots,j-1,i-1,a_{j+1},\ldots,a_{m+1})$ equals zero by B_{i-1}.

3. We prove that C_i holds by using the inductive statements A_{i-1} and C_{i-1}. Suppose that $a_{j+1}=i+1$ and $a_j \leqslant i-1$. If $i=j$ then all the terms $\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \ldots \Psi\overset{a_j-1}{\underset{j}{\downarrow}} \; \Psi\overset{i}{\underset{j+1}{\downarrow}}$

are trivial, so

$$y_i v(a_2,\dots,a_{m+1}) = y_i\Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda = \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} y_i z_\lambda = 0.$$

(i) Let i be odd. We have

$$\begin{aligned}
&y_i v(a_2,\dots,a_j,i+1,a_{j+2},\dots,a_{m+1})\\
&= y_i\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} \Psi \overset{i}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+1}-1}{\underset{j+1}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} y_i\psi_i\Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} \psi_i y_{i+1}\Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} \psi_i\Psi \overset{a_{j+2}-1}{\underset{i+2}{\downarrow}} y_{i+1}\Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{i+1}{\underset{j+2}{\downarrow}} \Psi \overset{a_{j+3}-1}{\underset{j+3}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} \psi_i\Psi \overset{a_{j+2}-1}{\underset{i+2}{\downarrow}} y_{i+1} v(2,\dots,j,i,i+2,a_{j+3},\dots,a_{m+1})
\end{aligned}$$

where $y_{i+1}v(2,\dots,j,i,i+2,a_{j+3},\dots,a_{m+1})$ equals zero by C_{i-1}.

(ii) Let i be even.

$$\begin{aligned}
&y_i v(a_2,\dots,a_j,i+1,a_{j+2},\dots,a_{m+1})\\
&= y_i\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} \Psi \overset{i}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} y_i\psi_i\Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} (\psi_i y_{i+1}-1)\Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda.
\end{aligned}$$

Now the first term of this expression becomes

$$\begin{aligned}
&\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} \psi_i\Psi \overset{a_{j+2}-1}{\underset{i+2}{\downarrow}} y_{i+1}\Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{i+1}{\underset{j+2}{\downarrow}} \Psi \overset{a_{j+3}-1}{\underset{j+3}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_j-1}{\underset{j}{\downarrow}} \psi_i\Psi \overset{a_{j+2}-1}{\underset{i+2}{\downarrow}} y_{i+1} v(2,\dots,j,i,i+2,a_{j+3},\dots,a_{m+1})
\end{aligned}$$

where $y_{i+1}v(2,\dots,j,i,i+2,a_{j+3},\dots,a_{m+1})$ equals zero by C_{i-1}.

Now the second term becomes

$$\begin{aligned}
&-\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi \overset{a_j-1}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= -\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi \overset{a_j-1}{\underset{i}{\downarrow}} \psi_{i-1}\Psi \overset{i-2}{\underset{j}{\downarrow}} \Psi \overset{i-1}{\underset{j+1}{\downarrow}} \Psi \overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda
\end{aligned}$$

$$= -\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{a_j-1}{\underset{i}{\downarrow}} \psi_{i-1}v(2,\dots,j-1,i-1,i,a_{j+2},\dots,a_{m+1})$$

where $\psi_{i-1}v(2,\dots,j-1,i-1,i,a_{j+2},\dots,a_{m+1})$ equals zero by A_{i-1}.

□

Lemma 3.4. *Let $e=2$. If both $i-1$ and $i+2$ lie in $\{a_2,\dots,a_{m+1}\}$, but neither i nor $i+1$ lie in $\{a_2,\dots,a_{m+1}\}$, then $\psi_i v(a_2,\dots,a_{m+1}) = 0$.*

Proof. Suppose that $a_j = i-1$ and $a_{j+1} = i+2$. Then

$$\psi_i v(a_2,\dots,a_{m+1}) = \psi_i\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} \Psi\overset{i+1}{\underset{j+1}{\downarrow}} \Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda.$$

We proceed by induction on the sum $a_2 + a_3 + \dots + a_{m+1}$.

If $i = j-1$ then the terms $\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} \Psi\overset{i+1}{\underset{j+1}{\downarrow}}$ are trivial, so

$$\psi_i v(a_2,\dots,a_{m+1}) = \psi_i\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda = \Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} \psi_i z_\lambda = 0.$$

We suppose that $i \geqslant j$.

(i) Let's first consider when i is odd. We have

$$\begin{aligned}
&\psi_i v(a_2,\dots,a_{m+1})\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} \psi_i\psi_{i+1}\psi_i\Psi\overset{i-1}{\underset{j+1}{\downarrow}} \Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} (\psi_{i+1}\psi_i\psi_{i+1} - 2y_{i+1} + y_i + y_{i+2})\Psi\overset{i-1}{\underset{j+1}{\downarrow}} \Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda.
\end{aligned}$$

The first term of this expression becomes

$$\begin{aligned}
&\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} \psi_{i+1}\psi_i\psi_{i+1}\Psi\overset{i-1}{\underset{j+1}{\downarrow}} \Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} \psi_{i+1}\psi_i\Psi\overset{a_{j+2}-1}{\underset{i+3}{\downarrow}} \psi_{i+1}\Psi\overset{i-1}{\underset{j+1}{\downarrow}} \Psi\overset{i+2}{\underset{j+2}{\downarrow}} \Psi\overset{a_{j+3}-1}{\underset{j+3}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} \psi_{i+1}\psi_i\Psi\overset{a_{j+2}-1}{\underset{i+3}{\downarrow}} \psi_{i+1}v(2,3,\dots,j,i,i+3,a_{j+3},\dots,a_{m+1})
\end{aligned}$$

where $\psi_{i+1}v(2,3,\dots,j,i,i+3,a_{j+3},\dots,a_{m+1})$ equals zero by the inductive hypothesis. Now the second term of the expression becomes

$$\begin{aligned}
&-2\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} y_{i+1}\Psi\overset{i-1}{\underset{j+1}{\downarrow}} \Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda\\
&= -2\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-1}-1}{\underset{j-1}{\downarrow}} \Psi\overset{i-2}{\underset{j}{\downarrow}} \Psi\overset{a_{j+2}-1}{\underset{i+2}{\downarrow}} y_{i+1}\Psi\overset{i-1}{\underset{j+1}{\downarrow}} \Psi\overset{i+1}{\underset{j+2}{\downarrow}} \Psi\overset{a_{j+3}-1}{\underset{j+3}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda
\end{aligned}$$

$$= -2\Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ \Psi \underset{i+2}{\overset{a_{j+2}-1}{\downarrow}}\ y_{i+1} v(2,3,\dots,j,i,i+2,a_{j+3},\dots,a_{m+1})$$

where $y_{i+1}v(2,3,\dots,j,i,i{+}2,a_{j+3},\dots,a_{m+1})$ equals zero by part three of Lemma 3.3. The third term of the expression becomes

$$\begin{aligned}
&\Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ y_i \Psi \underset{j+1}{\overset{i-1}{\downarrow}}\ \Psi \underset{j+2}{\overset{a_{j+2}-1}{\downarrow}} \dots \Psi \underset{m+1}{\overset{a_{m+1}-1}{\downarrow}}\ z_\lambda \\
&= \Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ y_i v(2,3,\dots,j,i,a_{j+2},\dots,a_{m+1})
\end{aligned}$$

where $y_iv(2,3,\dots,j,i,a_{j+2},\dots,a_{m+1})$ equals zero by part two of Lemma 3.3. The last term of the expression becomes

$$\begin{aligned}
&\Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ y_{i+2} \Psi \underset{j+1}{\overset{i-1}{\downarrow}}\ \Psi \underset{j+2}{\overset{a_{j+2}-1}{\downarrow}} \dots \Psi \underset{m+1}{\overset{a_{m+1}-1}{\downarrow}}\ z_\lambda \\
&= \Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ \Psi \underset{i+3}{\overset{a_{j+2}-1}{\downarrow}}\ y_{i+2} \Psi \underset{j+1}{\overset{i-1}{\downarrow}}\ \Psi \underset{j+2}{\overset{i+2}{\downarrow}}\ \Psi \underset{j+3}{\overset{a_{j+3}-1}{\downarrow}} \dots \Psi \underset{m+1}{\overset{a_{m+1}-1}{\downarrow}}\ z_\lambda \\
&= \Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ \Psi \underset{i+3}{\overset{a_{j+2}-1}{\downarrow}}\ y_{i+2} v(2,3,\dots,j,i,i+3,a_{j+3},\dots,a_{m+1})
\end{aligned}$$

where $y_{i+2}v(2,3,\dots,j,i,i{+}3,a_{j+3},\dots,a_{m+1})$ equals zero by part three of Lemma 3.3.

(ii) Now suppose that i is even. We have

$$\begin{aligned}
&\psi_i v(a_2,\dots,a_{m+1}) \\
&= \Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ \psi_i\psi_{i+1}\psi_i \Psi \underset{j+1}{\overset{i-1}{\downarrow}}\ \Psi \underset{j+2}{\overset{a_{j+2}-1}{\downarrow}} \dots \Psi \underset{m+1}{\overset{a_{m+1}-1}{\downarrow}}\ z_\lambda \\
&= \Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ \psi_{i+1}\psi_i\psi_{i+1} \Psi \underset{j+1}{\overset{i-1}{\downarrow}}\ \Psi \underset{j+2}{\overset{a_{j+2}-1}{\downarrow}} \dots \Psi \underset{m+1}{\overset{a_{m+1}-1}{\downarrow}}\ z_\lambda \\
&= \Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ \psi_{i+1}\psi_i \Psi \underset{i+3}{\overset{a_{j+2}-1}{\downarrow}}\ \psi_{i+1} \Psi \underset{j+1}{\overset{i-1}{\downarrow}}\ \Psi \underset{j+2}{\overset{i+2}{\downarrow}}\ \Psi \underset{j+3}{\overset{a_{j+3}-1}{\downarrow}} \dots \Psi \underset{m+1}{\overset{a_{m+1}-1}{\downarrow}}\ z_\lambda \\
&= \Psi \underset{2}{\overset{a_2-1}{\downarrow}} \dots \Psi \underset{j-1}{\overset{a_{j-1}-1}{\downarrow}}\ \Psi \underset{j}{\overset{i-2}{\downarrow}}\ \psi_{i+1}\psi_i \Psi \underset{i+3}{\overset{a_{j+2}-1}{\downarrow}}\ \psi_{i+1} v(2,3,\dots,j,i,i+3,a_{j+3},\dots,a_{m+1})
\end{aligned}$$

where $\psi_{i+1}v(2,3,\dots,j,i,i+3,a_{j+3},\dots,a_{m+1})$ equals zero by the inductive hypothesis.

□

Lemma 3.5. *Let $e = 2$ and i,k be such that i is odd and $k+1 \leqslant i \leqslant a_m - 1$. Then*

$$\begin{aligned}
&\Psi \underset{k}{\overset{i}{\downarrow}} v(a_2,\dots,a_{j-2},k,k+2,k+3,k+4,\dots,i+2,a_{j+i-k+1},a_{j+i-k+2},\dots,a_{m+1}) \\
&= v(a_2,\dots,a_{j-2},k,k+1,k+2,k+3,\dots,i+1,a_{j+i-k+1},a_{j+i-k+2},\dots,a_{m+1}).
\end{aligned}$$

Proof. We proceed by induction on i. If $i = k+1$, then

$$\begin{aligned}
&\Psi\overset{k+1}{\underset{k}{\downarrow}}\, v(a_2,\dots,a_{j-2},k,k+2,k+3,a_{j+2},\dots,a_{m+1})\\
&= \Psi\overset{k+1}{\underset{k}{\downarrow}}\,\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k+1}{\underset{j}{\downarrow}}\,\Psi\overset{k+2}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda\\
&= \psi_{k+1}\psi_k\psi_{k+1}\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,\Psi\overset{k+2}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda\\
&= (\psi_k\psi_{k+1}\psi_k + 2y_{k+1} - y_k - y_{k+2})\,\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,\Psi\overset{k+2}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda.
\end{aligned}$$

The first term becomes

$$\begin{aligned}
&\psi_k\psi_{k+1}\psi_k\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,\Psi\overset{k+2}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda\\
&= \psi_k\psi_{k+1}\psi_k v(a_2,\dots,a_{j-2},k,k+1,k+3,a_{j+2},\dots,a_{m+1})
\end{aligned}$$

where $\psi_k v(a_2,\dots,a_{j-2},k,k{+}1,k{+}3,a_{j+2},\dots,a_{m+1})$ equals zero by part one of Lemma 3.3. The second term becomes

$$\begin{aligned}
&2y_{k+1}\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,\Psi\overset{k+2}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda\\
&= 2y_{k+1}v(a_2,\dots,a_{j-2},k,k+1,k+3,a_{j+2},\dots,a_{m+1}),
\end{aligned}$$

which equals zero by part two of Lemma 3.3. The third term becomes

$$\begin{aligned}
&-y_k\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,\Psi\overset{k+2}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda\\
&= -y_k v(a_2,\dots,a_{j-2},k,k+1,k+3,a_{j+2},\dots,a_{m+1}),
\end{aligned}$$

which equals zero by part two of Lemma 3.3. The last term becomes

$$\begin{aligned}
&-y_{k+2}\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,\Psi\overset{k+2}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda\\
&= -\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\, y_{k+2}\psi_{k+2}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda\\
&= -\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,(\psi_{k+2}y_{k+3}-1)\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda.
\end{aligned}$$

Now, its first term becomes

$$\begin{aligned}
&-\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,\psi_{k+2}y_{k+3}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\,\Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}}\,\Psi\overset{a_{j+3}-1}{\underset{j+3}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda\\
&= -\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\,\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\,\Psi\overset{k}{\underset{j}{\downarrow}}\,\psi_{k+2}\Psi\overset{a_{j+2}-1}{\underset{k+4}{\downarrow}}\, y_{k+3}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\,\Psi\overset{k+3}{\underset{j+2}{\downarrow}}\,\Psi\overset{a_{j+3}-1}{\underset{j+3}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}\, z_\lambda
\end{aligned}$$

$$= -\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi\overset{k-1}{\underset{j-1}{\downarrow}} \Psi\overset{k}{\underset{j}{\downarrow}} \psi_{k+2}y_{k+3}v(2,\dots,j,k+2,k+4,a_{j+3},\dots,a_{m+1}),$$

where $y_{k+3}v(2,\dots,j,k+2,k+4,a_{j+3},\dots,a_{m+1})$ equals zero by part three of Lemma 3.3. Its second term is

$$\begin{aligned}&\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi\overset{k-1}{\underset{j-1}{\downarrow}} \Psi\overset{k}{\underset{j}{\downarrow}} \Psi\overset{k+1}{\underset{j+1}{\downarrow}} \Psi\overset{a_{j+2}-1}{\underset{j+2}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda \\ &= v(a_2,\dots,a_{j_2},k,k+1,k+2,a_{j+2},\dots,a_{m+1}),\end{aligned}$$

which is non-zero, as required.

Now let $i > k+1$. By induction, we have

$$\begin{aligned}
&\Psi\overset{i+2}{\underset{k}{\downarrow}} v(a_2,\dots,a_{j-2},k,k+2,k+3,k+4,\dots \\
&\qquad \dots,i+2,i+3,i+4,a_{j+i-k+3},a_{j+i-k+4},\dots,a_{m+1}) \\
&= \Psi\overset{i+2}{\underset{i+1}{\downarrow}} v(a_2,\dots,a_{j-2},k,k+1,k+2,k+3,\dots \\
&\qquad \dots,i+1,i+3,i+4,a_{j+i-k+3},a_{j+i-k+4},\dots,a_{m+1}) \\
&= \Psi\overset{i+2}{\underset{i+1}{\downarrow}} \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi\overset{k-1}{\underset{j-1}{\downarrow}} \Psi\overset{k}{\underset{j}{\downarrow}} \Psi\overset{k+1}{\underset{j+1}{\downarrow}} \Psi\overset{k+2}{\underset{j+2}{\downarrow}} \dots \\
&\qquad \dots \Psi\overset{i}{\underset{j+i-k}{\downarrow}} \Psi\overset{i+2}{\underset{j+i-k+1}{\downarrow}} \Psi\overset{i+3}{\underset{j+i-k+2}{\downarrow}} \Psi\overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda \\
&= \psi_{i+2}\psi_{i+1}\psi_{i+2}\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi\overset{k-1}{\underset{j-1}{\downarrow}} \Psi\overset{k}{\underset{j}{\downarrow}} \Psi\overset{k+1}{\underset{j+1}{\downarrow}} \Psi\overset{k+2}{\underset{j+2}{\downarrow}} \dots \\
&\qquad \dots \Psi\overset{i}{\underset{j+i-k}{\downarrow}} \Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}} \Psi\overset{i+3}{\underset{j+i-k+2}{\downarrow}} \Psi\overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda \\
&= (\psi_{i+1}\psi_{i+2}\psi_{i+1} + 2y_{i+2} - y_{i+1} - y_{i+3})\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi\overset{k-1}{\underset{j-1}{\downarrow}} \Psi\overset{k}{\underset{j}{\downarrow}} \Psi\overset{k+1}{\underset{j+1}{\downarrow}} \Psi\overset{k+2}{\underset{j+2}{\downarrow}} \dots \\
&\qquad \dots \Psi\overset{i}{\underset{j+i-k}{\downarrow}} \Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}} \Psi\overset{i+3}{\underset{j+i-k+2}{\downarrow}} \Psi\overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda.
\end{aligned}$$

The first term of this expression becomes

$$\begin{aligned}
&\psi_{i+1}\psi_{i+2}\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi\overset{k-1}{\underset{j-1}{\downarrow}} \Psi\overset{k}{\underset{j}{\downarrow}} \Psi\overset{k+1}{\underset{j+1}{\downarrow}} \Psi\overset{k+2}{\underset{j+2}{\downarrow}} \dots \\
&\qquad \dots \Psi\overset{i-1}{\underset{j+i-k-1}{\downarrow}} \psi_{i+1}\Psi\overset{i}{\underset{j+i-k}{\downarrow}} \Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}} \Psi\overset{i+3}{\underset{j+i-k+2}{\downarrow}} \Psi\overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}} \dots \Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda \\
&= \psi_{i+1}\psi_{i+2}\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi\overset{k-1}{\underset{j-1}{\downarrow}} \Psi\overset{k}{\underset{j}{\downarrow}} \Psi\overset{k+1}{\underset{j+1}{\downarrow}} \Psi\overset{k+2}{\underset{j+2}{\downarrow}} \dots \Psi\overset{i-1}{\underset{j+i-k-1}{\downarrow}} \psi_{i+3}\Psi\overset{a_{j+i-k+3}-1}{\underset{i+4}{\downarrow}} \Psi\overset{a_{j+i-k+4}-1}{\underset{i+5}{\downarrow}} \dots \\
&\qquad \dots \Psi\overset{a_{m+1}-1}{\underset{m-j+k-i+1}{\downarrow}} \psi_{i+1}\Psi\overset{i}{\underset{j+i-k}{\downarrow}} \Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}} \Psi\overset{i+2}{\underset{j+i-k+2}{\downarrow}} \Psi\overset{i+3}{\underset{j+i-k+3}{\downarrow}} \dots \Psi\overset{m-j+k-i}{\underset{m+1}{\downarrow}} z_\lambda \\
&= \psi_{i+1}\psi_{i+2}\Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi\overset{k-1}{\underset{j-1}{\downarrow}} \Psi\overset{k}{\underset{j}{\downarrow}} \Psi\overset{k+1}{\underset{j+1}{\downarrow}} \Psi\overset{k+2}{\underset{j+2}{\downarrow}} \dots \Psi\overset{i-1}{\underset{j+i-k-1}{\downarrow}} \psi_{i+3}\Psi\overset{a_{j+i-k+3}-1}{\underset{i+4}{\downarrow}} \Psi\overset{a_{j+i-k+4}-1}{\underset{i+5}{\downarrow}} \dots
\end{aligned}$$

$$\dots\Psi\overset{a_{m+1}-1}{\underset{m-j+k-i+1}{\downarrow}}\psi_{i+1}v(2,\dots,j+i-k,i+1,i+2,i+3,\dots,m-j+k-i+1)$$

where $\psi_{i+1}v(2,\dots,j+i-k,i+1,i+2,i+3,\dots,m-j+k-i+1)$ equals zero by part one of Lemma 3.3.

The second term of this expression becomes

$$\begin{aligned}
&2\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\Psi\overset{k}{\underset{j}{\downarrow}}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\Psi\overset{k+2}{\underset{j+2}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{i}{\underset{j+i-k}{\downarrow}}y_{i+2}\Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}}\Psi\overset{i+3}{\underset{j+i-k+2}{\downarrow}}\Psi\overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}}\Psi\overset{a_{j+i-k+4}-1}{\underset{j+i-k+4}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda\\
&=2\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\Psi\overset{k}{\underset{j}{\downarrow}}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\Psi\overset{k+2}{\underset{j+2}{\downarrow}}\dots\Psi\overset{i-1}{\underset{j+i-k-1}{\downarrow}}\psi_{i+3}\Psi\overset{a_{j+i-k+3}-1}{\underset{i+4}{\downarrow}}\Psi\overset{a_{j+i-k+4}-1}{\underset{i+5}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_{m+1}-1}{\underset{m-j+k+1}{\downarrow}}y_{i+2}\Psi\overset{i}{\underset{j+i-k}{\downarrow}}\Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}}\Psi\overset{i+2}{\underset{j+i-k+2}{\downarrow}}\Psi\overset{i+3}{\underset{j+i-k+3}{\downarrow}}\dots\Psi\overset{m-j+k}{\underset{m+1}{\downarrow}}z_\lambda\\
&=2\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\Psi\overset{k}{\underset{j}{\downarrow}}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\Psi\overset{k+2}{\underset{j+2}{\downarrow}}\dots\Psi\overset{i-1}{\underset{j+i-k-1}{\downarrow}}\psi_{i+3}\Psi\overset{a_{j+i-k+3}-1}{\underset{i+4}{\downarrow}}\Psi\overset{a_{j+i-k+4}-1}{\underset{i+5}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_{m+1}-1}{\underset{m-j+k+1}{\downarrow}}y_{i+2}v(2,\dots,j+i-k-1,i+1,i+2,i+3,i+4,\dots,m-j+k+1)
\end{aligned}$$

where $y_{i+2}v(2,\dots,j+i-k-1,i+1,i+2,i+3,i+4,\dots,m-j+k+1)$ equals zero by part two of Lemma 3.3.

The third term of this expression becomes

$$\begin{aligned}
&-\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\Psi\overset{k}{\underset{j}{\downarrow}}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\Psi\overset{k+2}{\underset{j+2}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{i-2}{\underset{j+i-k-2}{\downarrow}}y_{i+1}\Psi\overset{i-1}{\underset{j+i-k-1}{\downarrow}}\Psi\overset{i}{\underset{j+i-k}{\downarrow}}\Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}}\Psi\overset{i+3}{\underset{j+i-k+2}{\downarrow}}\Psi\overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda\\
&=-\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\Psi\overset{k}{\underset{j}{\downarrow}}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\Psi\overset{k+2}{\underset{j+2}{\downarrow}}\dots\Psi\overset{i-2}{\underset{j+i-k-2}{\downarrow}}\psi_{i+3}\Psi\overset{a_{j+i-l+3}-1}{\underset{i+4}{\downarrow}}\Psi\overset{a_{j+i-k+4}-1}{\underset{i+5}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_{m+1}-1}{\underset{m-j+k+1}{\downarrow}}y_{i+1}\Psi\overset{i-1}{\underset{j+i-k-1}{\downarrow}}\Psi\overset{i}{\underset{j+i-k}{\downarrow}}\Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}}\dots\Psi\overset{m-j+k}{\underset{m+1}{\downarrow}}z_\lambda\\
&=-\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\Psi\overset{k}{\underset{j}{\downarrow}}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\Psi\overset{k+2}{\underset{j+2}{\downarrow}}\dots\Psi\overset{i-2}{\underset{j+i-k-2}{\downarrow}}\psi_{i+3}\Psi\overset{a_{j+i-l+3}-1}{\underset{i+4}{\downarrow}}\Psi\overset{a_{j+i-k+4}-1}{\underset{i+5}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_{m+1}-1}{\underset{m-j+k+1}{\downarrow}}y_{i+1}v(2,\dots,j+i-k-2,i,i+1,i+2,\dots,m-j+k+1)
\end{aligned}$$

where $y_{i+1}v(2,\dots,j+i-k-2,i,i+1,i+2,\dots,m-j+k+1)$ equals zero by part two of Lemma 3.3.

The last term of this expression becomes

$$\begin{aligned}
&-\Psi\overset{a_2-1}{\underset{2}{\downarrow}}\dots\Psi\overset{a_{j-2}-1}{\underset{j-2}{\downarrow}}\Psi\overset{k-1}{\underset{j-1}{\downarrow}}\Psi\overset{k}{\underset{j}{\downarrow}}\Psi\overset{k+1}{\underset{j+1}{\downarrow}}\Psi\overset{k+2}{\underset{j+2}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{i}{\underset{j+i-k}{\downarrow}}\Psi\overset{i+1}{\underset{j+i-k+1}{\downarrow}}y_{i+3}\psi_{i+3}\Psi\overset{i+2}{\underset{j+i-k+2}{\downarrow}}\Psi\overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}}\dots\Psi\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}}z_\lambda
\end{aligned}$$

$$= -\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi \overset{k-1}{\underset{j-1}{\downarrow}} \Psi \overset{k}{\underset{j}{\downarrow}} \Psi \overset{k+1}{\underset{j+1}{\downarrow}} \Psi \overset{k+2}{\underset{j+2}{\downarrow}} \dots$$
$$\dots \Psi \overset{i}{\underset{j+i-k}{\downarrow}} \Psi \overset{i+1}{\underset{j+i-k+1}{\downarrow}} (\psi_{i+3}y_{i+4}-1) \Psi \overset{i+2}{\underset{j+i-k+2}{\downarrow}} \Psi \overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda.$$

The first term of this expression becomes

$$-\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi \overset{k-1}{\underset{j-1}{\downarrow}} \Psi \overset{k}{\underset{j}{\downarrow}} \Psi \overset{k+1}{\underset{j+1}{\downarrow}} \Psi \overset{k+2}{\underset{j+2}{\downarrow}} \dots$$
$$\dots \Psi \overset{i}{\underset{j+i-k}{\downarrow}} \Psi \overset{i+1}{\underset{j+i-k+1}{\downarrow}} \psi_{i+3}y_{i+4} \Psi \overset{i+2}{\underset{j+i-k+2}{\downarrow}} \Psi \overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda$$
$$= -\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi \overset{k-1}{\underset{j-1}{\downarrow}} \Psi \overset{k}{\underset{j}{\downarrow}} \Psi \overset{k+1}{\underset{j+1}{\downarrow}} \Psi \overset{k+2}{\underset{j+2}{\downarrow}} \dots$$
$$\dots \Psi \overset{i}{\underset{j+i-k}{\downarrow}} \Psi \overset{i+1}{\underset{j+i-k+1}{\downarrow}} \psi_{i+3} \Psi \overset{a_{j+i-k+3}-1}{\underset{i+5}{\downarrow}} y_{i+4} \Psi \overset{i+2}{\underset{j+i-k+2}{\downarrow}} \Psi \overset{i+4}{\underset{j+i-k+3}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda$$
$$= -\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi \overset{k-1}{\underset{j-1}{\downarrow}} \Psi \overset{k}{\underset{j}{\downarrow}} \Psi \overset{k+1}{\underset{j+1}{\downarrow}} \Psi \overset{k+2}{\underset{j+2}{\downarrow}} \dots$$
$$\dots \Psi \overset{i+1}{\underset{j+i-k+1}{\downarrow}} \psi_{i+3} \Psi \overset{a_{j+i-k+3}-1}{\underset{i+5}{\downarrow}} y_{i+4} v(2, \dots, j+i-k+1, i+3, i+5, a_{j+i-k+4}, \dots, a_{m+1}),$$

where $y_{i+4}v(2,\dots,j+i-k+1,i+3,i+5,a_{j+i-k+4},\dots,a_{m+1})$ equals zero by part three of Lemma 3.3.

Now the last expression of this expression becomes

$$\Psi \overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi \overset{a_{j-2}-1}{\underset{j-2}{\downarrow}} \Psi \overset{k-1}{\underset{j-1}{\downarrow}} \Psi \overset{k}{\underset{j}{\downarrow}} \Psi \overset{k+1}{\underset{j+1}{\downarrow}} \Psi \overset{k+2}{\underset{j+2}{\downarrow}} \dots \Psi \overset{i}{\underset{j+i-k}{\downarrow}} \Psi \overset{i+1}{\underset{j+i-k+1}{\downarrow}} \Psi \overset{i+2}{\underset{j+i-k+2}{\downarrow}} \Psi \overset{a_{j+i-k+3}-1}{\underset{j+i-k+3}{\downarrow}} \dots \Psi \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} z_\lambda$$
$$= v(a_2, \dots, a_{j-1}, k+1, k+2, k+3, \dots, i+1, i+2, i+3, a_{j+i-k+3}, a_{j+i-k+4}, \dots, a_{m+1}),$$

as required. □

We will show that there is a strong connection between Specht modules labelled by hook partitions and Specht modules labelled by two-part partitions $S_{(n-m,m)}$. We first give the presentation of $S_{(n-m,m)}$.

Remark 3.6. *For $\mu = (n-2k-2+2j, 2k+2-2j)$, S_μ is generated by z_μ, subject only to the relations*

- $e(\mathbf{i})z_\mu = \delta_{\mathbf{i},\mathbf{i}_\lambda} z_\mu$,
- $y_i z_\mu = 0$ *for all* $1 \leqslant i \leqslant n$,
- $\psi_i z_\mu = 0$ *for all* $i \in \{1, 3, 5, \dots, 4k-4j+3\}$,

together with the Garnir relations (by Section 1.3.1.1)

- $\psi_i z_\mu = 0$ *for all* $i \in \{4k-4j+5, \dots, n-1\}$,

- $\psi_i\psi_{i+1}z_\mu = 0$ *for all* $i \in \{1, 3, 5, \dots, 4k-4j+3\}$,
- $\psi_i\psi_{i-1}z_\mu = 0$ *for all* $i \in \{3, 5, \dots, 4k-4j+3\}$.

Theorem 3.7. *For* $\lambda = (n-2k, 1^{2k})$ *and* $n \geqslant 4k+1$, S_λ *has an increasing filtration*

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_j \subset \cdots \subset S_\lambda$$

whose factors are

$$M_j/M_{j-1} \cong S_{(n-2k-2+2j, 2k+2-2j)},$$

for all $j \in \{1, \dots, k+1\}$.

Proof of theorem. Recall that M_j is the $\mathscr{R}_n^\Lambda$-module generated by v_{A_j}. For $2 \leqslant j \leqslant k+1$, we observe that

$$v_{A_{j-1}} = \psi_{4k-2j+7}\Psi\overset{4k-4j+9}{\underset{4k-4j+8}{\downarrow}}\ \Psi\overset{4k-4j+10}{\underset{4k-4j+9}{\downarrow}} \dots \Psi\overset{4k-2j+4}{\underset{4k-2j+3}{\downarrow}} v_{A_j},$$

so that $M_{j-1} \subseteq M_j$.

We now show that M_j/M_{j-1} is isomorphic to a quotient of $S_{(n-2k-2+2j,2k+2-2j)}$. We take the generator v_{A_j} of M_j and show that modulo M_{j-1} it satisfies the relations that $z_{(n-2k-2+2j,2k+2-2j)}$ satisfies, which are given in Remark 3.6. We will draw particular attention to the relation $\psi_{4k-2j+3}z_{(n-2k-2+2j,2k+2-2j)} = 0$.

(i) We show that $y_iv_{A_j} = 0$ for all $1 \leqslant i \leqslant n$. Suppose that $i \in \{1, 2\}$. Then

$$\begin{aligned} y_iv_{A_j} &= y_i \prod_{i=3}^{2k-2j+3}\left(\Psi\overset{2i-3}{\underset{i}{\downarrow}}\right)\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\ &= \prod_{i=3}^{2k-2j+3}\left(\Psi\overset{2i-3}{\underset{i}{\downarrow}}\right)\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) y_iz_\lambda \\ &= 0. \end{aligned}$$

Suppose that $i \in \{4, 6, 8, \dots, 4k-4j+6\}$. Then

$$\begin{aligned} y_iv_{A_j} &= y_i\psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-3}{\underset{\frac{1}{2}i}{\downarrow}}\ \Psi\overset{i-1}{\underset{\frac{1}{2}(i+2)}{\downarrow}}\ \Psi\overset{i+1}{\underset{\frac{1}{2}(i+4)}{\downarrow}}\dots\Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\ &= \psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-3}{\underset{\frac{1}{2}i}{\downarrow}}\ y_i\Psi\overset{i-1}{\underset{\frac{1}{2}(i+2)}{\downarrow}}\ \Psi\overset{i+1}{\underset{\frac{1}{2}(i+4)}{\downarrow}}\dots\Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\ &= \psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-3}{\underset{\frac{1}{2}i}{\downarrow}}\ y_iv(2, 3, \dots, \frac{i}{2}, i, a_{\frac{1}{2}(i+4)}, \dots, a_{2k+1}) \end{aligned}$$

where $y_iv(2, 3, \dots, \frac{i}{2}, i, a_{\frac{1}{2}(i+4)}, \dots, a_{2k+1})$ equals zero by part two of Lemma 3.3.

Now suppose that $i \in \{3, 5, 7, \dots, 4k-4j+5\}$. Then

$$\begin{aligned}
y_i v_{A_j} &= y_i \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-4}{\underset{\frac{1}{2}(i-1)}{\downarrow}} \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} \Psi\overset{i}{\underset{\frac{1}{2}(i+3)}{\downarrow}} \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \dots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3} \left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}} \right) z_\lambda \\
&= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-4}{\underset{\frac{1}{2}(i-1)}{\downarrow}} y_i \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} \Psi\overset{i}{\underset{\frac{1}{2}(i+3)}{\downarrow}} \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \dots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3} \left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}} \right) z_\lambda \\
&= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-4}{\underset{\frac{1}{2}(i-1)}{\downarrow}} y_i v(2, 3, \dots, \tfrac{1}{2}(i-1), i-1, i+1, a_{\frac{1}{2}(i+5)}, \dots, a_{2k+1}),
\end{aligned}$$

where $y_i v(2, 3, \dots, \frac{1}{2}(i-1), i-1, i+1, a_{\frac{1}{2}(i+5)}, \dots, a_{2k+1})$ equals zero by part three of Lemma 3.3.

Suppose that $4k-4j+7 \leqslant i \leqslant 4k-2j+3$. Then we have

$$\begin{aligned}
y_i v_{A_j} = {} & y_i \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{4k-4j+1}{\underset{2k-2j+2}{\downarrow}} \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \Psi\overset{4k-4j+5}{\underset{2k-2j+4}{\downarrow}} \Psi\overset{4k-4j+6}{\underset{2k-2j+5}{\downarrow}} \dots \\
& \dots \Psi\overset{i-2}{\underset{i-2k+2j-3}{\downarrow}} \Psi\overset{i-1}{\underset{i-2k+2j-2}{\downarrow}} \Psi\overset{i}{\underset{i-2k+2j-1}{\downarrow}} \dots \Psi\overset{4k-2j+2}{\underset{2k+1}{\downarrow}} z_\lambda \\
= {} & \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{4k-4j+1}{\underset{2k-2j+2}{\downarrow}} \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \Psi\overset{4k-4j+5}{\underset{2k-2j+4}{\downarrow}} \Psi\overset{4k-4j+6}{\underset{2k-2j+5}{\downarrow}} \dots \\
& \dots \Psi\overset{i-3}{\underset{i-2k+2j-4}{\downarrow}} y_i \Psi\overset{i-2}{\underset{i-2k+2j-3}{\downarrow}} \Psi\overset{i-1}{\underset{i-2k+2j-2}{\downarrow}} \Psi\overset{i}{\underset{i-2k+2j-1}{\downarrow}} \dots \Psi\overset{4k-2j+2}{\underset{2k+1}{\downarrow}} z_\lambda \\
= {} & \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{4k-4j+1}{\underset{2k-2j+2}{\downarrow}} \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \Psi\overset{4k-4j+5}{\underset{2k-2j+4}{\downarrow}} \Psi\overset{4k-4j+6}{\underset{2k-2j+5}{\downarrow}} \dots \\
& \dots \Psi\overset{i-3}{\underset{i-2k+2j-4}{\downarrow}} y_i v(2, \dots, i-2k+2j-4, i-1, i, i+1, \dots, 4k-2j+3) \\
= {} & 0,
\end{aligned}$$

since $y_i v(2, \dots, i-2k+2j-4, i-1, i, i+1, \dots, 4k-2j+3)$ equals zero by part two of Lemma 3.3.

Finally suppose that $4k-2j+4 \leqslant i \leqslant n$. Then

$$\begin{aligned}
y_i v_{A_j} &= \prod_{i=3}^{2k-2j+3} \left(\Psi\overset{2i-3}{\underset{i}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}} \right) z_\lambda \\
&= \prod_{i=3}^{2k-2j+3} \left(\Psi\overset{2i-3}{\underset{i}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}} \right) y_i z_\lambda \\
&= 0.
\end{aligned}$$

(ii) We show that $\psi_i v_{A_j} = 0$ for all $i \in \{1, 3, 5, \dots, 4k-4j+3\} \cup \{4k-4j+5, \dots, n-1\}$.

Clearly $\psi_1 v_{A_j} = 0$. Now suppose that $i \in \{3, 5, 7, \ldots, 4k-4j+5\}$. Then

$$\begin{aligned}
\psi_i v_{A_j} &= \psi_i \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\; \Psi\overset{i}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\; \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \ldots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\
&= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\; \psi_i^2 \Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\; \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \ldots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\
&= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\; (y_{i+1} - y_i)(y_i - y_{i+1}) \Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\; \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \ldots \\
&\qquad \ldots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda.
\end{aligned}$$

Two of the terms of this expression become

$$\begin{aligned}
&\psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\; (y_{i+1} - y_i) y_i \Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\; \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \ldots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\
&= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\; (y_{i+1} - y_i) y_i v(2, 3, \ldots, \tfrac{1}{2}(i+1), i, a_{\frac{1}{2}(i+5)}, \ldots, a_{2k+1}),
\end{aligned}$$

where $y_i v(2, 3, \ldots, \frac{1}{2}(i+1), i, a_{\frac{1}{2}(i+5)}, \ldots, a_{2k+1})$ equals zero by part two of Lemma 3.3. Now the remaining terms of the expression become

$$\begin{aligned}
&\psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\; (y_{i+1} - y_i) y_{i+1} \Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\; \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \ldots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\
&= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\; (y_{i+1} - y_i) \psi_{i+2} y_{i+1} \Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\; \Psi\overset{i+1}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \ldots \\
&\qquad \ldots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\
&= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\; (y_{i+1} - y_i) \psi_{i+2} y_{i+1} v(2, \ldots, \tfrac{1}{2}(i+1), i, i+2, a_{\frac{1}{2}(i+7)}, \ldots, a_{2k+1}),
\end{aligned}$$

where $y_{i+1} v(2, \ldots, \frac{1}{2}(i+1), i, i+2, a_{\frac{1}{2}(i+7)}, \ldots, a_{2k+1})$ equals zero by part three of Lemma 3.3.

Now suppose that $4k - 4j + 6 \leqslant i \leqslant 4k - 2j + 2$. Then we have

$$\begin{aligned}
\psi_i v_{A_j} &= \psi_i \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{4k-4j+1}{\underset{2k-2j+2}{\downarrow}}\; \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\; \Psi\overset{4k-4j+5}{\underset{2k-2j+4}{\downarrow}}\; \Psi\overset{4k-4j+6}{\underset{2k-2j+5}{\downarrow}} \ldots \\
&\qquad \ldots \Psi\overset{i-2}{\underset{i-2k+2j-3}{\downarrow}}\; \Psi\overset{i-1}{\underset{i-2k+2j-2}{\downarrow}}\; \Psi\overset{i}{\underset{i-2k+2j-1}{\downarrow}}\; \Psi\overset{i+1}{\underset{i-2k+2j}{\downarrow}} \ldots \Psi\overset{4k-2j+2}{\underset{2k+1}{\downarrow}} z_\lambda \\
&= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}}\, \Psi\overset{7}{\underset{5}{\downarrow}} \ldots \Psi\overset{4k-4j+1}{\underset{2k-2j+2}{\downarrow}}\; \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\; \Psi\overset{4k-4j+5}{\underset{2k-2j+4}{\downarrow}}\; \Psi\overset{4k-4j+6}{\underset{2k-2j+5}{\downarrow}} \ldots \\
&\qquad \ldots \Psi\overset{i-2}{\underset{i-2k+2j-3}{\downarrow}}\; \psi_i \Psi\overset{i-1}{\underset{i-2k+2j-2}{\downarrow}}\; \Psi\overset{i}{\underset{i-2k+2j-1}{\downarrow}}\; \Psi\overset{i+1}{\underset{i-2k+2j}{\downarrow}} \ldots \Psi\overset{4k-2j+2}{\underset{2k+1}{\downarrow}} z_\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \psi_3 \Psi\underset{4}{\overset{5}{\downarrow}}\, \Psi\underset{5}{\overset{7}{\downarrow}} \dots \Psi \underset{2k-2j+2}{\overset{4k-4j+1}{\downarrow}}\ \Psi \underset{2k-2j+3}{\overset{4k-4j+3}{\downarrow}}\ \Psi \underset{2k-2j+4}{\overset{4k-4j+5}{\downarrow}}\ \Psi \underset{2k-2j+5}{\overset{4k-4j+6}{\downarrow}} \dots \\
&\qquad \dots \Psi \underset{i-2k+2j-3}{\overset{i-2}{\downarrow}}\ \psi_i v(2, \dots, i-2k+2j-3, i, i+1, i+2, \dots, 4k-2j+3) \\
&= 0,
\end{aligned}
$$

since $\psi_i v(2, \dots, i-2k+2j-3, i, i+1, i+2, \dots, 4k-2j+3)$ equals zero by part one of Lemma 3.3.

We now reach the most enlightening part of the proof, to show that $\psi_{4k-2j+3}$ kills v_{A_j}. By calling on Lemma 3.5, we have

$$
\begin{aligned}
&\Psi \underset{4k-4j+6}{\overset{4k-2j+1}{\downarrow}}\ \Psi \underset{4k-4j+8}{\overset{4k-2j+3}{\downarrow}}\ v_{A_{j-1}} \\
&= \Psi \underset{4k-4j+6}{\overset{4k-2j+1}{\downarrow}}\ \Psi \underset{4k-4j+8}{\overset{4k-2j+3}{\downarrow}} \prod_{i=3}^{2k-2j+5} \left(\Psi \underset{i}{\overset{2i-3}{\downarrow}} \right) \prod_{l=0}^{2j-5} \left(\Psi \underset{2k-2j+6+l}{\overset{4k-4j+9+l}{\downarrow}} \right) z_\lambda \\
&= \Psi \underset{4k-4j+6}{\overset{4k-2j+1}{\downarrow}}\ \Psi \underset{4k-4j+8}{\overset{4k-2j+3}{\downarrow}}\ v\,(2,4,6,\dots,4k-4j+6, 4k-4j+8, \\
&\qquad 4k-4j+10, 4k-4j+11, \dots, 4k-2j+4, 4k-2j+5) \\
&= \Psi \underset{4k-4j+6}{\overset{4k-2j+1}{\downarrow}}\ v\,(2,4,6,\dots,4k-4j+4, 4k-4j+6, \\
&\qquad 4k-4j+8, 4k-4j+9, 4k-4j+10, \dots, 4k-2j+3, 4k-2j+4) \\
&= v\,(2,4,6,\dots,4k-4j+4, 4k-4j+6, \\
&\qquad 4k-4j+7, 4k-4j+8, 4k-4j+9, \dots, 4k-2j+2, 4k-2j+4) \\
&= \psi_{4k-2j+3} v\,(2,4,6,\dots,4k-4j+4, 4k-4j+6, \\
&\qquad 4k-4j+7, 4k-4j+8, 4k-4j+9, \dots, 4k-2j+2, 4k-2j+3) \\
&= \psi_{4k-2j+3} \prod_{i=3}^{2k-2j+3} \left(\Psi \underset{i}{\overset{2i-3}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(\Psi \underset{2k-2j+4+l}{\overset{4k-4j+5+l}{\downarrow}} \right) z_\lambda \\
&= \psi_{4k-2j+3} v_{A_j}.
\end{aligned}
$$

Thus,

$$
\psi_{4k-2j+3} v_{A_j} = \Psi \underset{4k-4j+6}{\overset{4k-2j+1}{\downarrow}}\ \Psi \underset{4k-4j+8}{\overset{4k-2j+3}{\downarrow}}\ v_{A_{j-1}} \equiv 0 \pmod{M_{j-1}}.
$$

If $4k-2j+4 \leqslant i \leqslant n-1$, then

$$
\begin{aligned}
\psi_i v_{A_j} &= \psi_i \prod_{i=3}^{2k-2j+3} \left(\Psi \underset{i}{\overset{2i-3}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(\Psi \underset{2k-2j+4+l}{\overset{4k-4j+5+l}{\downarrow}} \right) z_\lambda \\
&= \prod_{i=3}^{2k-2j+3} \left(\Psi \underset{i}{\overset{2i-3}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(\Psi \underset{2k-2j+4+l}{\overset{4k-4j+5+l}{\downarrow}} \right) \psi_i z_\lambda
\end{aligned}
$$

$$= 0.$$

(iii) We now show that $\psi_i\psi_{i+1}v_{A_j} = 0$ for all $i \in \{1,3,5,\dots,4k-4j+3\}$. If $i=1$, then we have

$$\begin{aligned}
\psi_1\psi_2 v_{A_j} &= \psi_1\psi_2 \prod_{i=3}^{2k-2j+3}\left(\Psi\overset{2i-3}{\underset{i}{\downarrow}}\right)\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda \\
&= \psi_1\psi_2 \prod_{i=3}^{2k-2j+3}\left(\Psi\overset{2i-3}{\underset{i+1}{\downarrow}}\right)\prod_{i=3}^{2k-2j+3}(\psi_i)\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+5+l}{\downarrow}}\right)\prod_{l=0}^{2j-3}(\psi_l)\, z_\lambda \\
&= \prod_{i=3}^{2k-2j+3}\left(\Psi\overset{2i-3}{\underset{i+1}{\downarrow}}\right)\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+5+l}{\downarrow}}\right)\psi_1\psi_2\prod_{i=3}^{2k-2j+3}(\psi_i)\prod_{l=0}^{2j-3}(\psi_l)\, z_\lambda \\
&= \prod_{i=3}^{2k-2j+3}\left(\Psi\overset{2i-3}{\underset{i+1}{\downarrow}}\right)\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+5+l}{\downarrow}}\right)\psi_1\psi_2\psi_3\dots\psi_{2k}\psi_{2k+1}z_\lambda,
\end{aligned}$$

where $\psi_1\psi_2\psi_3\dots\psi_{2k}\psi_{2k+1}z_\lambda$ equals zero by the first Garnir relation given in Remark 3.2.

Now let $i \geqslant 3$. Then we have

$$\begin{aligned}
&\psi_i\psi_{i+1}v_{A_j} \\
&= \psi_i\psi_{i+1}\psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\,\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\,\Psi\overset{i}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\,\Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}}\dots\Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right)z_\lambda \\
&= \psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\,\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\,\psi_i\psi_{i+1}\psi_i\Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\,\Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}}\dots\Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right)z_\lambda \\
&= \psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\,\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\,(\psi_{i+1}\psi_i\psi_{i+1}-2y_{i+1}+y_i+y_{i+2})\,\Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\,\Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}}\dots \\
&\qquad \dots\Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right)z_\lambda.
\end{aligned}$$

The first term of this expression becomes

$$\begin{aligned}
&\psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\,\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\,\psi_{i+1}\psi_i\psi_{i+1}\Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\,\Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}}\dots\Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right)z_\lambda \\
&= \psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\,\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\,\psi_{i+1}\psi_i\psi_{i+1}v(2,3,\dots,\tfrac{1}{2}(i+1),i,i+3,a_{\frac{1}{2}(i+7)},\dots,a_{2k+1}),
\end{aligned}$$

where $\psi_{i+1}v(2,3,\dots,\frac{1}{2}(i+1),i,i+3,a_{\frac{1}{2}(i+7)},\dots,a_{2k+1})$ equals zero by Lemma 3.4. The second term of the expression becomes

$$-2\psi_3\Psi\overset{5}{\underset{4}{\downarrow}}\,\Psi\overset{7}{\underset{5}{\downarrow}}\dots\Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}}\,y_{i+1}\Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}}\,\Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}}\dots\Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}}\prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right)z_\lambda$$

$$= -2\psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} \psi_{i+2}y_{i+1}\Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}} \Psi\overset{i+1}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \dots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda$$
$$= -2\psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} \psi_{i+2}y_{i+1}v(2,3,\dots,\tfrac{1}{2}(i+1),i,i+2,a_{\frac{1}{2}(i+7)},\dots,a_{2k+1})$$

where $y_{i+1}v(2,3,\dots,\frac{1}{2}(i+1),i,i+2,a_{\frac{1}{2}(i+7)},\dots,a_{2k+1})$ equals zero by part three of Lemma 3.3. The third term of the expression becomes

$$\psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} y_i\Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}} \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \dots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda$$
$$= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} y_iv(2,3,\dots,\tfrac{1}{2}(i+1),i,i+3,a_{\frac{1}{2}(i+7)},\dots,a_{2k+1}),$$

where $y_iv(2,3,\dots,\frac{1}{2}(i+1),i,i+3,a_{\frac{1}{2}(i+7)},\dots,a_{2k+1})$ equals zero by part two of Lemma 3.3. The last term of the expression becomes

$$\psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} y_{i+2}\Psi\overset{i-1}{\underset{\frac{1}{2}(i+3)}{\downarrow}} \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \dots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=1}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right)$$
$$= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} y_{i+2}v(2,3,\dots,\tfrac{1}{2}(i+1),i,i+3,a_{\frac{1}{2}(i+7)},\dots,a_{2k+1}),$$

where $y_{i+2}v(2,3,\dots,\frac{1}{2}(i+1),i,i+3,a_{\frac{1}{2}(i+7)},\dots,a_{2k+1})$ equals zero by part three of Lemma 3.3.

(iv) We now show that $\psi_i\psi_{i-1}v_{A_j} = 0$ for all $i \in \{3,5,\dots,4k-4j+3\}$. Suppose that $i \in \{3,5,\dots,4k-4j+3\}$. Then we have

$$\psi_i\psi_{i-1}v(a_2,\dots,a_m)$$
$$= \psi_i\psi_{i-1}\psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-4}{\underset{\frac{1}{2}(i-1)}{\downarrow}} \Psi\overset{i-2}{\underset{\frac{1}{2}(i+1)}{\downarrow}} \Psi\overset{i}{\underset{\frac{1}{2}(i+3)}{\downarrow}} \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \dots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda$$
$$= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-4}{\underset{\frac{1}{2}(i-1)}{\downarrow}} \psi_i\Psi\overset{i-1}{\underset{\frac{1}{2}(i+1)}{\downarrow}} \Psi\overset{i}{\underset{\frac{1}{2}(i+3)}{\downarrow}} \Psi\overset{i+2}{\underset{\frac{1}{2}(i+5)}{\downarrow}} \dots \Psi\overset{4k-4j+3}{\underset{2k-2j+3}{\downarrow}} \prod_{l=0}^{2j-3}\left(\Psi\overset{4k-4j+5+l}{\underset{2k-2j+4+l}{\downarrow}}\right) z_\lambda$$
$$= \psi_3 \Psi\overset{5}{\underset{4}{\downarrow}} \Psi\overset{7}{\underset{5}{\downarrow}} \dots \Psi\overset{i-4}{\underset{\frac{1}{2}(i-1)}{\downarrow}} \psi_iv(2,3,\dots,\tfrac{1}{2}(i-1),i,i+1,a_{\frac{1}{2}(i+5)},\dots,a_{2k+1}),$$

where $\psi_iv(2,3,\dots,\frac{1}{2}(i-1),i,i+1,a_{\frac{1}{2}(i+5)},\dots,a_{2k+1})$ equals zero by part one of Lemma 3.3.

Thus, we have shown that the jth factor M_j/M_{j-1} in the filtration of λ is a quotient of $S_{(n-2k-2+2j,2k+2-2j)}$. By comparing dimensions, we ascertain that the jth factor in the filtration of λ is, in fact, equal to the entire Specht module $S_{(n-2k-2+2j,2k+2-2j)}$.

Using the Hook Length Formula given in Theorem 1.26 or by [P, Proposition 2],

observe that
$$\dim(S_\lambda) = \binom{n-1}{2k},$$
and by Proposition 2.1, we have
$$\dim\left(S_{(n-2k-2+2j,2k+2-2j)}\right) = \frac{n-4k+4j-3}{n+1}\binom{n+1}{2k+2-2j},$$
for all $1 \leqslant j \leqslant k+1$. To show that the factors M_j/M_{j-1} are, in fact, the entire Specht modules $S_{(n-2k-2+2j,2k+2-2j)}$, we want
$$\begin{aligned}
\dim(S_\lambda) &= \sum_{j=1}^{k+1}\left(\dim\left(S_{(n-2k-2+2j,2k+2-2j)}\right)\right)\\
\iff \quad (n+1)\binom{n-1}{2k} &= \sum_{j=1}^{k+1}(n-4k+4j-3)\binom{n+1}{2k+2-2j}\\
\iff \quad (n+1)\binom{n-1}{2k} &= (n-4k+1)\binom{n-1}{2k} + \sum_{j=1}^{k+1}(2n-8k-2+4j)\binom{n-1}{2k-j}\\
\iff \quad 4k\binom{n-1}{2k} &= \sum_{j=1}^{k+1}(2n-8k-2+4j)\binom{n-1}{2k-j}\\
\iff \quad 2k\binom{n-1}{2k} &= \sum_{j=1}^{k+1}(n-4k-1+2j)\binom{n-1}{2k-j},
\end{aligned}$$
which can be written as follows, using the binomial identity $\binom{m}{n} = \sum_{k=0}^{n}(-1)^{(n-k)}\binom{m+1}{k}$ for all $m \geqslant n \geqslant 0$. Hence we have
$$\begin{aligned}
&2k\left(\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{2k}\right)\\
&= (n-4k+1)\left(-\binom{n}{0} + \binom{n}{1} - \binom{n}{2} + \cdots + \binom{n}{2k-1}\right)\\
&\quad + (n-4k+3)\left(\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{2k-2}\right) + \cdots\\
&\quad \cdots + (n-5)\left(\binom{n}{0} - \binom{n}{1} + \binom{n}{2}\right) + (n-3)\left(-\binom{n}{0} + \binom{n}{1}\right) + (n-1)\binom{n}{0}.
\end{aligned}$$
Simplifying gives
$$\begin{aligned}
&(n-1)\binom{n}{1} + (n-3)\binom{n}{3} + \cdots + (n-2k+3)\binom{n}{2k-3} + (n-2k+1)\binom{n}{2k-1}\\
&= 2\binom{n}{2} + 4\binom{n}{4} + \cdots + (2k-2)\binom{n}{2k-2} + 2k\binom{n}{2k},
\end{aligned}$$

which clearly holds since

$$(n-2k+1+2j)\binom{n}{2k-1-2j} = (2k-2j)\binom{n}{2k-2j} \ \forall\ 0 \leqslant j \leqslant k-1.$$

$\square$

Thus, S_λ is factored by the graded modules $S_{(n-2k,2k)}$, $S_{(n-2k+2,2k-2)}$,...,$S_{(n)}$ from bottom to top.

Example 3.8. *By Theorem 3.7, $S_{(7,1^6)}$ has an increasing filtration*

$$0 \subset M_1 \subset M_2 \subset M_3 \subset S_{(7,1^6)},$$

where $M_1 \cong S_{(7,6)}$, $M_2/M_1 \cong S_{(9,4)}$, $M_3/M_2 \cong S_{(11,2)}$ and $S_{(7,1^6)}/M_3 \cong S_{(13)}$. Let $A_1 = s_3 s{\downarrow}^{5}_{4}\, s{\downarrow}^{7}_{5}\, s{\downarrow}^{9}_{6}\, s{\downarrow}^{11}_{7}\, T_{(7,1^6)}$, which is the standard $(7,1^6)$-tableau

$$A_1 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1} 8 \\ \cline{1-1} 10 \\ \cline{1-1} 12 \\ \cline{1-1} \end{array}\ .$$

By Equation (3.2.1), we know that M_1 is generated by $v_{A_1} = v(2,4,6,8,10,12)$.

Now, let $A_2 = s_3 s{\downarrow}^{5}_{4}\, s{\downarrow}^{7}_{5}\, s{\downarrow}^{9}_{6}\, s{\downarrow}^{10}_{7}\, T_{(7,1^6)}$, which is the standard $(7,1^6)$-tableau

$$A_2 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & 7 & 9 & 12 & 13 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1} 8 \\ \cline{1-1} 10 \\ \cline{1-1} 11 \\ \cline{1-1} \end{array}\ ,$$

where the consecutive entries in A_2 are shaded. By Equation (3.2.1), we know that $v_{A_2} = v(2,4,6,8,10,11)$ generates M_2/M_1 (satisfying the relations that $z_{(7,6)}$ satisfies modulo M_1).

In particular, by Lemma 3.5, we have that

$$\psi_{11} v(2,4,6,8,10,11) = v(2,4,6,8,10,12) \equiv 0 \pmod{M_1}.$$

For the next factor up, M_3/M_2, we let $A_3 = s_3 s{\downarrow}^{5}_{4}\, s{\downarrow}^{6}_{5}\, s{\downarrow}^{7}_{6}\, s{\downarrow}^{8}_{7}\, T_{(7,1^6)}$, which is the *standard* $(7,1^6)$*-tableau*

$$A_3 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & 10 & 11 & 12 & 13 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} 8 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array},$$

where the consecutive entries in A_3 *are shaded. By Equation* (3.2.1)*, we know that* M_3/M_2 *is generated by* $v_{A_3} = v(2,4,6,8,9)$ *(satisfying the relations that* $z_{(11,2)}$ *satisfies modulo* M_2*).*

In particular, by Lemma 3.5, we have that

$$\psi_9 v(2,4,6,7,8,9) = \Psi\overset{7}{\underset{6}{\downarrow}}\,\Psi\overset{9}{\underset{8}{\downarrow}}\, v(2,4,6,8,10,11) \equiv 0 \pmod{M_2}.$$

Now, for the top factor $S_{(7,1^6)}/M_3$*, we let* $A_4 = T_{(7,1^6)}$*. We know that* $S_{(7,1^6)}/M_3$ *is generated by* $z_{(7,1^6)} = v(2,3,4,5,6,7)$ *(satisfying the relations that* $z_{(13)}$ *satisfies modulo* M_3*), by Equation* (3.2.1)*. In particular, by Lemma 3.5, we have that*

$$\psi_7 v(2,3,4,5,6,7) = \Psi\overset{5}{\underset{2}{\downarrow}}\,\Psi\overset{7}{\underset{4}{\downarrow}}\, v(2,4,6,7,8,9) \equiv 0 \pmod{M_3}.$$

Finally, one can readily compute that $\dim S_{(7,1^6)} = 924 = 429 + 429 + 65 + 1 = \dim S_{(7,6)} + \dim S_{(9,4)} + \dim S_{(11,2)} + \dim S_{(13)}$*, by the Hook Length Formula. Thus, the factors in the filtration of* $S_{(7,1^6)}$ *are the whole Specht modules* $S_{(7,6)}$*,* $S_{(9,4)}$*,* $S_{(11,2)}$ *and* $S_{(13)}$*, from bottom to top.*

3.3 A Specht filtration of $S_{(n-2k-1,1^{2k+1})}$

We now investigate the structure of Specht modules labelled by hook partitions that have legs of odd length, that is, Specht modules with labels $(n-2k-1,1^{2k+1})$, for $k \geqslant 0$.

Definition 3.9. *For* $1 \leqslant j \leqslant k+1$*, we define the standard* $(n-2k-1,1^{2k+1})$*-tableau*

$$B_j := \prod_{i=3}^{2k-2j+4} \left(s \overset{2i-3}{\underset{i}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(s \overset{4k-4j+7+l}{\underset{2k-2j+5+l}{\downarrow}} \right) T_{(n-2k-1,1^{2k+1})}.$$

We describe B_j as the $(n-2k-1,1^{2k+1})$-tableau with $2k-2j+2$ even entries $2,4,\ldots,4k-4j+6$ lying in the first $2k-2j+2$ nodes in its arm, and with $2j-2$ consecutive entries $4k-4j+8, 4k-4j+9, \ldots, 4k-2j+5$ lying in the remaining $2j-2$ nodes in its arm. If we set $l = 4k-4j$, then B_j is the following standard

$(n-2k-1, 1^{2k+1})$-tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 3 & 5 & \cdots\cdots & l{+}7 & l{+}2j{+}6 & \cdots\cdots & n \\
\hline
2 \\
\cline{1-1}
4 \\
\cline{1-1}
\vdots \\
\cline{1-1}
l{+}6 \\
\cline{1-1}
l{+}8 \\
\cline{1-1}
l{+}9 \\
\cline{1-1}
\vdots \\
\cline{1-1}
l{+}2j{+}5 \\
\cline{1-1}
\end{array},$$

where the nodes containing the consecutive entries are shaded.

We now find the corresponding basis vector v_{B_j} of $S_{(n-2k-1,1^{2k+1})}$ to B_j.

For $1 \leqslant j \leqslant k+1$, we write

$$v_{B_j} := \prod_{i=3}^{2k-2j+4} \left(\Psi \overset{2i-3}{\underset{i}{\downarrow}} \right) \prod_{l=0}^{2j-3} \left(\Psi \overset{4k-4j+7+l}{\underset{2k-2j+5+l}{\downarrow}} \right) T_{(n-2k-1,1^{2k+1})}, \tag{3.3.1}$$

by setting $T = B_j$ in Equation (3.1.1).

That is, $v_{B_j} = (2, 4, \ldots, 4k-4j+6, 4k-4j+8, 4k-4j+9, \ldots, 4k-2j+5)$. We let N_j be the $\mathscr{R}_n^\Lambda$-module generated by v_{B_j}. Thus, we obtain the following analogous result to Theorem 3.7.

Theorem 3.10. *For $n \geqslant 4k+2$, $S_{(n-2k-1,1^{2k+1})}$ has an increasing Specht filtration*

$$0 \subset N_1 \subset N_2 \subset \cdots \subset N_j \subset \cdots \subset S_{(n-2k-1,1^{2k+1})}$$

whose factors are

$$N_j/N_{j-1} \cong S_{(n-2k-3+2j,2k+3-2j)},$$

for all $i \in \{1, \ldots, k+1\}$.

Proof. One shows that v_{B_j} satisfies the relations that $z_{(n-2k-3+2j,2k+3-2j)}$ satisfies. Similarly to Theorem 3.7, the main problem in doing this is showing that a particular relation is satisfied, that is, $\psi_{4k-2j+5}$ kills v_{B_j}. One finds, by using Lemma 3.5, that

$$\psi_{4k-2j+5} v_{B_j} = \Psi \overset{4k-2j+3}{\underset{4k-4j+8}{\downarrow}} \Psi \overset{4k-2j+5}{\underset{4k-2j+10}{\downarrow}} v_{B_{j-1}} \equiv \quad \pmod{N_{j-1}}.$$

$\square$

Thus, $S_{(n-2k-1,1^{2k+1})}$ is filtered by the Specht modules

$$S_{(n-2k-1,2k+1)}, S_{(n-2k-3,2k-1)}, \ldots, S_{(n-1,1)},$$

from bottom to top.

Example 3.11. *By Theorem 3.10, $S_{(6,1^5)}$ has an increasing filtration*

$$0 \subset M_1 \subset M_2 \subset S_{(6,1^5)},$$

where $M_1 \cong S_{(6,5)}$, $M_2/M_1 \cong S_{(8,3)}$ and $S_{(6,1^5)}/M_2 \cong S_{(10,1)}$. By Equation (3.3.1), $v_{A_1} = v(2,4,6,8,10)$, $v_{A_2} = v(2,4,6,8,9)$, and $v_{A_3} = v(2,4,5,6,7)$, where $M_1 = \langle v_{A_1} \rangle$, $M_2/M_1 = \langle v_{A_2} \rangle$ and $S_{(6,1^5)}/M_2 = \langle v_{A_3} \rangle$. Moreover, $\dim S_{(6,1^5)} = 252 = 132+110+10 = \dim S_{(6,5)} + \dim S_{(8,3)} + \dim S_{(10,1)}$.

3.4 Graded decomposition numbers

In this section we determine the graded multiplicities $[S_{(n-m,1^m)} : D]_v$, where D is a composition factor of $S_{(n-m,1^m)}$ for $\mathscr{R}_n^\Lambda$ over a field of characteristic zero. We first need to establish that the factors

$$S_{(n-m,m)}, S_{(n-m+2,m-2)}, S_{(n-m+4,m-4)}, \ldots,$$

arising in the filtrations of $S_{(n-m,1^m)}$ in Theorem 3.7 and Theorem 3.10, are the composition factors of $S_{(n-m,1^m)}$. We let $\mathbb{F}$ be arbitrary in this section unless otherwise stated.

The following is a special case of [JM2, Theorem 4.15], determining irreducibility of graded $\mathscr{R}_n^\Lambda$-modules when $l = 1$.

Theorem 3.12. *Let* $\operatorname{char}(\mathbb{F}) = 0$*. If n is odd, then $S_{(n-m,m)} = D_{(n-m,m)}$ is an irreducible $\mathscr{R}_n^\Lambda$-module.*

Over a field of arbitrary characteristic, we know from [JM2] that irreducible Specht modules S_λ for the Iwahori–Hecke algebra of type A, that is for $\mathscr{R}_n^\Lambda$ in level one with $p \nmid e$, are labelled by e-regular partitions. However, not all Specht modules S_λ, where λ is e-regular, remain irreducible in positive characteristic. When $e \neq 2$, the necessary and sufficient criterion for the irreducibility of Specht modules for the Iwahori–Hecke algebras of type A was conjectured by James and Mathas [M1, Conjecture 5.47], which was later proved over several papers [L, Fa1, Fa2].

Thus, if n is odd, then the filtrations of $S_{(n-m,1^m)}$ given in Theorem 3.7 and Theorem 3.10 are, in fact, composition series of $S_{(n-m,1^m)}$ for $\mathscr{R}_n^\Lambda$ over a field of characteristic zero.

Now, the following is a q-analogue of James's useful result [J3, Theorem 8.15], generalising from the symmetric algebra to the Khovanov–Lauda–Rouquier algebra.

Theorem 3.13. *[KMR, Theorem 8.5]Let e be arbitrary. For $\lambda \in \mathscr{P}_n^l$,*

$$S_\lambda^{\mathrm{sgn}} \cong (S_{\lambda'})^{\circledast} \langle \mathrm{wt}_e(\lambda) \rangle,$$

as graded $\mathscr{R}_n^\Lambda$-modules.

Corollary 3.14. *Let e be arbitrary. For $\lambda \in \mathscr{P}_n^l$,* $\operatorname{grdim}(S_{\lambda'}) = v^{\mathrm{wt}_e(\lambda)} \operatorname{grdim}\big(S_\lambda(v^{-1})\big)$.

Recall from Theorem 1.35 that irreducible $\mathscr{R}_n^\Lambda$-modules are self-dual. Moreover, in quantum characteristic two, $M^{\mathrm{sgn}} \cong M$ for any $\mathscr{R}_n^\Lambda$-module M. We also have from Theorem 3.13 the following well known consequence.

Corollary 3.15. *Let $e = 2$. For $\lambda \in \mathscr{P}_n^l$, S_λ and $S_{\lambda'}$ share the same ungraded composition factors, up to isomorphism.*

Hence, we know the composition factors of $S_{(n-m,1^m)}$ for all $m \in \{0, \dots, n-1\}$. In fact, we observe that the ladder numbers for the space $\mathbb{N} \times \mathbb{N}$ are

1	2	3	4	5	6	⋯
2	3	4	5	6	7	⋯
3	4	5	6	7	8	⋯
4	5	6	7	8	9	⋯
5	6	7	8	9	10	⋯
⋮	⋮	⋮	⋮	⋮	⋮	

,

which follows from Section 1.2.3. It is now easy to observe that the 2-regularisation of a hook partition $(n-m, 1^m)$ is $(n-m, m)$ if $m \leqslant \frac{n}{2}$, and hence $D_{(n-m,m)}$ is a composition factor of $S_{(n-m,1^m)}$. We thus can ascertain the following ungraded decomposition numbers for $\mathscr{R}_n^\Lambda$.

Proposition 3.16. *Let* $\operatorname{char}(\mathbb{F}) = 0$ *and $e = 2$.*

1. *If $n \equiv 3 \pmod 4$, then part of the decomposition matrix for $\mathscr{R}_n^\Lambda$ comprising rows*

corresponding to hook partitions is

$$
\begin{array}{c}
S_{(n)} \\
S_{(n-1,1)} \\
S_{(n-2,1^2)} \\
S_{(n-3,1^3)} \\
\vdots \\
S_{\left(n-\lfloor\frac{n}{2}\rfloor+1,1^{\lfloor\frac{n}{2}\rfloor-1}\right)} \\
S_{\left(n-\lfloor\frac{n}{2}\rfloor,1^{\lfloor\frac{n}{2}\rfloor}\right)} \\
S_{\left(n-\lfloor\frac{n}{2}\rfloor-1,1^{\lfloor\frac{n}{2}\rfloor+1}\right)} \\
\vdots \\
S_{(4,1^{n-4})} \\
S_{(3,1^{n-3})} \\
S_{(2,1^{n-2})} \\
S_{(1^n)}
\end{array}
\left(\begin{array}{ccccccc|c}
1 & & & & & & & \\
 & 1 & & & & & & \\
1 & & 1 & & & & & \\
 & 1 & & 1 & & & & \\
 & & \ddots & & \ddots & & & \\
 & 1 & & 1 & \cdots & 1 & & \\
1 & & 1 & & 1 & \cdots & 1 & 0 \\
 & 1 & & 1 & \cdots & 1 & & \\
 & & \iddots & & \iddots & & & \\
 & 1 & & 1 & & & & \\
1 & & 1 & & & & & \\
 & 1 & & & & & & \\
1 & & & & & & &
\end{array}\right),
$$

where the columns are labelled by $D_{(n)}, D_{(n-1,1)}, \ldots, D_{\left(n-\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor\right)}$ *from left to right.*

2. *If* $n \equiv 1 \pmod 4$, *then part of the decomposition matrix for* $\mathscr{R}_n^\Lambda$ *comprising rows corresponding to hook partitions is*

$$
\begin{array}{c}
S_{(n)} \\
S_{(n-1,1)} \\
S_{(n-2,1^2)} \\
\vdots \\
S_{\left(n-\lfloor\frac{n}{2}\rfloor+1,1^{\lfloor\frac{n}{2}\rfloor-1}\right)} \\
S_{\left(n-\lfloor\frac{n}{2}\rfloor,1^{\lfloor\frac{n}{2}\rfloor}\right)} \\
S_{\left(n-\lfloor\frac{n}{2}\rfloor-1,1^{\lfloor\frac{n}{2}\rfloor+1}\right)} \\
\vdots \\
S_{(3,1^{n-3})} \\
S_{(2,1^{n-2})} \\
S_{(1^n)}
\end{array}
\left(\begin{array}{ccccccc|c}
1 & & & & & & & \\
 & 1 & & & & & & \\
1 & & 1 & & & & & \\
 & \ddots & & \ddots & & & & \\
1 & & 1 & \cdots & 1 & & & \\
 & 1 & & 1 & & \cdots & 1 & 0 \\
1 & & 1 & \cdots & 1 & & & \\
 & \iddots & & \iddots & & & & \\
1 & & 1 & & & & & \\
 & 1 & & & & & & \\
1 & & & & & & &
\end{array}\right),
$$

where the columns are labelled by $D_{(n)}, D_{(n-1,1)}, \ldots, D_{(n-\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor)}$ *from left to right.*

Now that we have established the ungraded decomposition matrices for $\mathscr{R}_n^\Lambda$, whose rows correspond to hook representations with n odd, when the ground field has characteristic zero and the quantum characteristic is two, we seek graded analogues. For completeness, we first determine the graded dimensions of Specht modules labelled by hooks partitions in quantum characteristic two.

Lemma 3.17. *Let $e = 2$. If n is odd, then the 2-core of $(n-m, 1^m)$ is (1) or $(2,1)$ and its 2-weight is $\lfloor\frac{m}{2}\rfloor + \lfloor\frac{n-m-1}{2}\rfloor$.*

Proof. It is obvious that the 2-core of $(n-m, 1^m)$ is non-empty. Its 2-weight is the sum of the total number of consecutive rim 2-hooks we can remove from its arm $\lfloor\frac{n-m-1}{2}\rfloor$ and the total number of consecutive rim 2-hooks we can remove from its leg $\lfloor\frac{m}{2}\rfloor$. □

Lemma 3.18. *Let $e = 2$ and i, k be such that $2 \leqslant i \leqslant m+1$ and $i \leqslant k \leqslant n$. Then $(1, k-i+2)$ is an addable $(1-i)$-node of $(k-i+1, 1^{i-1})$ strictly above $(i,1)$ if and only if k is even.*

Proof. ($\Longleftarrow$) Follows from Lemma 2.4.
($\Longrightarrow$) Supposing that k is odd, we know that $\mathrm{res}(1, k-i+1) = \mathrm{res}(i,1)$. Hence $(1, k-i+1, 1)$ is a removable $(1-i)$-node of $(k-i+1, 1^{i-1})$ strictly above $(i,1)$. □

Proposition 3.19. *Let $e = 2$.*

1. *If $m \leqslant \lfloor\frac{n}{2}\rfloor$, then*

$$\operatorname{grdim}\left(S_{(n-m,1^m)}\right) = \sum_{i=0}^{m} \binom{\lfloor\frac{n}{2}\rfloor}{m-i}\binom{\lfloor\frac{n-1}{2}\rfloor}{i} v^{(m+\lfloor\frac{m}{2}\rfloor-2i)}.$$

2. *If $m > \lfloor\frac{n}{2}\rfloor$, then*

$$\operatorname{grdim}\left(S_{(n-m,1^m)}\right) = \sum_{i=0}^{m} \binom{\lfloor\frac{n}{2}\rfloor}{m-i}\binom{\lfloor\frac{n-1}{2}\rfloor}{i} v^{(n-m-1+\lfloor\frac{m}{2}\rfloor-2i)}.$$

Proof. 1. Let $T \in \mathrm{Std}((n-m, 1^m))$. Then there are $\lfloor\frac{n}{2}\rfloor$ even entries in T and $\lfloor\frac{n+1}{2}\rfloor$ odd entries in T, including 1 which lies in $(1,1)$. By Lemma 2.5 we know that

$$\deg(T) = \lfloor\tfrac{m}{2}\rfloor + \#\{i \mid T(i,1) \text{ is even}\} - \#\{i \mid T(i,1) \text{ is odd}\}.$$

Since $m \leqslant \lfloor\frac{n}{2}\rfloor$, entries in the leg of T can all be even or all be odd, so that the leading degree in the graded dimension of $S_{(n-m,1^m)}$ is $\lfloor\frac{m}{2}\rfloor + m$, and its trailing degree is $\lfloor\frac{m}{2}\rfloor - m$.

Also, by Lemma 2.5, the degree of T is determined by the entries in its leg. If we place i odd entries in the leg of T, then the remaining $m-i$ nodes in its leg contain even entries, and clearly, there are $\binom{\lfloor \frac{n}{e} \rfloor}{m-i}\binom{\lfloor \frac{n-1}{2} \rfloor}{i}$ possible standard $(n-m,1^m)$-tableaux with this number of odd and even entries in its leg.

2. By Corollary 3.14 and Lemma 3.17, we have

$$\operatorname{grdim}\left(S_{(m+1,1^{n-m-1})}\right) = v^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n-m-1}{2} \rfloor} \operatorname{grdim}\left(S_{(n-m,1^m)}(v^{-1})\right).$$

Hence, we need only determine the leading degree in the graded dimension of $S_{(n-m,1^m)}$, then the result is proven similarly to the previous part. The trailing degree in the graded dimension of $S_{(m+1,1^{n-m-1})}$ is $\lfloor \frac{m}{2} \rfloor - m$ by the previous part, and thus the leading degree in $\operatorname{grdim}\left(S_{(n-m,1^m)}\right)$ is $m - \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n-m-1}{2} \rfloor$, as required.

□

We now introduce a couple of useful results involving two-part partitions, which are proven similarly to Lemma 2.4 and Lemma 2.5, respectively.

Lemma 3.20. *Let $e = 2$, $T \in \operatorname{Std}(n-m,m)$ with $T(2,i) = k$, for some i,k, with $1 \leqslant i \leqslant m$ and $2i \leqslant k \leqslant n$.*

1. *If k is even, then $(1, k-i+1)$ is an addable k-node of $(k-i,i)$;*
2. *If k is odd, then $(1, k-i)$ is a removable k-node of $(k-i,i)$.*

Lemma 3.21. *Let $e = 2$, $T \in \operatorname{Std}(n-m,m)$ and $1 \leqslant i \leqslant m$. Then*

$$\deg(T) = \#\left\{i \mid T(2,i) \text{ is even}\right\} - \#\left\{i \mid T(2,i) \text{ is odd}\right\}.$$

Recall from Section 3.1 that $T \in \operatorname{Std}(n-m,1^m)$ is determined by $a_2,\ldots,a_{m+1}$ such that $1 < a_2 < \cdots < a_{m+1} \leqslant n$, where $T = s\overset{a_2-1}{\underset{2}{\downarrow}} \ldots s\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} T_{(n-m,1^m)}$.

Now let $T \in \operatorname{Std}(n-m,m)$ and write $b_j := T(2,j)$ for $1 \leqslant j \leqslant m$. Then T is completely determined by $b_1,\ldots,b_m$ such that $1 < b_1 < \cdots < b_m \leqslant n$ and $b_j \geqslant 2j$ for all $j \in \{1,\ldots,m\}$. We can thus write

$$T = s\overset{b_1-1}{\underset{2}{\downarrow}}\, s\overset{b_2-1}{\underset{4}{\downarrow}}\, s\overset{b_3-1}{\underset{6}{\downarrow}} \ldots s\overset{b_m-1}{\underset{2m}{\downarrow}} T_{(n-m,m)}.$$

Lemma 3.22. *Suppose that $n \geqslant 2m$ and let*

$$T = s\overset{a_2-1}{\underset{2}{\downarrow}}\, s\overset{a_3-1}{\underset{3}{\downarrow}} \ldots s\overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} T_{(n-m,1^m)}.$$

Then we have a bijection

$$f : \operatorname{Std}(n-m,1^m) \longrightarrow \operatorname{Std}(n-m,m) \cup \operatorname{Std}(n-m+2,1^{m-2}),$$

where,

1. *if* $T(r,1) \geqslant 2r-2$*, for all* $r \in \{2,\dots,m+1\}$*, then*

$$f(T) = s \overset{a_2-1}{\underset{2}{\downarrow}} s \overset{a_3-1}{\underset{4}{\downarrow}} \dots s \overset{a_{m+1}-1}{\underset{2m}{\downarrow}} T_{(n-m,m)};$$

2. *otherwise*

$$f(T) = s \overset{a_2-1}{\underset{2}{\downarrow}} \dots s \overset{a_{r-2}-1}{\underset{r-2}{\downarrow}} s \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots s \overset{a_{m+1}-1}{\underset{m-1}{\downarrow}} T_{(n-m+2,1^{m-2})},$$

where r *is minimal such that* $a_{r-1} = 2r-4$ *and* $a_r = 2r-3$*.*

Proof. If there exists an $r \in \{2,\dots,m+1\}$ such that $T(r,1) \neq 2r-2$, it is clear that $f(T)$ is indeed a standard $(n-m+2,1^{m-2})$-tableau.

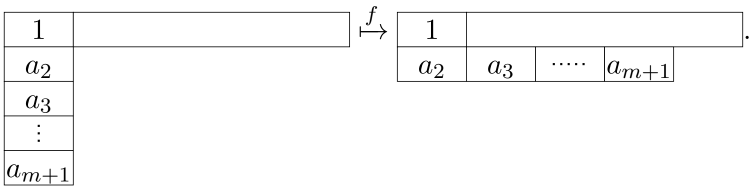
We instead assume otherwise and let S be an $(n-m,m)$-tableau. For S to be standard, we require $S(2,r) \geqslant 2r$, for all $r \in \{1,\dots,m\}$. By the action of f, we find that $f(T(r,1)) = S(2,r-1)$ where $T(r,1) \geqslant 2r-2$, for all $r \in \{2,\dots,m+1\}$. Thus $f(T) \in \mathrm{Std}(n-m,m)$, and moreover, f is well-defined.

We now observe that

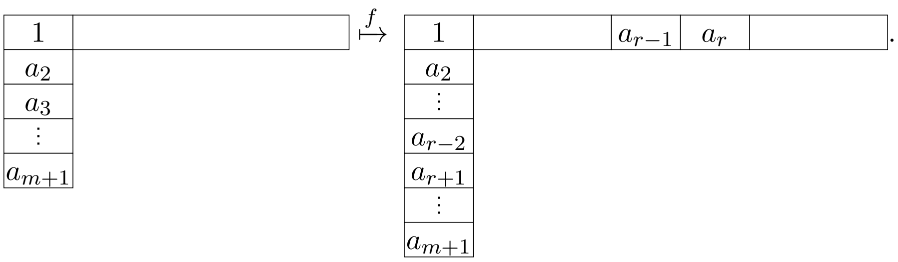
$$\begin{aligned}
&\mathrm{Std}(n-m,1^m) \\
&= \{T \in \mathrm{Std}(n-m,1^m) \mid T(r,1) \geqslant 2r-2,\ \forall r \in \{2,\dots,m+1\}\} \\
&\ + \{T \in \mathrm{Std}(n-m,1^m) \mid T(r,1) \geqslant 2r-3,\ \text{for some } r \in \{3,\dots,m+1\}\},
\end{aligned}$$

and hence f is bijective. □

That is, if $T(r,1) \geqslant 2r-2$, for all $r \in \{2,\dots,m+1\}$, then we can informally think of f acting on T by rotating its hook 90° anticlockwise as follows



Otherwise, we can think of f acting on T by moving the two nodes in the leg of T containing entries a_{r-1} and a_r to its arm as follows



Lemma 3.23. *Let $e = 2$,*

$$\mathscr{T} = \{T \in \mathrm{Std}(n-m, 1^m) \ \mid \ T(r,1) \geqslant 2r-2, \ \forall r \in \{2, \dots, m+1\}\}$$

and

$$\mathscr{S} = \{T \in \mathrm{Std}(n-m, 1^m) \ \mid \ T(r,1) \geqslant 2r-3, \ \textit{for some } r \in \{3, \dots, m+1\}\}.$$

Then

$$\deg(T) = \begin{cases} \deg(f(T)) + \lfloor \frac{m}{2} \rfloor & \textit{if } T \in \mathscr{T}; \\ \deg(f(T)) + 1 & \textit{if } T \in \mathscr{S}. \end{cases}$$

Proof. Let $T = s \overset{a_2-1}{\underset{2}{\downarrow}} s \overset{a_3-1}{\underset{3}{\downarrow}} \dots s \overset{a_{m+1}-1}{\underset{m+1}{\downarrow}} T_{(n-m,1^m)}$ and recall from Lemma 2.5 that

$$\deg(T) = \lfloor \tfrac{m}{e} \rfloor + \#\{i \mid T(i,1) \text{ is even}\} - \#\{i \mid T(i,1) \text{ is odd}\}, \tag{3.4.1}$$

for $2 \leqslant i \leqslant m+1$.

Let us first suppose that $T \in \mathscr{T}$, so that $f(T) \in \mathrm{Std}(n-m, m)$. Then, by Lemma 3.21,

$$\deg(f(T)) = \#\{i \mid T(2,i) \text{ is even}\} - \#\{i \mid T(2,i) \text{ is odd}\}, \tag{3.4.2}$$

for $1 \leqslant i \leqslant m$. We know that T contains $a_2, \dots, a_{m+1}$ in its leg, and that these entries are mapped under f, by Lemma 3.22, to the second row of $f(T)$. Hence on comparing Equation (3.4.1) and Equation (3.4.2), we have $\deg(T) = \deg(f(T)) + \lfloor \frac{m}{2} \rfloor$.

Now suppose that $T \in \mathscr{S}$, so that $f(T) \in \mathrm{Std}(n-m+2, 1^{m-2})$. By Lemma 2.5, we have

$$\deg(f(T)) = \lfloor \tfrac{m}{2} \rfloor - 1 + \#\{i \mid T(i,1) \text{ is even}\} - \#\{i \mid T(i,1) \text{ is odd}\}, \tag{3.4.3}$$

for $2 \leqslant i \leqslant m-1$. We know from Lemma 3.22 that the entries $a_2, \dots, a_{r-2}, a_{r+1} \dots, a_{m+1}$ are mapped under f to the leg of $f(T)$, whilst the entries a_{r-1} and a_r are mapped to the arm of $f(T)$. Since the parity of a_{r-1} is different to that of a_r, then by comparing Equation (3.4.1) and Equation (3.4.3), we have

$$\begin{aligned} \deg(T) &= \lfloor \tfrac{m}{2} \rfloor + \#\{i \mid T(i,1) \text{ is even}, \ i \neq r-1\} - \#\{i \mid T(i,1) \text{ is odd}, \ i \neq r\} \\ &= \deg(f(T)) + 1, \end{aligned}$$

for $2 \leqslant i \leqslant m+1$. □

We can now prove the main result of this subsection, by implicitly drawing on Proposition 3.16.

Theorem 3.24. *Let* $\mathrm{char}(\mathbb{F}) = 0$*,* $e = 2$ *and* n *be odd.*

1. *If* $m \leqslant \lfloor \frac{n}{2} \rfloor$*, then* $[S_{(n-m,1^m)} : S_{(n-m+2i,m-2i)}]_v = v^{\lfloor \frac{m}{2} \rfloor}$*, for* $i \geqslant 0$*.*

2. *If* $m > \lfloor \frac{n}{2} \rfloor$*, then* $[S_{(n-m,1^m)} : S_{(m+1+2i,n-m-1-2i)}]_v = v^{\lfloor \frac{m}{2} \rfloor}$*, for* $i \geqslant 0$*.*

Moreover, $[S_{(n-m,1^m)} : D] = 0$ *for any other graded irreducible* $\mathscr{R}_n^\Lambda$*-module.*

Proof. 1. Let $\alpha_{m-2i} \in \mathbb{Z}$ for $i \geqslant 0$. By the composition series of $S_{(n-m,1^m)}$ given in Proposition 3.16, we have

$$\mathrm{grdim}\left(S_{(n-m,1^m)}\right) = \begin{cases} v^{\alpha_m}\,\mathrm{grdim}\left(S_{(n-m,m)}\right) + v^{\alpha_{m-2}}\,\mathrm{grdim}\left(S_{(n-m+2,m-2)}\right) + \cdots \\ \qquad \cdots + v^{\alpha_1}\,\mathrm{grdim}\left(S_{(n-1,1)}\right) & \text{if } m \text{ is odd;} \\ v^{\alpha_m}\,\mathrm{grdim}\left(S_{(n-m,m)}\right) + v^{\alpha_{m-2}}\,\mathrm{grdim}\left(S_{(n-m+2,m-2)}\right) + \cdots \\ \qquad \cdots + v^{\alpha_0}\,\mathrm{grdim}\left(S_{(n)}\right) & \text{if } m \text{ is even.} \end{cases}$$

We want to show that $\alpha_m = \alpha_{m-2} = \alpha_{m-4} = \cdots = \lfloor \frac{m}{2} \rfloor$; we proceed by induction on m.

It is obvious that $\mathrm{grdim}\, S_{(n-1,1)} = v^{\lfloor \frac{1}{2} \rfloor}\,\mathrm{grdim}\, S_{(n-1,1)}$ and $\mathrm{grdim}\, S_{(n)} = v^{\lfloor \frac{0}{2} \rfloor}\,\mathrm{grdim}\, S_{(n)}$.

We assume that

$$\mathrm{grdim}\, S_{(n-m+2,1^{m-2})} = v^{\left(\lfloor \frac{m}{2} \rfloor - 1\right)} \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \mathrm{grdim}\, S_{(n-m+2i,m-2i)},$$

for $m > 3$.

Now let

$$\mathscr{T} = \{T \in \mathrm{Std}(n-m, 1^m) \mid T(r,1) \geqslant 2r-2,\ \forall r \in \{2, \ldots, m+1\}\}$$

and

$$\mathscr{S} = \{T \in \mathrm{Std}(n-m, 1^m) \mid T(r,1) \geqslant 2r-3,\ \text{for some } r \in \{3, \ldots, m+1\}\}.$$

We know from Lemma 3.22 that $\dim S_{(n-m,1^m)} = \dim S_{(n-m,m)} + \dim S_{(n-m+2,1^{m-2})}$. Moreover, we know from Lemma 3.23 that

$$\mathrm{grdim}\, S_{(n-m,1^m)} = v^{\lfloor \frac{m}{2} \rfloor}\,\mathrm{grdim}\, S_{(n-m,m)} + v\,\mathrm{grdim}\, S_{(n-m+2,1^{m-2})},$$

since $\lfloor \frac{m}{2} \rfloor = \deg(T) - \deg\left(f(T)\right)$ if $T \in \mathscr{T}$ and $1 = \deg(T) - \deg\left(f(T)\right)$ if $T \in \mathscr{S}$.

Hence, by the inductive hypothesis,

$$v^{\lfloor \frac{m}{2} \rfloor}\,\mathrm{grdim}\, S_{(n-m,m)} + v\,\mathrm{grdim}\, S_{(n-m+2,1^{m-2})}$$

$$= v^{\lfloor \frac{m}{2} \rfloor} \operatorname{grdim} S_{(n-m,m)} + v^{(\lfloor \frac{m}{2} \rfloor)} \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \operatorname{grdim} S_{(n-m+2i,m-2i)}$$

$$= v^{(\lfloor \frac{m}{2} \rfloor)} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \operatorname{grdim} S_{(n-m+2i,m-2i)},$$

for all $m \geqslant 0$, and we are done.

2. We know that $S_{(m+1,1^{n-m-1})}$ shares the same composition factors (up to isomorphism) as $S_{(n-m,1^m)}$ by Corollary 3.15. Thus, for $m < \lfloor \frac{n}{2} \rfloor$, $S_{(n-m,1^m)}$ has composition factors

$$S_{(m+1,n-m-1)}, S_{(m+3,n-m-3)}, S_{(m+5,n-m-5)}, \ldots,$$

from bottom to top. Following a similar inductive argument as in the first part of the proof, we obtain our desired result.

□

Recall that $(n-m,1^m)^R = (n-m,m)$ if $m \leqslant \frac{n}{2}$, and hence Theorem 3.24 gives an alternative approach to Chuang, Miyachi and Tan, whereby this result coincides with [CMT, Theorem 1(2)], for odd n.

Example 3.25. *Let* $\operatorname{char}(\mathbb{F}) = 0$ *and* $e = 2$. *Then part of the graded decomposition matrix for* $\mathscr{R}_{11}^{\Lambda}$ *comprising rows corresponding to hook partitions is*

$$\begin{array}{c} S_{(11)} \\ S_{(10,1)} \\ S_{(9,1^2)} \\ S_{(8,1^3)} \\ S_{(7,1^4)} \\ S_{(6,1^5)} \\ S_{(5,1^6)} \\ S_{(4,1^7)} \\ S_{(3,1^8)} \\ S_{(2,1^9)} \\ S_{(1^{11})} \end{array}
\left(\begin{array}{cccccc|c}
1 & & & & & & \\
 & 1 & & & & & \\
v & & v & & & & \\
 & v & & v & & & \\
v^2 & & v^2 & & v^2 & & \\
 & v^2 & & v^2 & & v^2 & \Large 0 \\
v^3 & & v^3 & & v^3 & & \\
 & v^3 & & v^3 & & & \\
v^4 & & v^4 & & & & \\
 & v^4 & & & & & \\
v^5 & & & & & &
\end{array}\right).$$

Chapter 4

Hook representations in quantum characteristic at least 3

We continue the study of the hook representations for the cyclotomic Khovanov–Lauda–Rouquier algebra $\mathscr{R}_n^\Lambda$ to $e \geqslant 3$. Recall that Peel [P] found the decomposition numbers for hook representations over the symmetric group algebra in odd characteristic. James [J4] developed this theory for the Iwahori–Hecke algebras over $\mathbb{C}$ at an eth root of unity. Using the recently introduced machinery of $\mathscr{R}_n$, we provide the analogous graded decomposition numbers corresponding to the hook representations. This is an alternative to Chuang–Miyachi–Tan [CMT, Theorem(1)], who discovered the corresponding v-decomposition numbers for the Iwahori–Hecke algebra of type A, prior to the development of graded representation theory of the Hecke algebras, via the Fock space. In fact, these v-decomposition numbers are shown to be equal to the graded decomposition numbers in [BK3], so our result is indeed equivalent to Chuang, Miyachi and Tan's result.

4.1 Ungraded decomposition numbers

Peel [P] obtained results for the decomposition of modular representations for the symmetric group in odd characteristic, corresponding to hook partitions. The following result is known as Peel's Theorem.

Theorem 4.1. *[P, Theorem 2] Let $\mathbb{F}$ be a field with characteristic p, where p is odd.*

1. *If $p \nmid n$, then the Specht modules $S_{(n-m,1^m)}$ for $0 \leqslant m \leqslant n-1$ are pairwise non-isomorphic and irreducible.*

2. *If $p \mid n$, then there are $n-1$ distinct p-regular partitions $\lambda_1, \ldots, \lambda_{n-1}$ of n, ordered*

lexicographically, such that

$$[S_{(n-m,1^m)} : D_\mu] = \begin{cases} 1 & \text{if } \mu = \lambda_m \text{ or } \mu = \lambda_{m+1}; \\ 0 & \text{otherwise.} \end{cases}$$

In [J4, Theorem 6.22], James states a q-analogous version of Peel's Theorem, where q is a primitive eth root of unity, $e > 2$ and the characteristic of the ground field $\mathbb{F}$ is arbitrary. Thus, James provides us with an analogous result for the Iwahori–Hecke algebras of type A, and hence for the Khovanov–Lauda–Rouquier algebras in level 1. We let $e \geqslant 3$ from now on.

4.2 Graded Specht modules $S_{(n-m,1^m)}$ with $e \nmid n$

Suppose that $n \not\equiv 0 \pmod e$. We deduce from Theorem 4.1 that $S_{(n-m,1^m)} \cong D_{(n-m,1^m)^R}$ as ungraded $\mathscr{R}_n^\Lambda$-modules. To determine the graded decomposition numbers corresponding to hook partitions, we need only find $S_{(n-m,1^m)}$ in terms of a grading shift on $D_{(n-m,1^m)^R}$.

Proposition 4.2. *Suppose $e \nmid n$. Then the leading and trailing term, respectively, in the graded dimension of $S_{(n-m,1^m)}$ are*

1. $\binom{\lfloor \frac{n}{e} \rfloor}{m} v^{m+\lfloor \frac{m}{e} \rfloor}$ *and* $\binom{\lfloor \frac{n}{e} \rfloor}{m} v^{-m+\lfloor \frac{m}{e} \rfloor}$, *if* $1 \leqslant m \leqslant \lfloor \frac{n}{e} \rfloor$,
2. $\binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor}$ *and* $\binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{-\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor}$, *if* $\lfloor \frac{n}{e} \rfloor < m < n - \lfloor \frac{n}{e} \rfloor - 1$,
3. $\binom{\lfloor \frac{n}{e} \rfloor}{n-m-1} v^{n-m-1+\lfloor \frac{m}{e} \rfloor}$ *and* $\binom{\lfloor \frac{n}{e} \rfloor}{n-m-1} v^{m-n+1+\lfloor \frac{m}{e} \rfloor}$, *if* $n - \lfloor \frac{n}{e} \rfloor - 1 \leqslant m \leqslant n-1$.

Moreover, $S_{(n-m,1^m)} \cong D_{(n-m,1^m)^R} \langle \lfloor \frac{m}{e} \rfloor \rangle$ as graded $\mathscr{R}_n^\Lambda$-modules.

Proof. Let $\mathscr{T} = \operatorname{Std}(n-m, 1^m)$ and $T \in \mathscr{T}$. We note that there are $\lfloor \frac{n}{e} \rfloor$ entries congruent to 0 modulo e, $\lfloor \frac{n}{e} \rfloor + 1$ entries congruent to 1 modulo e, and $n - 2\lfloor \frac{n}{e} \rfloor - 1$ entries congruent to neither 0 modulo e nor 1 modulo e lying in T. Since $S_{(n-m,1^m)}$ is irreducible, the coefficients of the leading and trailing terms in its graded dimension are equal. Moreover, if we suppose that $S, T \in \mathscr{T}$ such that $\operatorname{maxdeg}(\mathscr{T}) = \deg(T)$ and $\operatorname{mindeg}(\mathscr{T}) = \deg(S)$, then $\operatorname{mindeg}(\mathscr{T}) = \operatorname{maxdeg}(\mathscr{T}) - 2\max(A_{\mathscr{T}})$, where $A_{\mathscr{T}}$ is the set as defined in Equation (2.2.1).

1. By Lemma 2.5, T is formed from the Young diagram $[(n-m, 1^m)]$ by placing m of the $\lfloor \frac{n}{e} \rfloor$ entries congruent to 0 modulo e in its leg. Hence $\max(A_{\mathscr{T}}) = m$.

2. By Lemma 2.5, T is formed from the Young diagram $[(n-m,1^m)]$ by placing all of the $\lfloor \frac{n}{e} \rfloor$ congruent to 0 modulo e in the leg of T, together with $n-2\lfloor \frac{n}{e} \rfloor - 1$ entries congruent to neither 0 modulo e nor 1 modulo e in the remaining nodes in the leg of T. Hence $\max(A_{\mathscr{T}}) = \lfloor \frac{n}{e} \rfloor$.

3. By Lemma 2.5, T is formed from the Young diagram $[(n-m,1^m)]$ by placing all of the $\lfloor \frac{n}{e} \rfloor$ entries congruent to 0 modulo e and all of the $n-2\lfloor \frac{n}{e} \rfloor - 1$ entries congruent to neither 0 modulo e nor 1 modulo e in the leg of T, together with $m-n+\lfloor \frac{n}{e} \rfloor + 1$ entries congruent to 1 modulo e in the remaining nodes in the leg of T. Hence $\max(A_{\mathscr{T}}) = n-m-1$.

□

Example 4.3. *For $e = 3$, part of the graded decomposition matrix for $\mathscr{R}_7^\Lambda$ comprising rows corresponding to hook partitions has the form*

$$\begin{array}{c} \\ S_{(7)} \\ S_{(6,1)} \\ S_{(5,1^2)} \\ S_{(4,1^3)} \\ S_{(3,1^4)} \\ S_{(2,1^5)} \\ S_{(1^7)} \end{array} \begin{array}{c} \begin{array}{ccccccc} D_{(7)} & D_{(6,1)} & D_{(5,1^2)} & D_{(4,2,1)} & D_{(3,2^2)} & D_{(3^2,1)} & D_{(4,3)} \end{array} \\ \left(\begin{array}{ccccccc|c} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & v & & & & 0 \\ & & & & v & & & \\ & & & & & v & & \\ & & & & & & v^2 & \end{array}\right). \end{array}$$

4.3 Graded Specht modules $S_{(n-m,1^m)}$ with $e \mid n$

Suppose that $n \equiv 0 \pmod{e}$. It follows from Theorem 4.1 that we can order the rows and columns of the decomposition matrix for $\mathscr{R}_n^\Lambda$ so that the submatrix comprising the rows corresponding to hook partitions has the form

$$\begin{array}{c} S_{(n)} \\ S_{(n-1,1)} \\ S_{(n-2,1^2)} \\ \vdots \\ S_{(3,1^{n-3})} \\ S_{(2,1^{n-2})} \\ S_{(1^n)} \end{array} \left(\begin{array}{cccccc|c} 1 & & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & \ddots & \ddots & & & 0 \\ & & & 1 & 1 & & \\ & & & & 1 & 1 & \\ & & & & & 1 & \end{array}\right).$$

Further, the first $n-1$ columns correspond to the e-regular partitions $\lambda_1, \dots, \lambda_{n-1}$, from left to right. Thus, to obtain the analogous graded decomposition submatrix for $\mathscr{R}_n^\Lambda$, we need only find the graded dimensions of $S_{(n-m),(1^m)}$ and D_{λ_m}.

We know from Theorem 4.1 that the partition λ_m is the e-regularisation of a particular hook partition $(n-i, 1^i)$. The next result enables us to determine i, for $0 \leqslant m \leqslant n-1$.

Lemma 4.4. *If $e \mid n$, then*

$$\left(\frac{n+e}{e}, 1^{\left(\frac{e(n-1)-n}{e}\right)}\right)^R = \left(\frac{n}{e}, 1^{\left(\frac{en-n}{e}\right)}\right)^R.$$

Proof. Observe that

$$\left[\left(\frac{n+e}{e}, 1^{\frac{e(n-1)-n}{e}}\right)\right] = \left[\left(\frac{n}{e}, 1^{\left(\frac{e(n-1)-n}{e}\right)}\right)\right] \cup \left\{\left(1, \frac{n+e}{e}\right)\right\}$$

and

$$\left[\left(\frac{n}{e}, 1^{\frac{e(n-1)-n}{e}}\right)\right] = \left[\left(\frac{n}{e}, 1^{\left(\frac{e(n-1)-n}{e}\right)}\right)\right] \cup \left\{\left(1, \frac{e(n+1)-n}{e}\right)\right\}.$$

Now, notice that nodes $(1, \frac{n+e}{e})$ and $(\frac{e(n+1)-n}{e})$ share ladder number $n+1-\frac{n}{e}$, where $(1, \frac{n+e}{e})$ is the highest node in ladder $\mathscr{L}_{n+1-\frac{n}{e}}$. Thus,

$$\left[\left(\frac{n+e}{e}, 1^{\frac{e(n-1)-n}{e}}\right)^R\right] = \left[\left(\frac{n}{e}, 1^{\left(\frac{e(n-1)-n}{e}\right)}\right)^R\right] \cup \left\{\left(1, \frac{n+e}{e}\right)\right\},$$

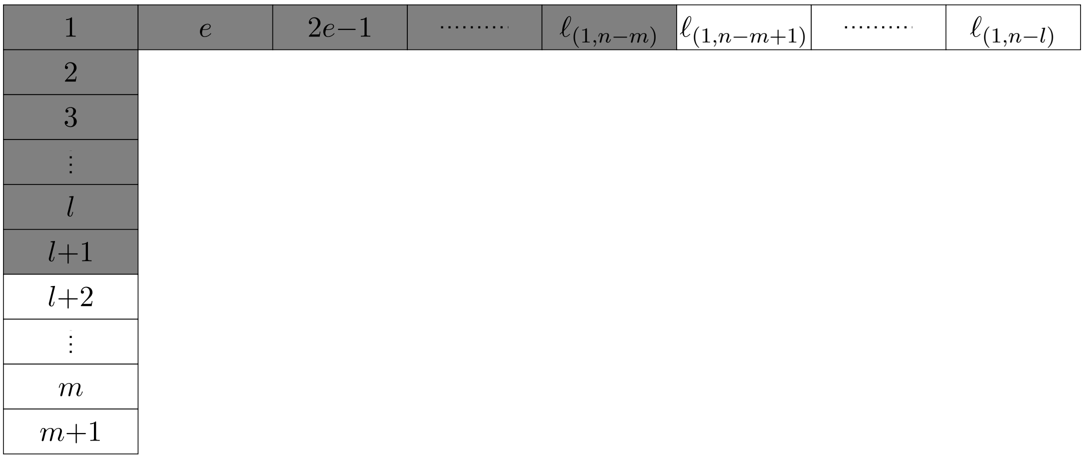
and by moving node $(\frac{e(n+1)-n}{e})$ to the highest node $(1, \frac{n+e}{e})$ in ladder $\mathscr{L}_{n+1-\frac{n}{e}}$, we have

$$\left[\left(\frac{n}{e}, 1^{\frac{e(n-1)-n}{e}}\right)^R\right] = \left[\left(\frac{n}{e}, 1^{\left(\frac{e(n-1)-n}{e}\right)}\right)^R\right] \cup \left\{\left(1, \frac{n+e}{e}\right)\right\}.$$

□

Lemma 4.5. *For $e \mid n$ and $l < m$, $(n-m, 1^m)^R = (n-l, 1^l)^R$ if and only if $l = n-1-\frac{n}{e}$ and $m = n - \frac{n}{e}$.*

Proof. It is clear that the e-regularisation of $(n-m, 1^m)^R$ can only equal the e-regularisation $(n-l, 1^l)^R$ when the set of ladder numbers contained in the Young diagrams of $(n-m, 1^m)$ and $(n-l, 1^l)$ are equal. The ladders numbers of nodes in $[(n-l, 1^l)] \cup [(n-m, 1^m)]$ are



with $\ell_{(1,n-m)} = 1 + (n-m-1)(e-1)$, $\ell_{(1,n-m+1)} = 1 + (n-m)(e-1)$ and $\ell_{(1,n-l)} = 1 + (n-l-1)(e-1)$. The ladder numbers that lie in both $[(n-l,1^l)]$ and $[(n-m,1^m)]$ are shaded, and the unshaded nodes lying in the arm (respectively, leg) of $[(n-l,1^l)] \cup [(n-m,1^m)]$ lie in $[(n-l,1^l)]$ (respectively, $[(n-m,1^m)]$). Hence, if the ladder numbers contained in the Young diagrams of $(n-m,1^m)$ and $(n-l,1^l)$ are equal, then $\ell_{(1,n-m)} = l+2, \ell_{(1,n-m+1)} = l+3, \ldots, \ell_{(1,n-l)} = m+1$. It is obvious that only one of these $m-l$ equalities holds, and thus $l = m-1$ and $\ell_{(1,n-l)} = m+1$, as required. □

Hence, we know that the only hook partitions which have the same e-regularisation are those given in Lemma 4.4. We can now define

$$\lambda_m := \begin{cases} (n-m,1^m)^R & \text{if } 0 \leqslant m \leqslant \frac{e(n-1)-n}{e}; \\ (n-m-1,1^{m+1})^R & \text{if } \frac{en-n}{e} \leqslant m \leqslant n-1. \end{cases} \tag{4.3.1}$$

Then $S_{(n)} \cong D_{(n)}$ and $S_{(1^n)} \cong D_{(1^n)^R}$, and for $1 \leqslant m \leqslant n-2$, $S_{(n-m,1^m)}$ has composition factors

- ◇ $D_{(n-m+1,1^{m-1})^R}$ and $D_{(n-m,1^m)^R}$, if $1 \leqslant m \leqslant \frac{e(n-1)-n}{e}$;
- ◇ $D_{(n-m,1^m)^R}$ and $D_{(n-m-1,1^{m+1})^R}$, if $\frac{en-n}{e} \leqslant m \leqslant n-2$.

Proposition 4.6. *Suppose $e \mid n$.*

1. *If $1 \leqslant m \leqslant \frac{n-e}{e}$, then the leading and trailing terms, respectively, of the graded dimension of $S_{(n-m,1^m)}$ are*

$$\binom{\frac{n}{e}}{m} v^{\left(m+\lfloor \frac{m}{e} \rfloor\right)} \text{ and } \binom{\frac{n-e}{e}}{m} v^{\left(-m+\lfloor \frac{m}{e} \rfloor\right)}.$$

2. *If $\frac{n}{e} \leqslant m \leqslant \frac{(e-1)n-e}{e}$, then the first two leading terms in the graded dimension of $S_{(n-m,1^m)}$ are*

$$\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\left(\frac{n}{e}+\lfloor \frac{m}{e} \rfloor\right)} \text{ and } \left(\frac{n}{e}\binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}} + \frac{n-e}{e}\binom{\frac{(e-2)n}{e}}{\frac{e(m-1)-n}{e}}\right) v^{\left(\frac{n}{e}+\lfloor \frac{m}{e} \rfloor -1\right)},$$

and the last two trailing terms in the graded dimension of $S_{(n-m,1^m)}$ are

$$\left(\frac{n-e}{e}\binom{\frac{(e-2)n}{e}}{\frac{e(m+2)-n}{e}}+\frac{n}{e}\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}}\right)v^{\left(2-\frac{n}{e}+\lfloor\frac{m}{e}\rfloor\right)} \text{ and } \binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}}v^{\left(1-\frac{n}{e}+\lfloor\frac{m}{e}\rfloor\right)}.$$

3. *If $\frac{(e-1)n}{e} \leqslant m \leqslant n-1$, then the leading and trailing terms, respectively, of the graded dimension of $S_{(n-m,1^m)}$ are*

$$\binom{\frac{n-e}{e}}{n-m-1}v^{\left(n-m+\lfloor\frac{m}{e}\rfloor\right)} \text{ and } \binom{\frac{n}{e}}{n-m-1}v^{\left(m-n+2+\lfloor\frac{m}{e}\rfloor\right)}.$$

Proof. Let $\mathscr{T}=\operatorname{Std}(n-m,1^m)$ and $T\in\mathscr{T}$. We note that there are $\frac{n}{e}$ entries congruent to 0 modulo e, $\frac{n}{e}$ entries congruent to 1 modulo e (including 1), and $\frac{(e-2)n}{e}$ entries congruent to neither 0 nor 1 modulo e lying in T. We now suppose that $S,T\in\mathscr{T}$ such that $\operatorname{maxdeg}(\mathscr{T})=\deg(T)$ and $\operatorname{mindeg}(\mathscr{T})=\deg(S)$.

1. By Lemma 2.5, T is formed from the Young diagram $[(n-m,1^m)]$ by placing m of the $\frac{n}{e}$ entries congruent to 0 modulo e in its leg. Hence, $\deg(T)=m+\lfloor\frac{m}{e}\rfloor$ by Equation (2.2.2). Similarly, we form S from $[(n-m,1^m)]$ by placing m of the $\frac{n-e}{e}$ entries congruent to 1 modulo e in its leg, and hence $\deg(S)=-m+\lfloor\frac{m}{e}\rfloor$.

2. By Lemma 2.5, T is formed from $[(n-m,1^m)]$ by placing all of the $\frac{n}{e}$ entries congruent to 0 modulo e in its leg, together with $m-\frac{n}{e}$ entries congruent to neither 0 nor 1 modulo e. Hence, $\deg(T)=\frac{n}{e}+\lfloor\frac{m}{e}\rfloor$, by Equation (2.2.2). Similarly, S is formed from $[(n-m,1^m)]$ by placing all of the $\frac{n}{e}-1$ entries congruent to 1 modulo e in its leg, together with $m-\frac{n}{e}+1$ entries congruent to neither 0 nor 1 modulo e. Hence, $\deg(S)=1-\frac{n}{e}+\lfloor\frac{m}{e}\rfloor$.

   Now let $S',T'\in\mathscr{T}$ such that $\deg(T')=\frac{n}{e}+\lfloor\frac{m}{e}\rfloor-1$ and $\deg(S')=1-\frac{n}{e}+\lfloor\frac{m}{e}\rfloor$. One can see that T' is formed from $[(n-m,1^m)]$ by either

   - placing $\frac{n}{e}-1$ entries congruent to 0 modulo e in its leg, together with $m-\frac{n}{e}+1$ entries congruent to neither 0 nor 1 modulo e,
   - or by placing $\frac{n}{e}$ entries congruent to 0 modulo e in its leg, together with $m-\frac{n}{e}-1$ entries congruent to neither 0 nor 1 modulo e, as well as 1 entry congruent to 1 modulo e.

   Further, S' is formed from $[(n-m,1^m)]$ by either

   - placing $\frac{n}{e}-1$ entries congruent to 1 modulo e in its leg, together with $m-\frac{n}{e}+2$ entries congruent to neither 0 nor 1 modulo e,
   - or by placing $\frac{n}{e}-1$ entries congruent to 1 modulo e in its leg, together with $m-\frac{n}{e}$ entries congruent to neither 0 nor 1 modulo e, as well as 1 entry congruent to 0 modulo e.

3. By Lemma 2.5, T is formed from $[(n-m,1^m)]$ by placing all of the $\frac{n}{e}$ entries congruent to 0 modulo e in its leg, together with all of the $n-\frac{2n}{e}$ entries congruent to neither 0 nor 1 modulo e, as well as $m-n+\frac{n}{e}$ entries congruent to 1 modulo e. In other words, there are $n-m-1$ entries congruent to 1 modulo e in the arm of T, and hence $\deg(T) = n-m+\lfloor \frac{m}{e} \rfloor$. Similarly, we form S from $[(n-m,1^m)]$ by placing all of the $\frac{n}{e}-1$ entries congruent to 1 modulo e in its leg, together with all of the $n-\frac{2n}{e}$ entries congruent to neither 0 nor 1 modulo e, as well as $m-n+\frac{n}{e}+1$ entries congruent to 0 modulo e. In other words, there are $n-m-1$ entries congruent to 0 modulo e in the arm of S, and hence $\deg(S) = m-n+2+\lfloor \frac{m}{e} \rfloor$.

□

Theorem 4.7. *1. If $0 \leqslant m \leqslant \frac{n-e}{e}$, then the leading term in the graded dimension of $D_{(n-m,1^m)^R}$ is*

$$\binom{\frac{n-e}{e}}{m} v^m,$$

2. If $\frac{n}{e} \leqslant m \leqslant \frac{(e-1)n-e}{e}$, then the two leading terms in the graded dimension of $D_{(n-m,1^m)^R}$ are

$$\binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}} v^{\frac{n-e}{e}} \text{ and } \frac{n-e}{e}\left(\binom{\frac{(e-2)n}{e}}{\frac{e(m+2)-n}{e}} + \binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}}\right) v^{\frac{n-2e}{e}},$$

3. If $n-\frac{n}{e} \leqslant m \leqslant n-1$, then the leading term in the graded dimension of $D_{(n-m,1^m)^R}$ is

$$\binom{\frac{n-e}{e}}{n-m-1} v^{n-m-1}.$$

Proof. 1. We proceed by induction on m. For $m=0$, it is obvious that $\operatorname{grdim}\left(D_{(n)}\right) = 1$. Now suppose that the leading term in the graded dimension of $D_{(n-m+1,1^{m-1})^R}$ is $\binom{\frac{n-e}{e}}{m-1} v^{m-1}$, for $m>0$. We have

$$\operatorname{grdim}\left(S_{(n-m,1^m)}\right) = v^{i_m} \operatorname{grdim}\left(D_{(n-m+1,1^{m-1})^R}\right) + v^{j_m} \operatorname{grdim}\left(D_{(n-m,1^m)^R}\right),$$

so that

$$\begin{aligned}\operatorname{grdim}\left(D_{(n-m,1^m)^R}\right) &= v^{-j_m}\left(\operatorname{grdim}\left(S_{(n-m,1^m)}\right) - v^{i_m}\operatorname{grdim}\left(D_{(n-m+1,1^{m-1})^R}\right)\right) \\ &= v^{-j_m}\left(\left(\binom{\frac{n}{e}}{m} v^{m+\lfloor \frac{m}{e} \rfloor} + \cdots + \binom{\frac{n-e}{e}}{m} v^{-m+\lfloor \frac{m}{e} \rfloor}\right.\right. \\ &\quad \left.\left. -v^{i_m}\left(\binom{\frac{n-e}{e}}{m-1} v^{m-1} + \cdots + \binom{\frac{n-e}{e}}{m-1} v^{1-m}\right)\right)\right),\end{aligned}$$

by Proposition 4.6. We observe that there are $2m+1$ and $2m-1$ terms, respectively, in the graded dimensions of $S_{(n-m,1^m)}$ and $D_{(n-m+1,1^{m-1})^R}$, and that the leading and trailing coefficients in the graded dimension of $D_{(n-m+1,1^{m-1})^R}$ are smaller than those in the graded dimension of $S_{(n-m,1^m)}$. Hence, there are $2m+1$ terms in the graded dimension of $D_{(n-m,1^m)^R}$. We require that the graded dimension of $D_{(n-m,1^m)^R}$ is symmetric in v and v^{-1}. Now, note that the difference of the leading coefficients in the graded dimensions of $S_{(n-m,1^m)}$ and $D_{(n-m+1,1^{m-1})^R}$ is $\binom{\frac{n}{e}}{m} - \binom{\frac{n-e}{e}}{m-1} = \binom{\frac{n-e}{e}}{m}$, which equals the trailing coefficient in the graded dimension of $S_{(n-m,1^m)}$, and thus is the leading coefficient in the graded dimension of $D_{(n-m,1^m)^R}$, as required.

2. We proceed by induction on m. For the base case we let $m = \frac{n}{e}$. By the previous part, together with Proposition 4.6, we have

$$\begin{aligned}
&\operatorname{grdim}\left(D_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)^R}\right)\\
&= v^{-j_{\frac{n}{e}}}\left(\operatorname{grdim}\left(S_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)}\right) - v^{i_{\frac{n}{e}}}\operatorname{grdim}\left(D_{\left(n-\frac{n}{e}+1,1^{\frac{n}{e}-1}\right)^R}\right)\right)\\
&= v^{-j_{\frac{n}{e}}}\left(v^{\left(\frac{n}{e}+\lfloor\frac{n/e}{e}\rfloor\right)} + \frac{(e-2)n^2}{e^2}v^{\left(\frac{n}{e}+\lfloor\frac{n/e}{e}\rfloor-1\right)} + \dots\right.\\
&\qquad \dots + \left(\frac{n-e}{e}\binom{\frac{(e-2)n}{e}}{2} + \frac{n}{e}\right)v^{\left(2-\frac{n}{e}+\lfloor\frac{n/e}{e}\rfloor\right)} + \frac{(e-2)n}{e}v^{\left(1-\frac{n}{e}+\lfloor\frac{n/e}{e}\rfloor\right)}\\
&\qquad \left.-v^{i_{\frac{n}{e}}}\left(v^{\frac{n-e}{e}} + \frac{(e-2)(n-e)n}{e^2}v^{\frac{n-2e}{e}} + \dots + \frac{(e-2)(n-e)n}{e^2}v^{\frac{2e}{e}} + v^{\frac{e-n}{e}}\right)\right).
\end{aligned}$$

We observe that the graded dimensions of $S_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)}$ and $D_{\left(n-\frac{n}{e}+1,1^{\frac{n}{e}-1}\right)^R}$ have $\frac{2n}{e}$ and $\frac{2n}{e}-1$ terms, respectively. For the graded dimension of $D_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)^R}$ to be symmetric in v and v^{-1}, then the leading coefficient of $D_{\left(n-\frac{n}{e}+1,1^{\frac{n}{e}-1}\right)^R}$ corresponds to the leading coefficient of $S_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)}$. Thus, the leading coefficient in the graded dimension of $D_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)^R}$ corresponds with the trailing coefficient $\frac{(e-2)n}{e}$ in the graded dimension of $S_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)}$. Now, the difference between the second trailing coefficient in the graded dimension of $S_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)}$ and the trailing coefficient in the graded dimension of $D_{\left(n-\frac{n}{e}+1,1^{\frac{n}{e}-1}\right)^R}$ is positive, and thus $D_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)^R}$ has $\frac{2n}{e}-1$ terms, with leading degree $\frac{n-e}{e}$. Further, this difference gives the second leading coefficient in the graded dimension of $D_{\left(n-\frac{n}{e},1^{\frac{n}{e}}\right)^R}$ to be $\frac{n-e}{e}\binom{\frac{(e-2)n}{e}}{2} + \frac{n}{e} - 1$, as required.

For $m > \frac{n}{e}$, we suppose that the first two leading terms in the graded dimension of $D_{(n-m+1,1^{m-1})^R}$ are $\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}}v^{\frac{n-e}{e}}$ and $\frac{n-e}{e}\left(\binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}} + \binom{\frac{(e-2)n}{e}}{\frac{e(m-1)-n}{e}}\right)v^{\frac{n-2e}{e}}$.

Then, by Proposition 4.6, we have

$$\begin{aligned}
&\operatorname{grdim}\left(D_{(n-m,1^m)^R}\right)\\
&= v^{-j_m}\left(\operatorname{grdim}\left(S_{(n-m,1^m)}\right) - v^{i_m}\operatorname{grdim}\left(D_{(n-m+1,1^{m-1})^R}\right)\right)\\
&= v^{-j_m}\Bigg(\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\left(\frac{n}{e}+\lfloor\frac{m}{e}\rfloor\right)}\\
&\quad+\left(\frac{n}{e}\binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}} + \frac{n-e}{e}\binom{\frac{(e-2)n}{e}}{\frac{e(m-1)-n}{e}}\right) v^{\left(\frac{n}{e}+\lfloor\frac{m}{e}\rfloor-1\right)} + \cdots\\
&\quad\cdots+\left(\frac{n-e}{e}\binom{\frac{(e-2)n}{e}}{\frac{e(m+2)-n}{e}} + \frac{n}{e}\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}}\right) v^{\left(2-\frac{n}{e}+\lfloor\frac{m}{e}\rfloor-1\right)}\\
&\quad+\binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}} v^{\left(1-\frac{n}{e}+\lfloor\frac{m}{e}\rfloor-1\right)}\\
&\quad- v^{i_m}\Bigg(\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\frac{n-e}{e}} + \frac{n-e}{e}\left(\binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}} + \binom{\frac{(e-2)n}{e}}{\frac{e(m-1)-n}{e}}\right) v^{\frac{n-2e}{e}} + \cdots\\
&\quad\cdots+\frac{n-e}{e}\left(\binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}} + \binom{\frac{(e-2)n}{e}}{\frac{e(m-1)-n}{e}}\right) v^{\frac{2e-n}{e}} + \binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\frac{e-n}{e}}\Bigg)\Bigg).
\end{aligned}$$

We note that the graded dimensions of $S_{(n-m,1^m)}$ and $D_{(n-m+1,1^{m-1})^R}$ have $\frac{2n}{e}$ and $\frac{2n}{e}-1$ terms, respectively. For the graded dimension of $D_{(n-m,1^m)^R}$ to be symmetric in v and v^{-1}, the leading coefficients $\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}}$ in the graded dimensions of $S_{(n-m,1^m)}$ and $D_{(n-m+1,1^{m-1})^R}$ must cancel out. Thus, the leading coefficient in the graded dimension of $D_{(n-m,1^m)^R}$ is $\binom{\frac{(e-2)n}{e}}{\frac{e(m+1)-n}{e}}$, which equals the difference in the second leading coefficients in the graded dimensions of $S_{(n-m,1^m)}$ and $D_{(n-m+1,1^{m-1})^R}$. Now, we observe that the difference between the second trailing coefficient in the graded dimension of $S_{(n-m,1^m)}$ and the trailing coefficient in the graded dimension of $D_{(n-m+1,1^{m-1})^R}$ is non-zero, and thus has $\frac{2n}{e}$ number of terms. We see that this difference is

$$\frac{n-e}{e}\binom{\frac{(e-2)n}{e}}{\frac{e(m+2)-n}{e}} + \frac{n}{e}\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} - \binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} = \frac{n-e}{e}\left(\binom{\frac{(e-2)n}{e}}{\frac{e(m+2)-n}{e}} + \binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}}\right),$$

which is the second leading coefficient in the graded dimension of $D_{(n-m,1^m)^R}$, as required.

3. We proceed by downwards induction on m. For $m = n-1$, it is obvious that $\operatorname{grdim}\left(D_{(1^n)^R}\right) = 1$. Now suppose that the leading term in the graded dimension of $D_{(n-m-1,1^{m+1})^R}$ is $\binom{\frac{n-e}{e}}{n-m} v^{n-m}$, for $m < n-1$. We have $\operatorname{grdim}\left(S_{(n-m,1^m)}\right) =$

$v^{j_n} \operatorname{grdim}\left(D_{(n-m,1^m)^R}\right) + v^{h_m} \operatorname{grdim}\left(D_{(n-m-1,1^{m+1})^R}\right)$, so that

$$\begin{aligned}
&\operatorname{grdim}\left(D_{(n-m,1^m)^R}\right)\\
&= v^{-j_m}\left(\operatorname{grdim}\left(S_{(n-m,1^m)}\right) - v^{h_m}\operatorname{grdim}\left(D_{(n-m-1,1^{m+1})^R}\right)\right)\\
&= v^{-j_m}\left(\left(\binom{\frac{n-e}{e}}{n-m-1} v^{\left(n-m+\lfloor \frac{m}{e}\rfloor\right)} + \cdots + \binom{\frac{n}{e}}{n-m-1} v^{\left(m-n+2+\lfloor \frac{m}{e}\rfloor\right)}\right.\right.\\
&\qquad \left.\left. - v^{h_m}\left(\binom{\frac{n-e}{e}}{n-m-2} v^{(n-m-2} + \cdots + \binom{\frac{n-e}{e}}{n-m-2} v^{(2+m-n)}\right)\right)\right),
\end{aligned}$$

by Proposition 4.6. We observe that there are $2n-2m-1$ and $2n-2n-3$ terms, respectively, in the graded dimensions of $S_{(n-m,1^m)}$ and $D_{(n-m-1,1^{m+1})^R}$, and that the leading and trailing coefficients in the graded dimension of $D_{(n-m-1,1^{m+1})^R}$ are smaller than those in the graded dimension of $S_{(n-m,1^m)}$. Hence, there are $2n-2m-1$ terms in the graded dimension of $D_{(n-m,1^m)^R}$. Now, the difference of the trailing coefficients in the graded dimensions of $S_{(n-m,1^m)}$ and $D_{(n-m-1,1^{m+1})^R}$ is $\binom{\frac{n}{e}}{n-m-1} - \binom{\frac{n-e}{e}}{n-m-2} = \binom{\frac{n-e}{e}}{n-m-1}$, which equals the leading coefficient in the graded dimension of $S_{(n-m,1^m)}$, and thus is the leading coefficient in the graded dimension of $D_{(n-m,1^m)^R}$, as required.

□

Corollary 4.8. *For an e-regular partition λ, $[S_{(n-m,1^m)} : D_\lambda]_v = 0$, except for the following cases:*

- ◇ $[S_{(n-m,1^m)} : D_{\lambda_{m-1}}]_v = v^{\lfloor \frac{m}{e}\rfloor+1}$ *if* $1 \leqslant m \leqslant n-1$;
- ◇ $[S_{(n-m,1^m)} : D_{\lambda_m}]_v = v^{\lfloor \frac{m}{e}\rfloor}$ *if* $0 \leqslant m \leqslant n-2$.

Proof. By Theorem 4.7, it is easy to see that

- ◇ $[S_{(n-m,1^m)} : D_{(n-m+1,1^{m-1})^R}]_v = v^{\lfloor \frac{m}{e}\rfloor+1}$ if $1 \leqslant m \leqslant n-1-\frac{n}{e}$;
- ◇ $[S_{(n-m,1^m)} : D_{(n-m,1^m)^R}]_v = \begin{cases} v^{\lfloor \frac{m}{e}\rfloor} & \text{if } 0 \leqslant m \leqslant n-1-\frac{n}{e};\\ v^{\lfloor \frac{m}{e}\rfloor+1} & \text{if } n-\frac{n}{e} \leqslant m \leqslant n-1;\end{cases}$
- ◇ $[S_{(n-m,1^m)} : D_{(n-m-1,1^{m+1})^R}]_v = v^{\lfloor \frac{m}{e}\rfloor}$ if $n-\frac{n}{e} \leqslant m \leqslant n-2$.

Now, the result follows from the definition of λ_m, given in Equation (4.3.1). □

Example 4.9. *For $e=3$, part of the graded decomposition matrix for $\mathscr{R}_6^\Lambda$ comprising* *rows corresponding to hook partitions has the form*

$$
\begin{array}{c} \\ S_{(6)} \\ S_{(5,1)} \\ S_{(4,1^2)} \\ S_{(3,1^3)} \\ S_{(2,1^4)} \\ S_{(1^6)} \end{array}
\begin{array}{c}
\begin{array}{ccccc} D_{(6)} & D_{(5,1)} & D_{(4,1^2)} & D_{(3,2,1)} & D_{(3,2)} \end{array} \\
\left(\begin{array}{ccccc|c}
v & & & & & \\
v & 1 & & & & \\
 & v & 1 & & & 0 \\
 & & v^2 & v & & \\
 & & & v^2 & v & \\
 & & & & v^2 &
\end{array}\right).
\end{array}
$$

Part III

Specht modules labelled by hook bipartitions

Chapter 5

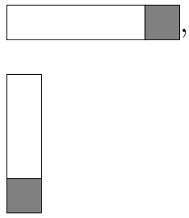
Introducing Specht modules labelled by hook bipartitions

From this chapter onwards, we begin the study of graded representation theory of the cyclotomic Khovanov–Lauda–Rouquier algebra $\mathscr{R}_n^\Lambda$ in level two, namely the Iwahori–Hecke algebra of type B, over $\mathbb{C}$ or a field of positive characteristic. For quantum characteristic at least three, we introduce a large class of representations in this setting, Specht modules $S_{((n-m),(1^m))}$ labelled by hook bipartitions, and we investigate the action of the generators of $\mathscr{R}_n^\Lambda$ on their basis vectors. We establish the ground work for our ultimate end of completely determining the graded decomposition matrices comprising rows labelled by hook bipartitions, inspired by the work of Peel [P] and Chuang, Miyachi and Tan [CMT]. We will eventually see that these matrices split into four separate cases, depending on κ and n; the ungraded decomposition matrices in one of these cases have the same form as given in Peel's Theorem 4.1. The Specht modules $S((n-m),(1^m))$ have a particularly nice presentation, and together with their connection with Peel's work, provide a suitable candidate to make progress on the Graded Decomposition Problem for graded Iwahori–Hecke algebras of type B. We set $l = 2$ and $e \geqslant 3$ from now on; we forget the $\mathbb{Z}$-grading on Specht modules labelled by hook bipartitions until Chapter 11.

In this chapter, we give a combinatorial description of the basis elements of Specht modules labelled by hook bipartitions and begin our study of the action of the generators of the cyclotomic Khovanov–Lauda–Rouquier algebra on these basis elements. The following results are motivated for the need to establish Specht module homomorphisms labelled by bipartitions in Chapter 6, where we later find the irreducible submodules of $S_{((n-m),(1^m))}$ arise as quotients of the kernels and images of these homomorphisms. We note that we work solely with *ungraded* cyclotomic Khovanov–Lauda–Rouquier modules up to and including Chapter 13.

We define a *hook bipartition* of n to be a bipartition $((n-m),(1^m))$, for all $m \in \{0,\dots,n\}$. We will refer to the first component $(n-m)$ of a hook bipartition as its *arm*, and similarly, to its second component (1^m) as its *leg*. We call the node $(n-m,1,1)$

lying at the end of its arm its *hand node* if $n-m>0$, and the node $(m,1,2)$ lying at the end of its leg its *foot node* if $m>0$. The Young diagram of a hook bipartition is



where the hand and foot nodes have been shaded.

5.1 Homogeneous basis elements of $S_{((n-m),(1^m))}$

Similarly to Chapter 3, we define the homogeneous basis elements of Specht modules labelled by hook bipartitions. Given a standard $((n-m),(1^m))$-tableau T, we write $a_j := T(j,1,2)$ for all $j\in\{1,\dots,m\}$. Then T is determined by $a_1,\dots,a_m$ and we can write

$$T = s\overset{a_1-1}{\underset{1}{\downarrow}}\, s\overset{a_2-1}{\underset{2}{\downarrow}} \dots s\overset{a_m-1}{\underset{m}{\downarrow}}\, T_{((n-m),(1^m))}.$$

If $a_i=i$ for some $i\in\{1,\dots,m\}$, then $T = s\overset{a_{i+1}-1}{\underset{i+1}{\downarrow}}\, s\overset{a_{i+2}-1}{\underset{i+2}{\downarrow}} \dots s\overset{a_m-1}{\underset{m}{\downarrow}}\, T_{((n-m),(1^m))}$, in particular, $T=T_{((n-m),(1^m))}$ if $a_i=i$ for all $i\in\{1,\dots,m\}$. Recalling that $\Psi\overset{j}{\underset{i}{\downarrow}} := \psi_j\psi_{j-1}\dots\psi_i$ from Section 1.3.3, we can now write

$$v_T = \Psi\overset{a_1-1}{\underset{1}{\downarrow}}\, \Psi\overset{a_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_{((n-m),(1^m))},$$

where v_T is completely determined by $a_1,\dots,a_m$. For brevity, we write $v(a_1,\dots,a_m) := v_T$. If $a_i=i$ for some $i\in\{1,\dots,m\}$, then $v_T = \Psi\overset{a_{i+1}-1}{\underset{i+1}{\downarrow}}\, \Psi\overset{a_{i+2}-1}{\underset{i+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_{((n-m),(1^m))}$, in particular, $v_T = z_{((n-m),(1^m))}$ if $a_i=i$ for all $i\in\{1,\dots,m\}$.

5.2 Presentation of Specht modules labelled by hook bipartitions

Recall that the Specht module presentation for the cyclotomic Khovanov–Lauda–Rouquier algebra was given in Section 1.3.2 and that we determined the Garnir elements of Specht modules labelled by hook bipartitions in Section 1.3.1.3.

A Specht module $S_{((n-m),(1^m))}$ labelled by a hook bipartition is generated by $z_{((n-m),(1^m))}$ with defining relations

◇ $e(\mathbf{i})z_{((n-m),(1^m))} = \delta_{\mathbf{i}_{((n-m),(1^m))},\mathbf{i}}\, z_{((n-m),(1^m))}$;

- $y_r z_{((n-m),(1^m))} = 0$ for all $r \in \{1, \dots, n\}$;
- $\psi_r z_{((n-m),(1^m))} = 0$ for all $r \in \{1, \dots, m-1\} \cup \{m+1, \dots, n-1\}$,

where $\mathbf{i}_{((n-m),(1^m))}$ is the e-residue sequence of the column-initial tableau $T_{((n-m),(1^m))}$.

5.3 The action of $\mathscr{R}_n^\Lambda$ on $S_{((n-m),(1^m))}$ I

We determine when the basis elements v_T of Specht modules labelled by hook bipartitions are killed by the generators $\psi_1, \dots, \psi_{n-1}$ of $\mathscr{R}_n^\Lambda$.

Lemma 5.1. *Suppose that $1 \leqslant a_1 < \cdots < a_m \leqslant n$, and $1 \leqslant i < n$.*

1. *Let $i \not\equiv 2 + \kappa_2 - \kappa_1 \pmod e$. Then $\psi_i v(a_1, \dots, a_m) = 0$ if both i and $i+1$ lie in $\{a_1, \dots, a_m\}$.*
2. *Let $i \not\equiv \kappa_2 - \kappa_1 \pmod e$. Then $\psi_i v(a_1, \dots, a_m) = 0$ if neither i nor $i+1$ lies in $\{a_1, \dots, a_m\}$.*

Proof. We proceed by induction on the sum $a_1 + \cdots + a_m$ to prove statements (1) and (2) simultaneously, as follows.

1. Let r be such that $a_r = i$ and $a_{r+1} = i+1$. If $i = r$, then all the terms $\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}$ are trivial, so we have

$$\begin{aligned}\psi_i v(a_1, \dots, a_m) &= \psi_i \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\ &= \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \psi_i z_{((n-m),(1^m))} = 0.\end{aligned}$$

Now assume that $i \geqslant r+1$, and observe that

$$\begin{aligned}&\psi_i v(a_1, \dots, a_m)\\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_i \psi_{i-1} \psi_i \Psi\overset{i-2}{\underset{r}{\downarrow}} \Psi\overset{i-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1} \psi_i \psi_{i-1} \Psi\overset{i-2}{\underset{r}{\downarrow}} \Psi\overset{i-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1} \psi_i \psi_{i-1} v(1, \dots, r-1, i-1, i, a_{r+2}, \dots, a_m),\end{aligned}$$

where $\psi_{i-1} v(1, \dots, r-1, i-1, i, a_{r+2}, \dots, a_m)$ equals zero by induction since $i-1 \not\equiv 2 + \kappa_2 - \kappa_1 \pmod e$. Now suppose that $i \equiv \kappa_2 - \kappa_1 + 3 \pmod e$. We note that the terms $\Psi\overset{i-2}{\underset{r}{\downarrow}} \Psi\overset{i-1}{\underset{r+1}{\downarrow}}$ are trivial if $i = r+1$, so that

$$\psi_{i-1} v(1, \dots, r-1, i-1, i, a_{r+2}, \dots, a_m) = \psi_{i-1} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}$$

$$= \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \psi_{i-1} z_{((n-m),(1^m))}$$
$$= 0.$$

Now supposing that $i \geqslant r+2$, the expression $\psi_i v(a_1, \dots, a_m)$ becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\psi_i\psi_{i-1}\psi_{i-2}\psi_{i-1}\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}$$
$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\psi_i(\psi_{i-2}\psi_{i-1}\psi_{i-2} - 1)\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots$$
$$\dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}.$$

By splitting this sum into its two terms, the first term becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\psi_i\psi_{i-2}\psi_{i-1}\psi_{i-2} v(1, \dots, r-1, i-2, i-1, a_{r+2}, \dots, a_m),$$

where $\psi_{i-2} v(1, \dots, r-1, i-2, i-1, a_{r+2}, \dots, a_m)$ is zero by induction since $i-2 \not\equiv 2 + \kappa_2 - \kappa_1 \pmod e$. The second term is

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\psi_i\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))},$$

which becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+2}{\downarrow}} \psi_i z_{((n-m),(1^m))} = 0$$

if $r+1 = m$. We now assume that $r+1 < m$, so that the second term becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\psi_i\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}} \Psi\overset{i+1}{\underset{i}{\downarrow}} \Psi\overset{i-1}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}$$
$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}} \psi_i\psi_{i+1}\psi_i\Psi\overset{i-1}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots$$
$$\dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}.$$

If $e \neq 3$, then this term becomes

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}} \psi_{i+1}\psi_i\psi_{i+1}\Psi\overset{i-1}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots$$
$$\dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))},$$

whereas if $e = 3$, then the term becomes

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-1}\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}} (\psi_{i+1}\psi_i\psi_{i+1} - 1)\Psi\overset{i-1}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots$$

$$\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_{((n-m),(1^m))}.$$

We first assume that e is arbitrary and see that the first term in its sum (and only term if $e\neq 3$) is

$$\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\,\psi_{i-1}\Psi\overset{i-3}{\underset{r}{\downarrow}}\,\Psi\overset{i-2}{\underset{r+1}{\downarrow}}\,\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\,\psi_{i+1}\psi_i\psi_{i+1}\Psi\overset{i-1}{\underset{r+2}{\downarrow}}\,\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_{((n-m),(1^m))}\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\,\psi_{i-1}\Psi\overset{i-3}{\underset{r}{\downarrow}}\,\Psi\overset{i-2}{\underset{r+1}{\downarrow}}\,\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\,\psi_{i+1}\psi_i\psi_{i+1}v(1,\dots,r+1,i,a_{r+3},\dots,a_m),
\end{aligned}$$

where $\psi_{i+1}v(1,\dots,r+1,i,a_{r+3},\dots,a_m)$ equals zero by the inductive hypothesis of part (2) of the lemma, since $i+1\not\equiv\kappa_2-\kappa_1 \pmod e$ if $e\neq 4$. Assuming that $e=4$, this term becomes

$$\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\,\Psi\overset{i-3}{\underset{r}{\downarrow}}\,\Psi\overset{i-1}{\underset{r+1}{\downarrow}}\,\Psi\overset{a_{r+2}-1}{\underset{i}{\downarrow}}\,\Psi\overset{i-1}{\underset{r+2}{\downarrow}}\,\Psi\overset{a_{r+3}-1}{\underset{i+3}{\downarrow}}\,\psi_{i+1}\psi_{i+2}\psi_{i+1}\Psi\overset{i}{\underset{r+3}{\downarrow}}\,\Psi\overset{a_{r+4}-1}{\underset{r+2}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_{((n-m),(1^m))}\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\,\Psi\overset{i-3}{\underset{r}{\downarrow}}\,\Psi\overset{i-1}{\underset{r+1}{\downarrow}}\,\Psi\overset{a_{r+2}-1}{\underset{i}{\downarrow}}\,\Psi\overset{i-1}{\underset{r+2}{\downarrow}}\,\Psi\overset{a_{r+3}-1}{\underset{i+3}{\downarrow}}\,\psi_{i+2}\psi_{i+1}\psi_{i+2}\Psi\overset{i}{\underset{r+3}{\downarrow}}\,\Psi\overset{a_{r+4}-1}{\underset{r+2}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_{((n-m),(1^m))}\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\,\Psi\overset{i-3}{\underset{r}{\downarrow}}\,\Psi\overset{i-1}{\underset{r+1}{\downarrow}}\,\Psi\overset{a_{r+2}-1}{\underset{i}{\downarrow}}\,\Psi\overset{i-1}{\underset{r+2}{\downarrow}}\,\Psi\overset{a_{r+3}-1}{\underset{i+3}{\downarrow}}\,\psi_{i+2}\psi_{i+1}\\
&\quad\cdot\psi_{i+2}v(1,\dots,r+2,i+1,a_{r+4},\dots,a_m),
\end{aligned}$$

where $\psi_{i+2}v(1,\dots,r+2,i+1,a_{r+4},\dots,a_m)$ equals zero by the inductive hypothesis of part (2) of the lemma, since $i+2\not\equiv\kappa_2-\kappa_1 \pmod 4$.

Now assume that $e=3$, where its second term becomes

$$-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\,\Psi\overset{i-3}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\,\psi_{i-1}\Psi\overset{i-2}{\underset{r+1}{\downarrow}}\,\Psi\overset{i-1}{\underset{r+2}{\downarrow}}\,\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_{((n-m),(1^m))}.$$

If $i=r+2$, then the two terms $\Psi\overset{i-2}{\underset{r+1}{\downarrow}}\,\Psi\overset{i-1}{\underset{r+2}{\downarrow}}$ are trivial, so this becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\,\Psi\overset{i-3}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\,\psi_{i-1}\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_{((n-m),(1^m))}\\
&=-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\,\Psi\overset{i-3}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\,\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\,\psi_{i-1}z_{((n-m),(1^m))}=0.
\end{aligned}$$

Now assuming that $i \geqslant r+3$, we can rewrite this expression to be

$$
\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\Psi\overset{i-3}{\underset{r}{\downarrow}}\psi_{i-1}\psi_{i-2}\psi_{i-1}\Psi\overset{i-3}{\underset{r+1}{\downarrow}}\Psi\overset{i-2}{\underset{r+2}{\downarrow}}\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_{((n-m),(1^m))}\\
&=-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\Psi\overset{i-3}{\underset{r}{\downarrow}}(\psi_{i-2}\psi_{i-1}\psi_{i-2}+1)\Psi\overset{i-3}{\underset{r+1}{\downarrow}}\Psi\overset{i-2}{\underset{r+2}{\downarrow}}\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_{((n-m),(1^m))},
\end{aligned}
$$

and again we consider its two summands. The first term is

$$
\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\Psi\overset{i-3}{\underset{r}{\downarrow}}\psi_{i-2}\psi_{i-1}\psi_{i-2}\Psi\overset{i-3}{\underset{r+1}{\downarrow}}\Psi\overset{i-2}{\underset{r+2}{\downarrow}}\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\\
&\quad\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_{((n-m),(1^m))}\\
&=-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\Psi\overset{i-3}{\underset{r}{\downarrow}}\psi_{i-2}\psi_{i-1}\psi_{i-2}v(1,\dots,r,i-2,i-1,a_{r+3},\dots,a_m),
\end{aligned}
$$

where $\psi_{i-2}v(1,\dots,r,i-2,i-1,a_{r+3},\dots,a_m)$ equals zero by induction since $i-2 \not\equiv 2+\kappa_2-\kappa_1 \pmod 3$. The second term becomes

$$
-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\psi_{i-3}\Psi\overset{i-4}{\underset{r}{\downarrow}}\Psi\overset{i-3}{\underset{r+1}{\downarrow}}\Psi\overset{i-2}{\underset{r+2}{\downarrow}}\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_{((n-m),(1^m))}.
$$

If $i=r+3$, then $\Psi\overset{i-4}{\underset{r}{\downarrow}}\Psi\overset{i-3}{\underset{r+1}{\downarrow}}\Psi\overset{i-2}{\underset{r+2}{\downarrow}}$ is trivial, so the expression becomes

$$
-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\psi_{i-3}z_{((n-m),(1^m))}=0.
$$

Now supposing that $i \geqslant r+4$, the expression becomes

$$
-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{i+2}{\downarrow}}\psi_{i-3}v(1,\dots,r-1,i-3,i-2,i-1,a_{r+3},\dots,a_m),
$$

where $\psi_{i-3}v(1,\dots,r-1,i-3,i-2,i-1,a_{r+3},\dots,a_m)$ equals zero by induction since $i-3 \not\equiv 2+\kappa_2-\kappa_1 \pmod e$. Thus, we have proved the first statement, as required.

2. Let r be such that $a_r \leqslant i-1$ and $a_{r+1} \geqslant i+2$. If $i=r-1$, then all the terms $\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}$, and further, if $a_{r+j}=i+j+1$ then $\Psi\overset{a_{r+j}-1}{\underset{r+j}{\downarrow}}$ is trivial, for $j>0$. Thus

$$
\psi_i v(a_1,\dots,a_m)=\psi_i\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_{((n-m),(1^m))}
$$

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \psi_i z_{((n-m),(1^m))} = 0.$$

So we can assume that $i \geqslant r$. Now we have

$$\begin{aligned}
&\psi_i v(a_1,\dots,a_m)\\
&= \psi_i \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}} \psi_i\psi_{i+1}\psi_i \Psi\overset{i-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}} \psi_{i+1}\psi_i\psi_{i+1} \Psi\overset{i-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}} \psi_{i+1}\psi_i\psi_{i+1} v(1,\dots,r,i,a_{r+2},\dots,a_m)
\end{aligned}$$

If $i+1 \not\equiv \kappa_2-\kappa_1 \pmod e$, then $\psi_{i+1}v(1,\dots,r,i,a_{r+2},\dots,a_m)$ is zero by induction. So we assume that $i \equiv \kappa_2-\kappa_1-1 \pmod e$. Now the expression $\psi_i v(a_1,\dots,a_m)$ becomes

$$\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \psi_{i+1} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+3}{\downarrow}} \psi_{i+1}\psi_{i+2}\psi_{i+1} \Psi\overset{i}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+3}{\downarrow}} (\psi_{i+2}\psi_{i+1}\psi_{i+2}-1) \Psi\overset{i}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots\\
&\quad \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}
\end{aligned}$$

Splitting this sum into two terms, its first term is

$$\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+3}{\downarrow}} \psi_{i+2}\psi_{i+1}\psi_{i+2} \Psi\overset{i}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+3}{\downarrow}} \psi_{i+2}\psi_{i+1}\psi_{i+2} v(1,\dots,r+1,i+1,a_{r+3},\dots,a_m),
\end{aligned}$$

where $\psi_{i+2}v(1,\dots,r+1,i+1,a_{r+3},\dots,a_m)$ equals zero by induction since $i+2 \equiv \kappa_2-\kappa_1+1 \pmod e$. Its second term is

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+3}{\downarrow}} \Psi\overset{i}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+3}{\downarrow}} \psi_i \Psi\overset{i-1}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{i+3}{\downarrow}} \psi_i v(1,\dots,r+1,i,a_{r+3},\dots,a_m),
\end{aligned}$$

where $\psi_i v(1,\ldots,r+1,i,a_{r+3},\ldots,a_m)$ equals zero by induction since $i \equiv \kappa_2-\kappa_1-1 \pmod e$. Thus, we have proved the second statement, as required, by assuming its inductive hypothesis. □

Remark 5.2. *We note that these are not the only cases when ψ_i kills basis vector $v(a_1,\ldots,a_m)$, for $1 \leqslant i < n$. For example, let $e = 3$, $\kappa = (0,0)$, $i = 3$, and $S_{((3),(1^3))}$. Then $\psi_3 v(1,2,4) = 0$ where $3 \notin \{a_1,a_2,a_3\}$. In Chapter 7, we will expand on the previous lemma and give an explicit description of the action of the cyclotomic Khovanov–Lauda–Rouquier generators $\psi_1,\ldots,\psi_{n-1}$ on basis elements of Specht modules labelled by hook bipartitions.*

We now show when the generators $y_1,\ldots,y_n$ of $\mathscr{R}_n^\Lambda$ act trivially on the elements v_T of Specht modules labelled by hook bipartitions.

Lemma 5.3. *1. Let $i \equiv 1+\kappa_2-\kappa_1 \pmod e$. Then $y_i v(a_1,\ldots,a_m) = 0$ if and only if either $i \in \{a_1,\ldots,a_m\}$ or $i+1 \notin \{a_1,\ldots,a_m\}$.*

2. Let $i \equiv 2+\kappa_2-\kappa_1 \pmod e$. Then $y_i v(a_1,\ldots,a_m) = 0$ if and only if either $i-1 \in \{a_1,\ldots,a_m\}$ or $i \notin \{a_1,\ldots,a_m\}$.

3. Let $i \not\equiv 1,2+\kappa_2-\kappa_1 \pmod e$ Then $y_i v(a_1,\ldots,a_m) = 0$.

Proof. We first proceed by simultaneous induction on the sum $a_1+\cdots+a_m$ to show that $y_i v(a_1,\ldots,a_m)$ equals zero in the following six cases:

- ◇ $i \equiv 1+\kappa_2-\kappa_1 \pmod e$ and $i \in \{a_1,\ldots,a_m\}$;
- ◇ $i \equiv 1+\kappa_2-\kappa_1 \pmod e$ and $i+1 \notin \{a_1,\ldots,a_m\}$;
- ◇ $i \equiv 2+\kappa_2-\kappa_1 \pmod e$ and $i-1 \in \{a_1,\ldots,a_m\}$;
- ◇ $i \equiv 2+\kappa_2-\kappa_1 \pmod e$ and $i \notin \{a_1,\ldots,a_m\}$;
- ◇ $i-\kappa_2+\kappa_1 \not\equiv 1,2 \pmod e$ and $i \in \{a_1,\ldots,a_m\}$;
- ◇ $i-\kappa_2+\kappa_1 \not\equiv 1,2 \pmod e$ and $i \notin \{a_1,\ldots,a_m\}$.

We label these cases A, A', B, B', C and C', respectively, from top to bottom.

1. (a) Suppose that $i \in \{a_1,\ldots,a_m\}$ and let $a_r = i$. If $i = r$ then $\Psi\overset{i-1}{\underset{r}{\downarrow}}$ is trivial, so

$$\begin{aligned} y_i v(a_1,\ldots,a_m) &= y_i \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \ldots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \ldots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} y_i z_{((n-m),(1^m))} = 0. \end{aligned}$$

So suppose that $i \geqslant r+1$. Then

$$y_i v(a_1,\ldots,a_m)$$

$$
\begin{aligned}
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; y_i\psi_{i-1}\Psi\overset{i-2}{\underset{r}{\downarrow}} \; \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \; z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \psi_{i-1}y_{i-1}\Psi\overset{i-2}{\underset{r}{\downarrow}} \; \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \; z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \psi_{i-1}y_{i-1}v(1,\dots,r-1,i-1,a_{r-1},\dots,a_m)
\end{aligned}
$$

where $y_{i-1}v(1,\dots,r-1,i-1,a_{r-1},\dots,a_m)$ equals zero by the inductive hypothesis of C.

(b) Suppose that $i+1 \notin \{a_1,\dots,a_m\}$, so let $a_r \leqslant i$ and $a_{r+1} \geqslant i+2$. If $i=r$ then $y_iv(a_1,\dots,a_m)$ is trivial by part (a). So let $i \geqslant r+1$.

i. Suppose that $a_r = i$. Then $y_iv(a_1,\dots,a_m) = 0$ by part (a).

ii. Suppose that $a_r \leqslant i-1$. Then we have

$$
\begin{aligned}
&y_iv(a_1,\dots,a_m)\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \; y_i\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \; \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \; z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \; \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}} \; y_i\psi_i\Psi\overset{i-1}{\underset{r+1}{\downarrow}} \; \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \; z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \; \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}} \; (\psi_iy_{i+1}-1)\Psi\overset{i-1}{\underset{r+1}{\downarrow}} \; \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots\\
&\quad \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \; z_{((n-m),(1^m))}.
\end{aligned}
$$

The first term becomes

$$
\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \; \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}} \; \psi_iy_{i+1}\Psi\overset{i-1}{\underset{r+1}{\downarrow}} \; \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \; z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \; \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}} \; \psi_iy_{i+1}v(1,\dots,r,i,a_{r+2},\dots,a_m)
\end{aligned}
$$

where $y_{i+1}v(1,\dots,r,i,a_{r+2},\dots,a_m)$ equals zero by the inductive hypothesis of B. The second term becomes

$$
\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \; \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}} \; \psi_{i+1}\Psi\overset{i-1}{\underset{r+1}{\downarrow}} \; \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} \; z_{((n-m),(1^m))}\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \; \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \; \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}} \; \psi_{i+1}v(1,\dots,r,i,a_{r+2},\dots,a_m)
\end{aligned}
$$

where $\psi_{i+1}v(1,\dots,r,i,a_{r+2},\dots,a_m)$ equals zero by part two of Lemma 5.1 since $a_{r+2} \geqslant i+3$ and $i+1 \not\equiv \kappa_2-\kappa_1 \pmod e$.

2. (a) Suppose that $i-1 \in \{a_1,\dots,a_m\}$ and let $a_r = i-1$.

i. Suppose that $a_{r+1} = i$. If $i = r+1$ then $\Psi\overset{i-1}{\underset{r+1}{\downarrow}}$ is trivial. Then

$$\begin{aligned} y_i v(a_1,\dots,a_m) &= y_i \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} y_i z_{((n-m),(1^m))} = 0. \end{aligned}$$

Now suppose that $i \geqslant r+2$. Then

$$\begin{aligned} &y_i v(a_1,\dots,a_m) \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{i-2}{\underset{r}{\downarrow}} y_i\psi_{i-1}\Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{i-2}{\underset{r}{\downarrow}} (\psi_{i-1}y_{i-1}+1)\Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}. \end{aligned}$$

The first term becomes

$$\begin{aligned} &\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{i-2}{\underset{r}{\downarrow}} \psi_{i-1}y_{i-1}\Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{i-2}{\underset{r}{\downarrow}} \psi_{i-1}y_{i-1}v(1,\dots,r,i-1,a_{r+2},\dots,a_m) \end{aligned}$$

where $y_{i-1}v(1,\dots,r,i-1,a_{r+2},\dots,a_m)$ equals zero by the inductive hypothesis of A. The second term becomes

$$\begin{aligned} &\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-2}\Psi\overset{i-3}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{i-2}v(1,\dots,r-1,i-2,i-1,a_{r+2},\dots,a_m) \end{aligned}$$

where $\psi_{i-2}v(1,\dots,r-1,i-2,i-1,a_{r+2},\dots,a_m)$ equals zero by the first part of Lemma 5.1 since $i-2 \not\equiv 2+\kappa_2-\kappa_1 \pmod{e}$.

ii. Suppose that $a_{r+1} \geqslant i+1$. If $i = r$ then $\Psi\overset{i}{\underset{r+1}{\downarrow}}$ is trivial. So we have

$$\begin{aligned} y_i v(a_1,\dots,a_m) &= \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}} y_i \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} y_i z_{((n-m),(1^m))} \\ &= 0. \end{aligned}$$

So suppose that $i \geqslant r+1$. Then

$$\begin{aligned} &y_i v(a_1,\dots,a_m) \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{i-2}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}} y_i\psi_i\Psi\overset{i-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \end{aligned}$$

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{i-2}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}}\ \psi_i y_{i+1} \Psi\overset{i-1}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}$$
$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{i-2}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+1}{\downarrow}}\ \psi_i y_{i+1} v(1,\dots,r,i,a_{r+2},\dots,a_m)$$

where $y_{i+1}v(1,\dots,r,i,a_{r+2},\dots,a_m)$ equals zero by the inductive hypothesis of C'.

(b) Suppose that $i \notin \{a_1,\dots,a_m\}$, so that $a_r \leqslant i-1$ and $a_{r+1} \geqslant i+1$. Then $y_i v(a_1,\dots,a_m) = 0$ by part the previous part 2.$(a)ii$..

3. (a) Suppose that $a_r = i$. If $i = r$ then $\Psi\overset{i-1}{\underset{r}{\downarrow}}$ is trivial. Then

$$\begin{aligned} y_i v(a_1,\dots,a_m) &= y_i \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))} \\ &= \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ y_i z_{((n-m),(1^m))} = 0. \end{aligned}$$

So suppose that $i \geqslant r+1$. Then

$$\begin{aligned} &y_i v(a_1,\dots,a_m) \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ y_i \psi_{i-1} \Psi\overset{i-2}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{i-1} y_{i-1} \Psi\overset{i-2}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{i-1} y_{i-1} v(1,\dots,r-1,i-1,a_{r+1},\dots,a_m) \end{aligned}$$

where $y_{i-1}v(1,\dots,r-1,i-1,a_{r+1},\dots,a_m)$ equals zero by the inductive hypothesis of C if $i \not\equiv 3+\kappa_2-\kappa_1 \pmod e$. Now suppose that $i \equiv 3+\kappa_2-\kappa_1 \pmod e$. Then we have

$$\begin{aligned} &\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{i-1} y_{i-1} \psi_{i-2} \Psi\overset{i-3}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{i-1} (\psi_{i-2} y_{i-2} + 1) \Psi\overset{i-3}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}. \end{aligned}$$

The first term becomes

$$\begin{aligned} &\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{i-1} \psi_{i-2} y_{i-2} \Psi\overset{i-3}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{i-1} \psi_{i-2} y_{i-2} v(1,\dots,r-1,i-2,a_{r+1},\dots,a_m) \end{aligned}$$

where $y_{i-2}v(1,\dots,r-1,i-2,a_{r+1},\dots,a_m)$ equals zero by the inductive hypothesis of A. The second term becomes

$$\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{i-1}\Psi\overset{i-3}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{i-1}v(1,\dots,r-1,i-2,a_{r+1},\dots,a_m)
\end{aligned}$$

where $\psi_{i-1}v(1,\dots,r-1,i-2,a_{r+1},\dots,a_m)$ equals zero by part two of Lemma 5.1 since $i-1\not\equiv\kappa_2-\kappa_1 \pmod e$ and $a_{r+1}\geqslant i+1$.

(b) Suppose that $a_r\geqslant i+1$ and $a_{r-1}\leqslant i-1$. If $i=r-1$ then $\Psi\overset{i}{\underset{r}{\downarrow}}$ is trivial. So we have

$$\begin{aligned}
y_iv(a_1,\dots,a_m)&=\Psi\overset{a_{r-1}-1}{\underset{i+1}{\downarrow}}\ y_i\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&=\Psi\overset{a_{r-1}-1}{\underset{i+1}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ y_iz_{((n-m),(1^m))}\\
&=0.
\end{aligned}$$

So suppose that $i\geqslant r$. Then we have

$$\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \Psi\overset{a_r-1}{\underset{i+1}{\downarrow}}\ y_i\psi_i\Psi\overset{i-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \Psi\overset{a_r-1}{\underset{i+1}{\downarrow}}\ \psi_iy_{i+1}\Psi\overset{i-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \Psi\overset{a_r-1}{\underset{i+1}{\downarrow}}\ \psi_iy_{i+1}v(1,\dots,r-1,i,a_{r+1},\dots,a_m)
\end{aligned}$$

where $y_{i+1}v(1,\dots,r-1,i,a_{r+1},\dots,a_m)$ equals zero by the inductive hypothesis if $i\not\equiv\kappa_2-\kappa_1 \pmod e$. Now suppose that $i\equiv\kappa_2-\kappa_1 \pmod e$. Then we have

$$\begin{aligned}
&y_iv(a_1,\dots,a_m)\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}}\ y_{i+1}\psi_{i+1}\Psi\overset{i}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}}\ (\psi_{i+1}y_{i+2}-1)\Psi\overset{i}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}.
\end{aligned}$$

The first term becomes

$$\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}}\ \psi_{i+1}y_{i+2}\Psi\overset{i}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}}\ \psi_{i+1}y_{i+2}v(1,\dots,r,i+1,a_{r+1},\dots,a_m)
\end{aligned}$$

where $y_{i+2}v(1,\dots,r,i+1,a_{r+1},\dots,a_m)$ equals zero by the inductive hypothesis of B. Firstly, suppose that $a_{r+2} = i+3$. The the second term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+2}{\downarrow}}\ \Psi\overset{i}{\underset{r+1}{\downarrow}}\ \Psi\overset{i+2}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+3}{\downarrow}}\ \Psi\overset{i}{\underset{r+1}{\downarrow}}\ \psi_{i+2}^2\Psi\overset{i+1}{\underset{r+2}{\downarrow}}\ \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+3}{\downarrow}}\ \Psi\overset{i}{\underset{r+1}{\downarrow}}\ (y_{i+2}-y_{i+3})\Psi\overset{i+1}{\underset{r+2}{\downarrow}}\ \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}.
\end{aligned}$$

The first term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+3}{\downarrow}}\ y_{i+2}\Psi\overset{i}{\underset{r+1}{\downarrow}}\ \Psi\overset{i+1}{\underset{r+2}{\downarrow}}\ \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+3}{\downarrow}}\ y_{i+2}v(1,\dots,r,i+1,i+2,a_{r+3},\dots,a_m)
\end{aligned}$$

where $y_{i+2}v(1,\dots,r,i+1,i+2,a_{r+3},\dots,a_m)$ equals zero by the inductive hypothesis of B. The second term clearly equals zero by the inductive hypothesis of C.

Secondly, suppose that $a_{r+2} \geqslant i+4$. Then the second term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+3}{\downarrow}}\ \psi_{i+2}\Psi\overset{i}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{i+3}{\downarrow}}\ \psi_{i+2}v(1,\dots,r,i+1,a_{r+2},\dots,a_m)
\end{aligned}$$

where $\psi_{i+2}v(1,\dots,r,i+1,a_{r+2},\dots,a_m)$ equals zero by part two of Lemma 5.1 since $i+2 \not\equiv \kappa_2-\kappa_1 \pmod e$ and $a_{r+2} \geqslant i+4$.

We have thus shown that $y_iv(a_1,\dots,a_m)$ equals zero in the six cases given above.

We now show that y_i kills $v(a_1,\dots,a_m)$ only in the above six cases. We first suppose that $i \equiv \kappa_2-\kappa_1+1 \pmod e$, $i \notin \{a_1,\dots,a_m\}$ and $i+1 \in \{a_1,\dots,a_m\}$. Let $a_r \leqslant i-1$ and $a_{r+1} = i+1$. We have

$$\begin{aligned}
&y_iv(a_1,\dots,a_m)\\
&= y_i\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \Psi\overset{i}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ y_i\Psi\overset{i}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ (\psi_iy_{i+1}-1)\Psi\overset{i-1}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}.
\end{aligned}$$

Its first term becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_r-1}{\underset{r}{\downarrow}}\ \psi_iy_{i+1}\Psi\overset{i-1}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_{((n-m),(1^m))}$$

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \psi_i y_{i+1} v(1,\dots,r,i,a_{r+2},\dots,a_m),$$

where $y_{i+1}v(1,\dots,r,i,a_{r+2},\dots,a_m)$ equals zero by B' since $i+1 \notin \{a_1,\dots,a_m\}$. Now its second term is

$$-\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{i-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} = -v(a_1,\dots,a_r,i,a_{r+2},\dots,a_m),$$

which is clearly non-zero.

Finally, suppose that $i \equiv \kappa_2 - \kappa_1 + 2 \pmod e$, $i-1 \notin \{a_1,\dots,a_m\}$ and $i \in \{a_1,\dots,a_m\}$. We let $a_r \leqslant i-2$ and $a_{r+1} = i$. Then

$$\begin{aligned} & y_i v(a_1,\dots,a_m) \\ &= y_i \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{i-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} y_i \Psi\overset{i-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} (\psi_{i-1}y_{i-1}+1) \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}. \end{aligned}$$

The first term is

$$\begin{aligned} & \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \psi_{i-1}y_{i-1} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &\quad = \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \psi_{i-1}y_{i-1} v(1,\dots,r,i-1,a_{r+2},\dots,a_m), \end{aligned}$$

where $y_{i-1}v(1,\dots,r,i-1,a_{r+2},\dots,a_m)$ equals zero by A since $i-1 \in \{a_1,\dots,a_m\}$. Its second term is

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_r-1}{\underset{r}{\downarrow}} \Psi\overset{i-2}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} = v(a_1,\dots,a_r,i-1,a_{r+2},\dots,a_m),$$

which is clearly non-zero. □

Chapter 6

Homomorphisms between Specht modules in level two of $\mathscr{R}_n^\Lambda$

We introduce certain Specht module homomorphisms over $\mathscr{R}_n^\Lambda$, involving the Specht modules $S_{((n-m),(1^m))}$, $S_{((n-m,1^m),\varnothing)}$ and $S_{(\varnothing,(n-m,1^m))}$. The action of $\mathscr{R}_n^\Lambda$ on basis elements of Specht modules, determined in the previous chapter, allows us to to establish these homomorphisms. We will see that these Specht module homomorphisms are instrumental in finding the composition factors of Specht modules labelled by hook bipartitions, for they arise as quotient modules of the images and kernels of these homomorphisms.

6.1 Specht modules labelled by hook partitions in level two of $\mathscr{R}_n^\Lambda$

We consider Specht modules labelled by bipartitions $\lambda \in \mathscr{P}_n^2$, where one component of λ is a hook partition $(n-m, 1^m)$ and the other component is the empty partition $\varnothing$, and define their homogeneous basis elements, similarly to Section 3.1 and Section 5.1.

For a standard $((n-m,1^m),\varnothing)$-tableau S, we write $b_j := S(1,j,1)$ for all $i \in \{2,\dots,m+1\}$. Then S is determined by $b_2,\dots,b_{m+1}$. Analogously to the homogeneous elements of $((n-m),(1^m))$, we write

$$v_S = \Psi \overset{b_2-1}{\underset{2}{\downarrow}} \Psi \overset{b_3-1}{\underset{3}{\downarrow}} \dots \Psi \overset{b_{m+1}-1}{\underset{m+1}{\downarrow}} z_{((n-m,1^m),\varnothing)}.$$

Then v_S is completely determined by $b_2,\dots,b_{m+1}$, written $v(b_2,\dots,b_{m+1}) := v_S$.

Whereas, for $T \in \mathrm{Std}(((\varnothing,(n-m,1^m)))$, we write $c_j := T(j,1,2)$ for all $i \in \{2,\dots,m+1\}$. Then T is determined by $c_2,\dots,c_{m+1}$, and we can write

$$v_T = \Psi \overset{c_2-1}{\underset{2}{\downarrow}} \Psi \overset{c_3-1}{\underset{3}{\downarrow}} \dots \Psi \overset{c_{m+1}-1}{\underset{m+1}{\downarrow}} z_{(\varnothing,(n-m,1^m))}.$$

Hence, v_T is completely determined by $c_2, \dots, c_{m+1}$, and we write $v(c_2, \dots, c_{m+1}) := v_T$.

By Section 1.3.1.2, we found the Garnir relations for $S_{((n-m,1^m),\varnothing)}$, and hence, we also know the Garnir relations for $S_{(\varnothing,(n-m,1^m))}$.

Remark 6.1. *$S_{((n-m,1^m),\varnothing)}$ is generated by $z_{((n-m,1^m),\varnothing)}$ subject only to the defining relations*

- $e(\mathbf{i})z_{((n-m,1^m),\varnothing)} = \delta_{\mathbf{i}_{((n-m,1^m),\varnothing)},\mathbf{i}} z_{((n-m,1^m),\varnothing)}$;
- $y_r z_{((n-m,1^m),\varnothing)} = 0$ *for all* $r \in \{1, \dots, n\}$;
- $\psi_r z_{((n-m,1^m),\varnothing)} = 0$ *for all* $r \in \{1, \dots, m\} \cup \{m+2, \dots, n-1\}$;
- $\psi_1 \dots \psi_{m+1} z_{((n-m,1^m),\varnothing)} = 0$.

Similarly, $S_{(\varnothing,(n-m,1^m))}$ is generated by $z_{(\varnothing,(n-m,1^m))}$ subject only to the last three defining relations that $z_{((n-m,1^m),\varnothing)}$ satisfies together with the relation $e(\mathbf{i})z_{(\varnothing,(n-m,1^m))} = \delta_{\mathbf{i}_{(\varnothing,(n-m,1^m))},\mathbf{i}} z_{(\varnothing,(n-m,1^m))}$.

6.2 Specht module homomorphisms

For $\lambda, \mu \in \mathscr{P}_n^2$, we consider Specht module $\mathscr{R}_n^\Lambda$-homomorphisms $H : S_\lambda \to S_\mu$. It will become apparent that, for all $T \in \mathrm{Std}(\lambda)$, the homomorphism H maps $v_T \in \mathscr{R}_n^\Lambda$ to either 0 or a standard basis element v_S, for $S \in \mathrm{Std}(\mu)$. It is obvious that we will then have $\mathbf{i}_T = \mathbf{i}_S$ when H is a non-trivial $\mathscr{R}_n^\Lambda$-homomorphism. In fact, we arrived at the following homomorphisms by first observing the residue sequences of standard λ-tableaux for $\lambda \in \{((n-m),(1^m)), ((n-m-1),(1^{m+1})), ((n-m,1^m),\varnothing), (\varnothing,(n-m+1,1^{m-1})), ((n-m-1,1^{m+1}),\varnothing), ((n-m-1,1^{m+1}),\varnothing), (\varnothing,(n-m,1^m)), ((n-m+1,1^{m-1}),\varnothing)\}$.

Proposition 6.2. 1. *If $n \equiv \kappa_2 - \kappa_1 + 1 \pmod e$ and $0 \leqslant m \leqslant n-1$, then there exists the following non-zero homomorphism of Specht modules*

$$\begin{aligned}\gamma_m : S_{((n-m),(1^m))} &\to S_{((n-m-1),(1^{m+1}))}\\ z_{((n-m),(1^m))} &\mapsto v(1, \dots, m, n).\end{aligned}$$

2. *If $\kappa_2 \equiv \kappa_1 - 1 \pmod e$, then there exist the following two non-zero homomorphisms of Specht modules.*

   (a) For $1 \leqslant m \leqslant n-1$,

$$\begin{aligned}\chi_m : S_{((n-m,1^m),\varnothing)} &\to S_{((n-m),(1^m))}\\ z_{((n-m,1^m),\varnothing)} &\mapsto v(2, 3, \dots, m+1).\end{aligned}$$

(b) *For* $1 \leqslant m \leqslant n$,

$$\begin{aligned} \tau_m : S_{((n-m),(1^m))} &\to S_{(\varnothing,(n-m+1,1^{m-1}))} \\ z_{((n-m),(1^m))} &\mapsto z_{(\varnothing,(n-m+1,1^{m-1}))}. \end{aligned}$$

3. *If* $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ *and* $n \equiv \kappa_2 - \kappa_1 + 1 \pmod e$, *then there exist the following three non-zero homomorphisms of Specht modules.*

(a) *For* $0 \leqslant m \leqslant n-2$,

$$\begin{aligned} \alpha_m : S_{((n-m,1^m),\varnothing)} &\to S_{((n-m-1,1^{m+1}),\varnothing)} \\ z_{((n-m,1^m),\varnothing)} &\mapsto v(1,\dots,m+1,n). \end{aligned}$$

(b) *For* $0 \leqslant m \leqslant n-2$,

$$\begin{aligned} \beta_m : S_{(\varnothing,(n-m,1^m))} &\to S_{(\varnothing,(n-m-1,1^{m+1}))} \\ z_{(\varnothing,(n-m,1^m))} &\mapsto v(1,\dots,m+1,n). \end{aligned}$$

(c) *For* $1 \leqslant m \leqslant n-1$,

$$\begin{aligned} \phi_m : S_{((n-m+1,1^{m-1}),\varnothing)} &\to S_{((n-m),(1^m))} \\ z_{(n-m+1,1^{m-1})} &\mapsto v(2,3,\dots,m,n). \end{aligned}$$

Proof. Residues are taken modulo e throughout.

1. Firstly, let $m < n-1$. We see that $s \overset{n-1}{\underset{m+1}{\downarrow}} T_{((n-m-1),(1^{m+1}))}$ is the non-zero $((n-m-1),(1^{m+1}))$-tableau

$$s \overset{n-1}{\underset{m+1}{\downarrow}} T_{((n-m),(1^m))} = \left(\begin{array}{|c|c|c|c|} \hline m{+}1 & m{+}2 & \cdots\cdots & n-1 \\ \hline \end{array}\, , \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline m \\ \hline n \\ \hline \end{array} \right).$$

Hence $\Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),(1^{m+1}))} \neq 0$.

Recall the presentation of $S_{((n-m),(1^m))}$ as given in Section 5.2. We show that $\Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))}$ satisfies the defining relations that $z_{((n-m),(1^m))}$ satisfies.

Entries $1, 2, \dots, n-1$ lie in the same nodes in $T_{((n-m),(1^m))}$ and $s \overset{n-1}{\underset{m+1}{\downarrow}} T_{((n-m-1),(1^{m+1}))}$. Now n lies in node $(1, 1, n-m)$ in $T_{((n-m),(1^m))}$, whereas n lies in node $(2, m+1, 1)$

in $s \overset{n-1}{\underset{m+1}{\downarrow}} T_{((n-m-1),(1^{m+1}))}$. Since

$$\mathrm{res}(1,1,n-m) = \kappa_1 + n - m - 1 = \kappa_2 - m = \mathrm{res}(2, m+1, 1),$$

$T_{((n-m),(1^m))}$ and $s \overset{n-1}{\underset{m+1}{\downarrow}} T_{((n-m-1),(1^{m+1}))}$ share the same e-residue sequence, say $\mathbf{i}$, and thus $e(\mathbf{i}) z_{((n-m-1),(1^{m+1}))} = z_{((n-m-1),(1^{m+1}))}$.

It is clear that $y_i \Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} = 0$ for all $i \in \{1, \ldots, m\}$ and $\psi_i \Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} = 0$ for all $i \in \{1, \ldots, m-1\}$.

(a) Let $i \in \{m+1, \ldots, n-1\}$. Then

$$y_i \Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} = \Psi \overset{n-1}{\underset{i+1}{\downarrow}} y_i \Psi \overset{i}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))},$$

which clearly equals zero by part 3 of Lemma 5.3 if $i \not\equiv 1, 2 + \kappa_2 - \kappa_1 \pmod e$. So suppose that $i \equiv 1 + \kappa_2 - \kappa_1 \pmod e$. Then $i \leqslant n-3$, so we have

$$\begin{aligned}
&\Psi \overset{n-1}{\underset{i+1}{\downarrow}} y_i \psi_i \Psi \overset{i-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} \\
&= \Psi \overset{n-1}{\underset{i+1}{\downarrow}} (\psi_i y_{i+1} - 1) \Psi \overset{i-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} \\
&= \Psi \overset{n-1}{\underset{i+1}{\downarrow}} \psi_i y_{i+1} \Psi \overset{i-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} - \Psi \overset{n-1}{\underset{i+1}{\downarrow}} \Psi \overset{i-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} \\
&= \Psi \overset{n-1}{\underset{i+1}{\downarrow}} \psi_i \Psi \overset{i-1}{\underset{m+1}{\downarrow}} y_{i+1} z_{((n-m-1),1^{m+1}))} - \Psi \overset{n-1}{\underset{i+2}{\downarrow}} \Psi \overset{i-1}{\underset{m+1}{\downarrow}} \psi_{i+1} z_{((n-m-1),1^{m+1}))} \\
&= 0.
\end{aligned}$$

Now suppose that $i \equiv 2 + \kappa_2 - \kappa_1 \pmod e$. Then

$$\begin{aligned}
\Psi \overset{n-1}{\underset{i+1}{\downarrow}} y_i \psi_i \Psi \overset{i-1}{\underset{m+1}{\downarrow}} \Psi \overset{n-1}{\underset{i+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} &= \Psi \overset{n-1}{\underset{i+1}{\downarrow}} \psi_i y_{i+1} \Psi \overset{i-1}{\underset{m+1}{\downarrow}} \Psi \overset{n-1}{\underset{i+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} \\
&= \Psi \overset{n-1}{\underset{i+1}{\downarrow}} \psi_i \Psi \overset{i-1}{\underset{m+1}{\downarrow}} \Psi \overset{n-1}{\underset{i+1}{\downarrow}} y_{i+1} z_{((n-m-1),1^{m+1}))} \\
&= 0.
\end{aligned}$$

Finally,

$$y_n \Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} = y_n v(1, \ldots, m, n),$$

which is zero by part 1 of Lemma 5.3 since $n \equiv 1 + \kappa_2 - \kappa_1 \pmod e$.

(b) Let $i \in \{m+1, \ldots, n-2\}$. Firstly, suppose that $i \not\equiv \kappa_2 - \kappa_1 \pmod e$. Then

$$\psi_i \Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} = \Psi \overset{n-1}{\underset{i+2}{\downarrow}} \psi_i \psi_{i+1} \psi_i \Psi \overset{i-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))}$$

$$
\begin{aligned}
&= \Psi\overset{n-1}{\underset{i+2}{\downarrow}} \psi_{i+1}\psi_i\psi_{i+1}\Psi\overset{i-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))}\\
&= \Psi\overset{n-1}{\underset{m+1}{\downarrow}} \psi_{i+1} z_{((n-m-1),1^{m+1}))}\\
&= 0.
\end{aligned}
$$

Now suppose that $i \equiv \kappa_2 - \kappa_1 \pmod{e}$. Then

$$
\begin{aligned}
&\psi_i\Psi\overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))}\\
&= \Psi\overset{n-1}{\underset{i+2}{\downarrow}} \psi_i\psi_{i+1}\psi_i\Psi\overset{i-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))}\\
&= \Psi\overset{n-1}{\underset{i+2}{\downarrow}} (\psi_{i+1}\psi_i\psi_{i+1} - 1)\Psi\overset{i-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))}\\
&= \Psi\overset{n-1}{\underset{m+1}{\downarrow}} \psi_{i+1} z_{((n-m-1),1^{m+1}))} - \Psi\overset{n-1}{\underset{i+3}{\downarrow}}\ \Psi\overset{i-1}{\underset{m+1}{\downarrow}} \psi_{i+2} z_{((n-m-1),1^{m+1}))}\\
&= 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\psi_{n-1}\Psi\overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))} &= \psi_{n-1}^2\Psi\overset{n-2}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))}\\
&= (y_{n-1} - y_n)\Psi\overset{n-2}{\underset{m+1}{\downarrow}} z_{((n-m-1),1^{m+1}))}.
\end{aligned}
$$

The second term of this expression is clearly zero, and the first term is zero by part 3 of Lemma 5.3 since $n-1 \equiv \kappa_2 - \kappa_1 \pmod{e}$.

Now let $m = n-1$. Clearly, $z_{((n-m-1),(1^{m+1}))}$ is non-zero. We see that $T_{((n-m),(1^m))}$ and $T_{((n-m-1),(1^{m+1}))}$ share the same e-residue sequence since

$$\mathrm{res}(1,1,1) = \kappa_1 = \kappa_2 + 1 - n = \mathrm{res}(2,n,1).$$

The remaining relations are trivial.

2. (a) We show that $\psi_1 \dots \psi_m z_{((n-m),(1^m))}$ satisfies the defining relations that $z_{((n-m,1^m),\varnothing)}$ satisfies in Section 6.1. Observe that we have the following standard tableaux

$$
s\overset{m}{\underset{1}{\uparrow}} T_{((n-m),(1^m))} = \left(\begin{array}{|c|c|c|c|} \hline 1 & m{+}2 & \cdots\cdots & n \\ \hline \end{array}\,,\ \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline m{+}1 \\ \hline \end{array} \right)
$$

and

$$T_{((n-m,1^m),\varnothing)} = \begin{array}{|c|c|c|c|}\hline 1 & m{+}2 & \cdots\cdots & n \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \vdots \\ \cline{1-1} m{+}1 \\ \cline{1-1} \end{array}.$$

Hence $\Psi\overset{m}{\underset{1}{\uparrow}} z_{((n-m),(1^m))} \neq 0$. Entries 1 and $m+2,\dots,n$ lie in the same nodes in $s\overset{m}{\underset{1}{\uparrow}} T_{((n-m),(1^m))}$ and $T_{((n-m,1^m),\varnothing)}$. For $2 \leqslant j \leqslant m+1$, we see that j lies in node $(j-1,1,2)$ in $s\overset{m}{\underset{1}{\uparrow}} T_{((n-m),(1^m))}$, whereas j lies in node $(j,1,1)$ in $T_{((n-m,1^m),\varnothing)}$. We have

$$\mathrm{res}(j-1,1,2) = \kappa_2 + 2 - i = \kappa_1 + 1 - i = \mathrm{res}(j,1,1),$$

that is, $s\overset{m}{\underset{1}{\uparrow}} T_{((n-m),(1^m))}$ and $T_{((n-m,1^m),\varnothing)}$ share the same e-residue sequence, say $\mathbf{i}$, and hence $e(\mathbf{i})\Psi\overset{m}{\underset{1}{\uparrow}} z_{((n-m),(1^m))} = \Psi\overset{m}{\underset{1}{\uparrow}} z_{((n-m),(1^m))}$, as required.

i. Firstly, by part two of Lemma 5.3, we have

$$y_1\psi_1\dots\psi_m z_{((n-m),(1^m))} = y_1 v(2,\dots,m+1) = 0.$$

For all $i \in \{2,\dots,m+1\}$,

$$\begin{aligned} y_i\left(\psi_1\dots\psi_m z_{((n-m),(1^m))}\right) =& \psi_1\dots\psi_{i-2}y_i\psi_{i-1}\psi_i\dots\psi_m z_{((n-m),(1^m))} \\ =& \psi_1\dots\psi_{i-2}y_i v(1,\dots,i-2,i,i+1,\dots,m+1) \\ =& 0 \end{aligned}$$

since $y_i v(1,\dots,i-2,i,i+1,\dots,m+1)$ equals zero by parts one and three of Lemma 5.3 if $i \not\equiv 2+\kappa_2-\kappa_1 \pmod{e}$. Now suppose that $i \equiv 2+\kappa_2-\kappa_1 \pmod{e}$. Then

$$\begin{aligned} & y_i\left(\psi_1\dots\psi_m z_{((n-m),(1^m))}\right) \\ &= \psi_1\dots\psi_{i-2}y_i\psi_{i-1}\psi_i\dots\psi_m z_{((n-m),(1^m))} \\ &= \psi_1\dots\psi_{i-2}(\psi_{i-1}y_{i-1}+1)\psi_i\dots\psi_m z_{((n-m),(1^m))} \\ &= \psi_1\dots\psi_{i-2}\psi_{i-1}y_{i-1}\psi_i\dots\psi_m z_{((n-m),(1^m))} \\ &\qquad + \psi_1\dots\psi_{i-2}\psi_i\dots\psi_m z_{((n-m),(1^m))} \\ &= \psi_1\dots\psi_{i-2}\psi_{i-1}\psi_i\dots\psi_m y_{i-1} z_{((n-m),(1^m))} \\ &\qquad + \psi_1\dots\psi_{i-3}\psi_i\dots\psi_m\psi_{i-2} z_{((n-m),(1^m))} \\ &= 0. \end{aligned}$$

Clearly, for all $i \in \{m+2, \ldots, n\}$, $y_i \left(\psi_1 \ldots \psi_m z_{((n-m),(1^m))}\right) = 0$.

ii. Observe

$$\begin{aligned}
&\psi_1^2 \psi_2 \ldots \psi_m z_{((n-m),(1^m))} \\
&= (y_2 - y_1)\psi_2 \ldots \psi_m z_{((n-m),(1^m))} \\
&= y_2 \psi_2 \ldots \psi_m z_{((n-m),(1^m))} - y_1 \psi_2 \ldots \psi_m z_{((n-m),(1^m))} \\
&= y_2 v(3, \ldots, m+1) - \psi_2 \ldots \psi_m y_1 z_{((n-m),(1^m))} \\
&= 0,
\end{aligned}$$

since $y_2 v(3, \ldots, m+1)$ equals zero by part three of Lemma 5.3.
For all $i \in \{2, \ldots, m\}$, we have

$$\begin{aligned}
&\psi_i \left(\psi_1 \ldots \psi_m z_{((n-m),(1^m))}\right) \\
&= \psi_1 \ldots \psi_{i-2} \psi_i \psi_{i-1} \ldots \psi_m z_{((n-m),(1^m))} \\
&= \psi_1 \ldots \psi_{i-2} \psi_i v(1, \ldots, i-2, i, i+1, \ldots, m+1) \\
&= 0,
\end{aligned}$$

since $\psi_i v(1, \ldots, i-2, i, i+1, \ldots, m+1)$ equals zero by part one of Lemma 5.1 if $i \not\equiv 2 + \kappa_2 - \kappa_1 \pmod e$. So suppose that $i \equiv 2 + \kappa_2 - \kappa_1 \pmod e$. Then

$$\begin{aligned}
&\psi_i \left(\psi_1 \ldots \psi_m z_{((n-m),(1^m))}\right) \\
&= \psi_1 \ldots \psi_{i-2} \psi_i \psi_{i-1} \psi_i \psi_{i+1} \ldots \psi_m z_{((n-m),(1^m))} \\
&= \psi_1 \ldots \psi_{i-2} (\psi_{i-1} \psi_i \psi_{i-1} + 1) \psi_{i+1} \ldots \psi_m z_{((n-m),(1^m))} \\
&= \psi_1 \ldots \psi_{i-2} \psi_{i-1} \psi_i \psi_{i-1} \psi_{i+1} \ldots \psi_m z_{((n-m),(1^m))} \\
&\qquad + \psi_1 \ldots \psi_{i-2} \psi_{i+1} \ldots \psi_m z_{((n-m),(1^m))} \\
&= \psi_1 \ldots \psi_{i-2} \psi_{i-1} \psi_i \psi_{i+1} \ldots \psi_m \psi_{i-1} z_{((n-m),(1^m))} \\
&\qquad + \psi_1 \ldots \psi_{i-3} \psi_{i+1} \ldots \psi_m \psi_{i-2} z_{((n-m),(1^m))} \\
&= 0.
\end{aligned}$$

For all $i \in \{m+2, \ldots, n-1\}$, it is trivial that $\psi_i \left(\psi_1 \ldots \psi_m z_{((n-m),(1^m))}\right) = 0$.

iii. By part two of Lemma 5.1,

$$\begin{aligned}
\psi_1 \psi_2 \ldots \psi_{m+1} \left(\psi_1 \ldots \psi_m z_{((n-m),(1^m))}\right) &= \psi_1 \overset{2}{\underset{1}{\Psi\!\downarrow}} \ldots \overset{m+1}{\underset{m}{\Psi\!\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 v(3, \ldots, m+2) \\
&= 0.
\end{aligned}$$

(b) We know from Remark 6.1 that $z_{(\varnothing,(n-m+1,1^{m-1}))}$ satisfies the second and third defining relations that $z_{((n-m),(1^m))}$ satisfies, given in Section 5.2. Thus, one need only check that $e(\mathbf{i}_{((n-m),(1^m))})z_{(\varnothing,(n-m+1,1^{m-1}))} = z_{(\varnothing,(n-m+1,1^{m-1}))}$, that is, $T_{((n-m),(1^m))}$ and $T_{(\varnothing,(n-m+1,1^{m-1}))}$ share the same e-residue sequence. Observe that we have the following non-zero tableaux

$$T_{((n-m),(1^m))} = \begin{array}{|c|c|c|}\hline m{+}1 & \cdots\cdots & n \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline m \\ \hline \end{array}$$

and

$$T_{(\varnothing,(n-m+1,1^{m-1}))} = \begin{array}{|c|c|c|c|}\hline 1 & m{+}1 & \cdots\cdots & n \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \vdots \\ \cline{1-1} m \\ \cline{1-1} \end{array}.$$

Entries $1,\ldots,m$ lie in the same nodes in $T_{((n-m),(1^m))}$ and $T_{(\varnothing,(n-m+1,1^{m-1}))}$. For $1 \leqslant j \leqslant n-m$, $m+j$ lies in node $(1,j,1)$ in $T_{((n-m),(1^m))}$, whereas $m+j$ lies in node $(1,j,2)$ in $T_{(\varnothing,(n-m+1,1^{m-1}))}$. We see that $T_{((n-m),(1^m))}$ and $T_{(\varnothing,(n-m+1,1^{m-1}))}$ share the same e-residue sequence, say $\mathbf{i}$, since $\mathrm{res}(1,j,1) = \kappa_1 + j - 1 = \kappa_2 + j = \mathrm{res}(1,j+1,2)$, and hence $e(\mathbf{i})z_{(\varnothing,(n-m+1,1^{m-1}))} = z_{(\varnothing,(n-m+1,1^{m-1}))}$.

3. (a) We see that $s\overset{n-1}{\underset{m+2}{\downarrow}} T_{((n-m-1,1^{m+1}),\varnothing)}$ is the standard $((n-m-1,1^{m+1}),\varnothing)$-tableau

$$s\overset{n-1}{\underset{m+2}{\downarrow}} T_{((n-m-1,1^{m+1}),\varnothing)} = \begin{array}{|c|c|c|c|}\hline 1 & m{+}2 & \cdots\cdots & n-1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \vdots \\ \cline{1-1} m{+}1 \\ \cline{1-1} n \\ \cline{1-1} \end{array}.$$

Hence $\Psi\overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)} \neq 0$.

We show that $\Psi\overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}$ satisfies the defining relations that $z_{((n-m,1^m),\varnothing)}$ satisfies in Section 6.1.

Entries $1,2,\ldots,n-1$ lie in the same nodes in tableaux $T_{((n-m,1^m),\varnothing)}$ and $s\overset{n-1}{\underset{m+2}{\downarrow}} T_{((n-m-1,1^{m+1}),\varnothing)}$. Now n lies in node $(1,n-m,1)$ in $T_{((n-m,1^m),\varnothing)}$, whereas n lies in node $(m+2,1,1)$ in $s \overset{n-1}{\underset{m+2}{\downarrow}} T_{((n-m-1,1^{m+1}),\varnothing)}$. We have

$$\mathrm{res}(1,n-m,1)=\kappa_1+n-m-1=\kappa_2-m=\kappa_1-m-1=\mathrm{res}(m+2,1,1).$$

So $T_{((n-m,1^m),\varnothing)}$ and $s \overset{n-1}{\underset{m+2}{\downarrow}} T_{((n-m-1,1^{m+1}),\varnothing)}$ share the same e-residue sequence, say $\mathbf{i}$, and hence $e(\mathbf{i})\Psi \overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)} = \Psi \overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}$, as required.

i. Clearly, $y_i\left(\Psi \overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}\right)=0$ for all $i\in\{1,\dots,m\}$.
Let $m+2\leqslant i\leqslant n-1$. Then

$$y_i\left(\Psi \overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}\right)=\Psi \overset{n-1}{\underset{i+1}{\downarrow}} y_i\Psi \overset{i}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)},$$

which equals zero by Lemma 5.3 if $i\not\equiv 1+\kappa_2-\kappa_1 \pmod e$. So suppose that $i\equiv 1+\kappa_2-\kappa_1 \pmod e$. Then

$$\begin{aligned}
&y_i\left(\Psi \overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}\right)\\
&=\Psi \overset{n-1}{\underset{i+1}{\downarrow}} y_i\psi_i\Psi \overset{i-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}\\
&=\Psi \overset{n-1}{\underset{i+1}{\downarrow}} (\psi_iy_{i+1}-1)\Psi \overset{i-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}\\
&=\Psi \overset{n-1}{\underset{i}{\downarrow}} y_{i+1}\Psi \overset{i-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}-\Psi \overset{n-1}{\underset{i+1}{\downarrow}} \Psi \overset{i-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}\\
&=\Psi \overset{n-1}{\underset{i}{\downarrow}} \Psi \overset{i-1}{\underset{m+2}{\downarrow}} y_{i+1}z_{((n-m-1,1^{m+1}),\varnothing)}-\Psi \overset{n-1}{\underset{i+2}{\downarrow}} \Psi \overset{i-1}{\underset{m+2}{\downarrow}} \psi_{i+1}z_{((n-m-1,1^{m+1}),\varnothing)}\\
&=0.
\end{aligned}$$

Further, $y_n\Psi \overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}=y_nv(1,\dots,m+1,n)$, which equals zero by Lemma 5.3 part one.

ii. Clearly, $\psi_i\left(\Psi \overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}\right)=0$ for all $i\in\{1,\dots,m\}$.
Let $m+2\leqslant i\leqslant n-2$. Then

$$\begin{aligned}
&\psi_i\left(\Psi \overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}\right)\\
&=\Psi \overset{n-1}{\underset{i+2}{\downarrow}} \psi_i\psi_{i+1}\psi_i\Psi \overset{i-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}.
\end{aligned}$$

This expression equals

$$\Psi \overset{n-1}{\underset{i+2}{\downarrow}} \psi_{i+1}\psi_i\psi_{i+1}\Psi \overset{i-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}$$

if $i \not\equiv \kappa_2 - \kappa_1 \pmod{e}$, and equals

$$\Psi \overset{n-1}{\underset{i+2}{\downarrow}} (\psi_{i+1}\psi_i\psi_{i+1} - 1)\Psi \overset{i-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)}$$

if $i \equiv \kappa_2 - \kappa_1 \pmod{e}$, which is zero in both cases.
Further,

$$\begin{aligned}
&\psi_{n-1}^2 \Psi \overset{n-2}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)} \\
&= (y_{n-1} - y_n)\Psi \overset{n-2}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)} \\
&= y_{n-1}\Psi \overset{n-2}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)} - y_n \Psi \overset{n-2}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)} \\
&= y_{n-1} v(1, \ldots, m+1, n-1) - \Psi \overset{n-2}{\underset{m+2}{\downarrow}} y_n z_{((n-m-1,1^{m+1}),\varnothing)} \\
&= 0
\end{aligned}$$

by part three of Lemma 5.3.

(b) We similarly check that the homomorphism β_m holds as α_m does in the previous part. In particular, we see that n lies in node $(1, n-m, 2)$ in $T_{(\varnothing,(n-m,1^m))}$, whereas n lies in node $(m+2, 1, 2)$ in $s \overset{n-1}{\underset{m+2}{\downarrow}} T_{(\varnothing,(n-m-1,1^{m+1}))}$. Thus, $T_{(\varnothing,(n-m,1^m))}$ and $s \overset{n-1}{\underset{m+2}{\downarrow}} T_{(\varnothing,(n-m-1,1^{m+1}))}$ share the same e-residue sequence by observing that

$$\operatorname{res}(1, n-m, 2) = \kappa_2 + n - m - 1 = \kappa_2 - m - 1 = \operatorname{res}(m+2, 1, 2).$$

(c) We show that $\psi_1\psi_2 \ldots \psi_{m-1}\Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}$ satisfies the defining relations that $z_{((n-m+1,1^{m-1}),\varnothing)}$ satisfies in Section 6.1.
Observe that we have the non-zero tableaux

$$s \overset{m-1}{\underset{1}{\uparrow}} s \overset{n-1}{\underset{m}{\downarrow}} T_{((n-m),(1^m))} = \begin{array}{|c|c|c|c|c|} \hline 1 & m+1 & m+2 & \cdots\cdots & n-1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline m \\ \hline n \\ \hline \end{array}$$

and

$$T_{((n-m+1,1^{m-1}),\varnothing)} = \begin{array}{|c|c|c|c|c|}\hline 1 & m{+}1 & m{+}2 & \cdots\cdots & n \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \vdots \\ \cline{1-1} m \\ \cline{1-1}\end{array}.$$

Entries 1 and $m+1,\dots,n-1$ lie in the same node in $s \overset{m-1}{\underset{1}{\uparrow}} s \overset{n-1}{\underset{m}{\downarrow}} T_{((n-m),(1^m))}$ and $T_{((n-m+1,1^{m-1}),\varnothing)}$. For $2 \leqslant j \leqslant m$,

$$s \overset{m-1}{\underset{1}{\uparrow}} s \overset{n-1}{\underset{m}{\downarrow}} T_{((n-m),(1^m))}(j-1,1,2) = j = T_{((n-m+1,1^{m-1}),\varnothing)}(j,1,1)$$

where $\mathrm{res}(j-1,1,2) = \kappa_2 - j + 2 = \kappa_1 + 1 - j = \mathrm{res}(j,1,1)$. Now $s \overset{m-1}{\underset{1}{\uparrow}} s \overset{n-1}{\underset{m}{\downarrow}} T_{((n-m),(1^m))}(m,1,2) = n = T_{((n-m+1,1^{m-1}),\varnothing)}(1,n-m+1,1)$ where $\mathrm{res}(m,1,2) = \kappa_2 + 1 - m = \kappa_1 + n - m = \mathrm{res}(1,n-m+1,1)$. So $s \overset{m-1}{\underset{1}{\uparrow}} s \overset{n-1}{\underset{m}{\downarrow}} T_{((n-m),(1^m))}$ and $T_{((n-m+1,1^{m-1}),\varnothing)}$ share the same e-residue sequence, say $\mathbf{i}$, and hence $e(\mathbf{i})\Psi \overset{m-1}{\underset{1}{\uparrow}} \Psi \overset{n-1}{\underset{m}{\downarrow}} T_{((n-m),(1^m))} = \Psi \overset{m-1}{\underset{1}{\uparrow}} \Psi \overset{n-1}{\underset{m}{\downarrow}} T_{((n-m),(1^m))}$.

i. Firstly, by part two of Lemma 5.3, we have

$$y_1\left(\psi_1 \dots \psi_{m-1}\Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) = y_1 v(2,\dots,m,n) = 0.$$

For all $i \in \{2,\dots,m\}$, we have

$$\begin{aligned} &y_i\left(\psi_1 \dots \psi_{m-1}\Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) \\ &= \psi_1 \dots \psi_{i-2} y_i \psi_{i-1}\psi_i \dots \psi_{m-1}\Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \psi_1 \dots \psi_{i-2} y_i v(1,\dots,i-2,i,i+1,\dots,m,n) \\ &= 0 \end{aligned}$$

since $y_i v(1,\dots,i-2,i,i+1,\dots,m,n)$ equals zero by parts one and three of Lemma 5.3 if $i \not\equiv 2 + \kappa_2 - \kappa_1 \pmod e$. So suppose that $i \equiv 2 + \kappa_2 - \kappa_1 \pmod e$. Then

$$\begin{aligned} &y_i\left(\psi_1 \dots \psi_{m-1}\Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) \\ &= \psi_1 \dots \psi_{i-2} y_i \psi_{i-1}\psi_i \dots \psi_{m-1}\Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= \psi_1 \dots \psi_{i-2} (\psi_{i-1} y_{i-1} + 1)\psi_i \dots \psi_{m-1}\Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \end{aligned}$$

$$\begin{aligned}
&= \psi_1 \dots \psi_{i-2}\psi_{i-1}y_{i-1}\psi_i \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&\quad + \psi_1 \dots \psi_{i-2}\psi_i \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} y_{i-1}z_{((n-m),(1^m))} \\
&\quad + \psi_1 \dots \psi_{i-3}\psi_i \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} \psi_{i-2}z_{((n-m),(1^m))} \\
&= 0.
\end{aligned}$$

For all $i \in \{m+1, \dots, n-1\}$, we have

$$\begin{aligned}
&y_i \left(\psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+1}{\downarrow}} y_i\Psi\overset{i}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+1}{\downarrow}} y_i v(1, \dots, m-1, i+1) \\
&= 0
\end{aligned}$$

since $y_i v(1, \dots, m-1, i+1)$ equals zero by parts two and three of Lemma 5.3 if $i \not\equiv 1 + \kappa_2 - \kappa_1 \pmod e$. Now suppose that $i \equiv 1 + \kappa_2 - \kappa_1 \pmod e$. Then

$$\begin{aligned}
&y_i \left(\psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+1}{\downarrow}} y_i\psi_i\Psi\overset{i-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+1}{\downarrow}} (\psi_i y_{i+1} - 1)\Psi\overset{i-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+1}{\downarrow}} \psi_i y_{i+1}\Psi\overset{i-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&\quad - \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+1}{\downarrow}} \Psi\overset{i-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+1}{\downarrow}} \psi_i\Psi\overset{i-1}{\underset{m}{\downarrow}} y_{i+1}z_{((n-m),(1^m))} \\
&\quad - \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+2}{\downarrow}} \Psi\overset{i-1}{\underset{m}{\downarrow}} \psi_{i+1}z_{((n-m),(1^m))} \\
&= 0.
\end{aligned}$$

Finally, $y_n \left(\psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}}\right)$ equals zero by part one of Lemma 5.3.

ii. Firstly, we have

$$\psi_1^2\psi_2 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}$$

$$
\begin{aligned}
&= (y_2 - y_1)\psi_2 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= y_2\psi_2 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} - y_1\psi_2 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= y_2 v(1,3,\dots,m,n) - \psi_2 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} y_1 z_{((n-m),(1^m))} \\
&= 0
\end{aligned}
$$

since $y_2 v(1,3,\dots,m,n)$ equals zero by part three of Lemma 5.3.

For all $i \in \{2,\dots,m-1\}$, $\psi_i\left(\psi_1\psi_2 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) = 0$ if $i \not\equiv 2 + \kappa_2 - \kappa_1 \pmod{e}$. So suppose $i \equiv 2 + \kappa_2 - \kappa_1 \pmod{e}$. Then

$$
\begin{aligned}
&\psi_i\left(\psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) \\
&= \psi_1 \dots \psi_{i-1}\psi_i^2\psi_{i+1} \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{i-1}(y_{i+1} - y_i)\psi_{i+1} \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{i-1}y_{i+1}v(1,\dots,i,i+2,\dots,m,n) \\
&\qquad - \psi_1 \dots \psi_{i-1}y_i v(1,\dots,i-1,i,i+2,\dots,m,n) \\
&= 0
\end{aligned}
$$

since $y_{i+1}v(1,\dots,i,i{+}2,\dots,m,n)$ equals zero by part three of Lemma 5.3 and $y_i v(1,\dots,i{-}1,i,i{+}2,\dots,m,n)$ equals zero by part two of Lemma 5.3.

Let $i \in \{m+1,\dots,n-2\}$. Then

$$
\begin{aligned}
&\psi_i\left(\psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+2}{\downarrow}} \psi_i v(1,\dots,m-1,i+2) \\
&= 0
\end{aligned}
$$

since $\psi_i v(1,\dots,m-1,i+2)$ equals zero by part two of Lemma 5.1 if $i \not\equiv \kappa_2 - \kappa_1 \pmod{e}$. Instead, suppose that $i \equiv \kappa_2 - \kappa_1 \pmod{e}$. Then

$$
\begin{aligned}
&\psi_i\left(\psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}\right) \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+2}{\downarrow}} (\psi_i\psi_{i+1}\psi_i)\Psi\overset{i-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+2}{\downarrow}} (\psi_{i+1}\psi_i\psi_{i+1} - 1)\Psi\overset{i-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1}\Psi\overset{n-1}{\underset{i+2}{\downarrow}} \psi_{i+1}\psi_i\psi_{i+1}\Psi\overset{i-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))}
\end{aligned}
$$

$$
\begin{aligned}
&- \psi_1 \dots \psi_{m-1} \Psi \overset{n-1}{\underset{i+2}{\downarrow}} \Psi \overset{i-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1} \Psi \overset{n-1}{\underset{i+2}{\downarrow}} \psi_{i+1} \psi_i \Psi \overset{i-1}{\underset{m}{\downarrow}} \psi_{i+1} z_{((n-m),(1^m))} \\
&- \psi_1 \dots \psi_{m-1} \Psi \overset{n-1}{\underset{i+3}{\downarrow}} \Psi \overset{i-1}{\underset{m}{\downarrow}} \psi_{i+2} z_{((n-m),(1^m))} \\
&= 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\psi_{n-1} \left(\psi_1 \dots \psi_{m-1} \Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \right) \\
&= \psi_1 \dots \psi_{m-1} \psi_{n-1}^2 \Psi \overset{n-2}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1} (y_{n-1} - y_n) \Psi \overset{n-2}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1} y_{n-1} \Psi \overset{n-2}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} - \psi_1 \dots \psi_{m-1} y_n \Psi \overset{n-2}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 \dots \psi_{m-1} y_{n-1} v(1, \dots, m-1, n-1) - \psi_1 \dots \psi_{m-1} \Psi \overset{n-2}{\underset{m}{\downarrow}} y_n z_{((n-m),(1^m))} \\
&= 0
\end{aligned}
$$

since $y_{n-1} v(1, \dots, m-1, n-1)$ equals zero by part three of Lemma 5.3.

iii. Observe

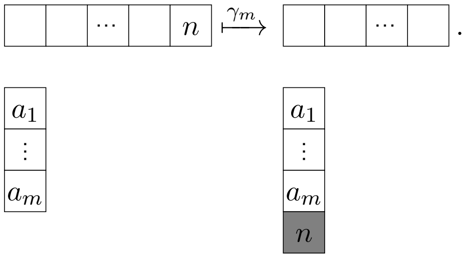
$$
\begin{aligned}
&\psi_1 \psi_2 \dots \psi_m \left(\psi_1 \dots \psi_{m-1} \Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \right) \\
&= \psi_1 \Psi \overset{2}{\underset{1}{\downarrow}} \dots \Psi \overset{m}{\underset{m-1}{\downarrow}} \Psi \overset{n-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\
&= \psi_1 v(3, \dots, m+1, n) \\
&= 0,
\end{aligned}
$$

by part two of Lemma 5.1.

□

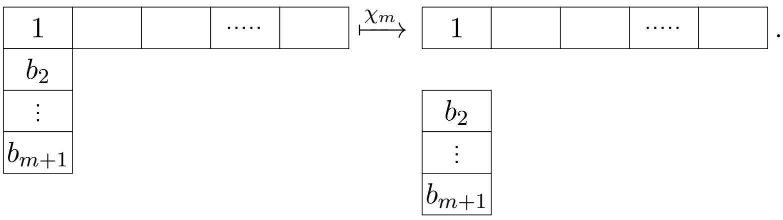
Let $v(a_1, \dots, a_m) \in S_{((n-m),(1^m))}$ and $v(b_2, \dots, b_{m+1}) \in S_{((n-m,1^m),\varnothing)}$, where $1 \leqslant a_1 < \dots < a_m < n$ and $1 < b_2 < \dots < b_{m+1} < n$. Then $v(a_1, \dots, a_m)$ corresponds to the standard $((n-m),(1^m))$-tableau with $a_1, \dots, a_m$ lying in its leg, and $v(b_2, \dots, b_{m+1})$ corresponds to the standard $((n-m,1^m),\varnothing)$-tableau with $b_2, \dots, b_{m+1}$ lying in its leg.

Informally, we can think of the action of γ_m on $v(a_1, \dots, a_m)$ by its corresponding action on the standard $((n-m),(1^m))$-tableau determined by $a_1, \dots, a_m$, which moves its hand node containing entry n to the addable node at the end of its leg as follows

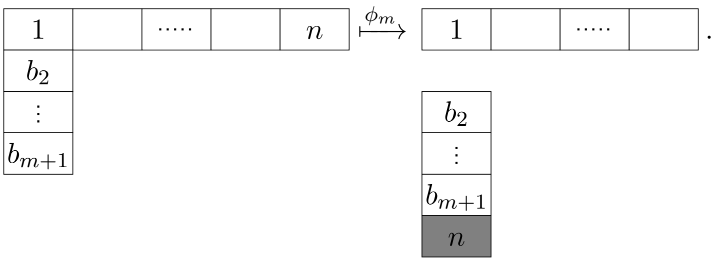


Homomorphisms α_m and β_m act similarly on standard $((n-m, 1^m), \varnothing)$- and $(\varnothing, (n-m, 1^m))$-tableaux, respectively.

The corresponding action of χ_m on $v(b_2, \dots, b_{m+1})$ essentially splits its first row of entries and its remaining rows of entries into two separate components as follows



Finally, the corresponding action of ϕ_m acts on $v(b_2, \dots, b_{m+1})$ both by the corresponding action of γ_m and by the corresponding action of χ_m as follows



We determine when the aforementioned Specht module homomorphisms act non-trivially.

Lemma 6.3. *Let $S \in \mathrm{Std}((n-m), (1^m))$, $T \in \mathrm{Std}((n-m, 1^m), \varnothing)$ and $U \in \mathrm{Std}(\varnothing, (n-m, 1^m))$, where S, T and U are determined by $\{a_1, \dots, a_m\}$, $\{b_2, \dots, b_{m+1}\}$ and $\{c_2, \dots, c_{m+1}\}$, respectively.*

1. *Let $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$. Then $\gamma_m(v_S) \neq 0$ if only if $a_m < n$. Moreover, if $a_m < n$, then*
$$\gamma_m(v_S) = v(a_1, \dots, a_m, n) \in S_{((n-m-1),(1^{m+1}))}.$$

2. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$.*

   (a) *Then*
$$0 \neq \chi_m(v_T) = v(b_2, \dots, b_{m+1}) \in S_{((n-m),(1^m))}.$$

(b) $\tau_m(v_T) \neq 0$ *if only if* $a_1 = 1$. *Moreover, if* $a_1 = 1$, *then*

$$\tau_m(v_T) = v(1, a_2, \dots, a_m) \in S_{(\varnothing,(n-m+1,1^{m-1}))}.$$

3. *Let* $n \equiv \kappa_2 - \kappa_1 + 1 \pmod e$ *and* $\kappa_2 \equiv \kappa_1 - 1 \pmod e$.

(a) $\alpha_m(v_T) \neq 0$ *if only if* $b_{m+1} < n$. *Moreover, if* $b_{m+1} < n$, *then*

$$\alpha_m(v_T) = v(b_2, \dots, b_{m+1}, n) \in S_{((n-m-1,1^{m+1}),\varnothing)}.$$

(b) $\beta_m(v_U) \neq 0$ *if only if* $c_{m+1} < n$. *Moreover, if* $c_{m+1} < n$, *then*

$$\beta_m(v_U) = v(c_2, \dots, c_{m+1}, n) \in S_{(\varnothing,(n-m-1,1^{m+1}))}.$$

(c) $\phi_{m+1}(v_T) \neq 0$ *if only if* $b_{m+1} < n$. *Moreover, if* $b_{m+1} < n$, *then*

$$\phi_{m+1}(v_T) = v(b_2, \dots, b_{m+1}, n) \in S_{((n-m-1),(1^{m+1}))}.$$

Proof. We write $\psi_{w_S} = \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}$ and $\psi_{w_T} = \Psi\overset{b_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{b_{m+1}-1}{\underset{m+1}{\downarrow}}$.

1. Let $a_m < n$. Then

$$\gamma_m\left(\psi_{w_S} z_{((n-m),(1^m))}\right) = \psi_{w_S} \Psi\overset{n-1}{\underset{m}{\downarrow}} z_{((n-m-1),(1^{m+1}))} = v(a_1, \dots, a_m, n) \neq 0.$$

Now, supposing that $a_m = n$, we have

$$\begin{aligned}\gamma_m\left(\psi_{w_S} z_{((n-m),(1^m))}\right) &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}} \Psi\overset{n-1}{\underset{m}{\downarrow}} \Psi\overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),(1^{m+1}))} \\ &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}} \psi_{n-1} v(1, \dots, m-1, n-1, n) \\ &= 0,\end{aligned}$$

since $\psi_{n-1}v(1, \dots, m-1, n-1, n)$ equals zero by the first part of Lemma 5.1.

2. (a) We have

$$\begin{aligned}\chi_m\left(\psi_{w_T} z_{((n-m,1^m),\varnothing)}\right) &= \psi_{w_T} \Psi\overset{m}{\underset{1}{\uparrow}} z_{((n-m),(1^m))} \\ &= \Psi\overset{b_2-1}{\underset{2}{\downarrow}} \Psi\overset{b_3-1}{\underset{3}{\downarrow}} \dots \Psi\overset{b_{m+1}-1}{\underset{m+1}{\downarrow}} \psi_1\psi_2 \dots \psi_m z_{((n-m),(1^m))} \\ &= \Psi\overset{b_2-1}{\underset{1}{\downarrow}} \Psi\overset{b_3-1}{\underset{2}{\downarrow}} \dots \Psi\overset{b_{m+1}-1}{\underset{m}{\downarrow}} z_{((n-m),(1^m))} \\ &= v(b_2, \dots, b_{m+1}) \\ &\neq 0.\end{aligned}$$

(b) We have

$$\begin{aligned}\tau_m\left(\psi_{w_S} z_{((n-m),(1^m))}\right) &= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_{(\varnothing,(n-m+1,1^{m-1}))} \\ &= v(a_1,\dots,a_m) \in S_{(\varnothing,(n-m+1,1^{m-1}))},\end{aligned}$$

which is clearly non-zero if $a_1 = 1$, that is, $S(1,1,2) = 1$. However, if $a_1 \neq 1$, then the corresponding tableau $w_s T_{(\varnothing,(n-m+1,1^{m-1}))}$ is not standard. Hence $\tau_m\left(\psi_{w_S} z_{((n-m),(1^m))}\right) = 0$ since $\psi_{w_S} z_{((n-m),(1^m))}$ is not a standard basis element of $S_{(\varnothing,(n-m+1,1^{m-1}))}$.

3. (a) Let $b_{m+1} < n$. Then, for $m < n-1$,

$$\alpha_m\left(\psi_{w_T} z_{((n-m,1^m),\varnothing)}\right) = \psi_{w_T} \Psi\overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}),\varnothing)} = v(b_2,\dots,b_{m+1},n) \neq 0.$$

Instead, suppose that $b_{m+1} = n$. Then

$$\begin{aligned}\alpha_m\left(\psi_{w_T} z_{((n-m,1^m),\varnothing)}\right) &= \Psi\overset{b_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{b_m-1}{\underset{m}{\downarrow}} \Psi\overset{n-1}{\underset{m}{\downarrow}} \Psi\overset{n-1}{\underset{m+2}{\downarrow}} z_{((n-m-1,1^{m+1}))} \\ &= \Psi\overset{b_2-1}{\underset{2}{\downarrow}} \dots \Psi\overset{b_m-1}{\underset{m}{\downarrow}} \psi_{n-1} v(b_2,\dots,b_m,n-1,n) \\ &= 0,\end{aligned}$$

as $\psi_{n-1} v(b_2,\dots,b_m,n-1,n)$ is zero by part one of Lemma 5.1.

(b) Similar to the previous part.

(c) Suppose that $b_{m+1} < n$. If $m < n-1$, then

$$\begin{aligned}&\phi_{m+1}\left(\psi_{w_T} z_{((n-m,1^m),\varnothing)}\right) \\ &= \psi_{w_T} \Psi\overset{m}{\underset{1}{\uparrow}} \Psi\overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),(1^{m+1}))} \\ &= \Psi\overset{b_2-1}{\underset{2}{\downarrow}} \Psi\overset{b_3-1}{\underset{3}{\downarrow}} \dots \Psi\overset{b_{m+1}-1}{\underset{m+1}{\downarrow}} \psi_1\psi_2\dots\psi_m \Psi\overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),(1^{m+1}))} \\ &= \Psi\overset{b_2-1}{\underset{1}{\downarrow}} \Psi\overset{b_3-1}{\underset{2}{\downarrow}} \dots \Psi\overset{b_{m+1}-1}{\underset{m}{\downarrow}} \Psi\overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),(1^{m+1}))} \\ &= v(b_2,\dots,b_{m+1},n) \\ &\neq 0.\end{aligned}$$

If $m = n-1$, then

$$\begin{aligned}\phi_{n-1}\left(\psi_{w_T} z_{((2,1^{n-1}),\varnothing)}\right) &= \Psi\overset{b_2-1}{\underset{2}{\downarrow}} \Psi\overset{b_3-1}{\underset{3}{\downarrow}} \dots \Psi\overset{b_{n-1}-1}{\underset{n-1}{\downarrow}} \psi_1\psi_2\dots\psi_{n-2} z_{((1),(1^{n-1}))} \\ &= \Psi\overset{b_2-1}{\underset{1}{\downarrow}} \Psi\overset{b_3-1}{\underset{2}{\downarrow}} \dots \Psi\overset{b_{n-1}-1}{\underset{n-2}{\downarrow}} z_{((1),(1^{n-1}))}\end{aligned}$$

$$= v(b_2, \dots, b_{n-1})$$
$$\neq 0.$$

Instead, suppose that $b_{m+1} = n$. Then

$$\begin{aligned}
&\phi_{m+1}\left(\psi_{w_T} z_{((n-m,1^m),\varnothing)}\right) \\
&= \Psi \overset{b_2-1}{\underset{1}{\downarrow}} \Psi \overset{b_3-1}{\underset{2}{\downarrow}} \dots \Psi \overset{b_m-1}{\underset{m-1}{\downarrow}} \Psi \overset{n-1}{\underset{m}{\downarrow}} \Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),(1^{m+1}))} \\
&= \Psi \overset{b_2-1}{\underset{1}{\downarrow}} \Psi \overset{b_3-1}{\underset{2}{\downarrow}} \dots \Psi \overset{b_m-1}{\underset{m-1}{\downarrow}} \psi_{n-1} v(1, \dots, m-1, n-1, n) \\
&= 0,
\end{aligned}$$

as $\psi_{n-1} v(1, \dots, m-1, n-1, n)$ equals zero by part one of Lemma 5.1.

□

6.3 Exact sequences of Specht modules

The following result introduces a useful bijection between sets of basis elements of Specht modules, which is a restriction of the Specht module homomorphisms γ_m given above.

Lemma 6.4. *Let $n \equiv \kappa_2 - \kappa_1 + 1 \pmod e$. Define*

$$M := \{v_T \mid T \in \mathrm{Std}\left((n-m),(1^m)\right), T(1, n-m, 1) = n\}$$

and

$$N := \{v_T \mid T \in \mathrm{Std}\left((n-m-1),(1^{m+1})\right), T(m+1, 1, 2) = n\}.$$

Then γ_m restricts to a bijection from M to N.

Proof. Let $\psi_\omega \in \mathscr{R}_n^\Lambda$, not including ψ_{n-1}, for a reduced expression $\omega \in \mathfrak{S}_n$. It follows, by the definition of the homomorphism for γ_m in Proposition 6.2, that

$$\gamma_m\left(\psi_\omega z_{((n-m),(1^m))}\right) = \psi_\omega \Psi \overset{n-1}{\underset{m+1}{\downarrow}} z_{((n-m-1),(1^{m+1}))}.$$

That is, under γ_m we remove node $(1, n-m, 1)$ containing entry n from T and move it to node $(m+1, 1, 2)$ in S, where $v_T \in M$, $v_S \in N$. By performing the opposite action, the inverse map between M and N is well defined, and we arrive back at T, as expected. □

We now determine standard basis elements of the kernels and images of the Specht modules homomorphisms given in Proposition 6.2.

Lemma 6.5. *1. If $n \equiv \kappa_2 - \kappa_1 + 1 \pmod e$, then*

(a) $\mathrm{im}(\gamma_m) = \mathrm{span}\left\{v_T \;\mid\; T \in \mathrm{Std}\left((n-m-1),(1^{m+1})\right), T(m+1,1,2) = n\right\}$*;*

(b) and $\ker(\gamma_m) = \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), T(m,1,2) = n\}$.

2. If $\kappa_2 \equiv \kappa_1 - 1 \pmod e$, then

   (a) i. $\operatorname{im}(\chi_m) = \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), T(1,1,1) = 1\}$;

   ii. $\ker(\chi_m) = 0$;

   (b) i. $\operatorname{im}(\tau_m) = S_{(\varnothing,(n-m+1,1^{m-1}))}$;

   ii. $\ker(\tau_m) = \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), T(1,1,1) = 1\}$.

3. If $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \equiv \kappa_2 - \kappa_1 + 1 \pmod e$, then

   (a) $\operatorname{im}(\alpha_m) = \operatorname{span}\left\{v_T \mid T \in \operatorname{Std}((n-m-1,1^{m+1}),\varnothing),\ T(m+2,1,1) = n\right\}$;

   (b) $\operatorname{im}(\beta_m) = \operatorname{span}\left\{v_T \mid T \in \operatorname{Std}(\varnothing,(n-m-1,1^{m+1}),\ T(m+2,1,2) = n\right\}$;

   (c)
$$\operatorname{im}(\phi_m) = \begin{cases} \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), \\ \qquad\qquad T(1,1,1) = 1, T(m,1,2) = n\} & \text{if } m < n-1; \\ \operatorname{span}\{v_T \mid T \in \operatorname{Std}\left((1),(1^{n-1})\right), T(1,1,1) = 1\} & \text{if } m = n-1. \end{cases}$$

Proof. The images of the Specht module homomorphisms γ_m, χ_m, α_m, β_m and ϕ_m are immediate from Lemma 6.3, where we determined where every standard basis element maps to under each of these homomorphisms. In particular, recall that the standard generators are mapped under these Specht module homomorphisms as follows:

- ◇ $\gamma_m\left(z_{((n-m),(1^m))}\right) = v(1,\ldots,m,n)$;
- ◇ $\chi_m\left(z_{((n-m,1^m),\varnothing)}\right) = v(2,3,\ldots,m+1)$;
- ◇ $\tau_m\left(z_{((n-m),(1^m))}\right) = z_{(\varnothing,(n-m+1,1^{m-1}))}$;
- ◇ $\alpha_m\left(z_{((n-m,1^m),\varnothing)}\right) = v(1,\ldots,m+1,n)$;
- ◇ $\beta_m\left(z_{(\varnothing,(n-m,1^m))}\right) = v(1,\ldots,m+1,n)$
- ◇ $\phi_m\left(z_{((n-m+1,1^{m-1}))}\right) = v(2,3,\ldots,m,n)$.

We subsequently know which basis vectors are killed under these homomorphisms, thus determining the spanning sets of the respective kernels.

- ◇ The kernel of γ_m is obvious since we know the image of γ_m.
- ◇ Checking dimensions, $\dim(\operatorname{im}(\chi_m)) = \dim\left(S_{((n-m,1^m),\varnothing)}\right)$, and thus $\ker(\chi_m) = 0$.
- ◇ By comparing the residue sequences of a standard basis vector v_T of $S_{((n-m),(1^m))}$ and $\tau_m(v_T)$, we know that $\tau_m(v_T) \neq 0$ if 1 lies in the leg of T, and hence $v_T \in \ker(\tau_m)$ if 1 lies in the arm of T.

□

An immediate consequence is the following result, which aids us in finding the composition factors of Specht modules labelled by hook bipartitions.

Lemma 6.6. *1. If $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$, then we have the following exact sequence*

$$0 \to S_{((n),\varnothing)} \xrightarrow{\gamma_0} S_{((n-1),(1))} \xrightarrow{\gamma_1} S_{((n-2),(1^2))} \xrightarrow{\gamma_2} \cdots$$
$$\cdots \xrightarrow{\gamma_{n-2}} S_{((1),(1^{n-1}))} \xrightarrow{\gamma_{n-1}} S_{(\varnothing,(1^n))} \to 0.$$

2. If $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$, then the following sequence is exact

$$0 \to S_{((n-m,1^m),\varnothing)} \xrightarrow{\chi_m} S_{((n-m),(1^m))} \xrightarrow{\tau_m} S_{(\varnothing,(n-m+1,1^{m-1}))} \to 0.$$

3. If $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$ and $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$, then the following sequences are exact:

(a)
$$0 \to S_{((n),\varnothing)} \xrightarrow{\alpha_0} S_{((n-1,1),\varnothing)} \xrightarrow{\alpha_1} S_{((n-2,1^2),\varnothing)} \xrightarrow{\alpha_2} \cdots$$
$$\cdots \xrightarrow{\alpha_{n-3}} S_{((2,1^{n-2}),\varnothing)} \xrightarrow{\alpha_{n-2}} S_{((1^n),\varnothing)} \to 0;$$

(b)
$$0 \to S_{(\varnothing,(n))} \xrightarrow{\beta_0} S_{(\varnothing,(n-1,1))} \xrightarrow{\beta_1} S_{(\varnothing,(n-2,1^2))} \xrightarrow{\beta_2} \cdots$$
$$\cdots \xrightarrow{\beta_{n-3}} S_{(\varnothing,(2,1^{n-2}))} \xrightarrow{\beta_{n-2}} S_{(\varnothing,(1^n))} \to 0.$$

We show that our exact sequences fit into a commutative diagram under certain conditions, in particular, we see that ϕ_{m+1} is a composition of two other homomorphisms given in Proposition 6.2.

Lemma 6.7. *If $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$, then*

1. *$\beta_{m-1} \circ \tau_m = \tau_{m+1} \circ \gamma_m$, and*
2. *$\gamma_m \circ \chi_m = \chi_{m+1} \circ \alpha_m = \phi_{m+1}$.*

Proof. To check the above equalities, we show that they are satisfied on the generator $z_{((n-m),(1^m),\varnothing)}$.

1. We first observe that

$$\beta_{m-1}\left(\tau_m\left(z_{((n-m),(1^m))}\right)\right) = \beta_{m-1}\left(v(2,3,\ldots,m)\right) = v(2,3,\ldots,m,n),$$

which equals zero if and only if $m < n$ by part $(3b)$ of Lemma 6.3. We now observe that

$$\tau_{m+1}\left(\gamma_m\left(z_{((n-m),(1^m))}\right)\right) = \tau_{m+1}\left(v(1,\ldots,m,n)\right) = v(2,3,\ldots,m,n),$$

which equals zero if and only if $m < n$ by the first part of Lemma 6.3, as required.

2. We first observe that

$$\gamma_m\left(\chi_m\left(z_{((n-m),(1^m))}\right)\right) = \gamma_m\left(v(2,3,\ldots,m+1)\right) = v(2,3,\ldots,m+1,n),$$

which is non-zero if and only if $m+1 < n$ by the first part of Lemma 6.3. We now observe that

$$\chi_{m+1}\left(\alpha_m\left(z_{((n-m,1^m),\varnothing)}\right)\right) = \chi_{m+1}\left(v(1,\ldots,m+1,n)\right) = v(2,3,\ldots,m+1,n),$$

which is non-zero if and only if $m+1 < n$ by the part $(3a)$ of Lemma 6.3. Finally, we observe that $\phi_{m+1}\left(z_{((n-m,1^m),\varnothing)}\right) = v(2,\ldots,m+1,n)$, which is non-zero if and only if $m+1 < n$ by part $(3c)$ of Lemma 6.3, as required.

□

This leads us to the following result.

Lemma 6.8. *If* $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ *and* $n \equiv 0 \pmod{e}$*, then the following diagram consists entirely of exact sequences where every square commutes.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & S_{((n),\varnothing)} & \xrightarrow{\chi_0} & S_{((n),\varnothing)} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \alpha_0} & \searrow{\scriptstyle \phi_1} & \downarrow{\scriptstyle \gamma_0} & & \downarrow & & \\
0 & \longrightarrow & S_{((n-1,1),\varnothing)} & \xrightarrow{\chi_1} & S_{((n-1),(1))} & \xrightarrow{\tau_1} & S_{(\varnothing,(n))} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & \searrow{\scriptstyle \phi_2} & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \beta_0} & & \\
0 & \longrightarrow & S_{((n-2,1^2),\varnothing)} & \xrightarrow{\chi_2} & S_{((n-2),(1^2))} & \xrightarrow{\tau_2} & S_{(\varnothing,(n-1,1))} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \beta_1} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \gamma_{n-1}} & & \downarrow{\scriptstyle \beta_{n-2}} & & \\
0 & \longrightarrow & S_{((2,1^{n-2}),\varnothing)} & \xrightarrow{\chi_{n-2}} & S_{((2),(1^{n-2}))} & \xrightarrow{\tau_{n-2}} & S_{(\varnothing,(3,1^{n-3}))} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{n-2}} & \searrow{\scriptstyle \phi_{n-1}} & \downarrow{\scriptstyle \gamma_{n-2}} & & \downarrow{\scriptstyle \beta_{n-3}} & & \\
0 & \longrightarrow & S_{((1^n),\varnothing)} & \xrightarrow{\chi_{n-1}} & S_{((1),(1^{n-1}))} & \xrightarrow{\tau_{n-1}} & S_{(\varnothing,(2,1^{n-2}))} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \gamma_{n-1}} & & \downarrow{\scriptstyle \beta_{n-2}} & & \\
 & & 0 & \longrightarrow & S_{(\varnothing,(1^n))} & \xrightarrow{\tau_n} & S_{(\varnothing,(1^n))} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Chapter 7

Composition series of $S_{((n-m),(1^m))}$

In this chapter, we completely determine the composition factors of Specht modules labelled by hook bipartitions up to isomorphism with quantum characteristic at least three, recalling that we are still forgetting the grading on these Specht modules. The composition series of Specht modules labelled by hook bipartitions split into four distinct cases, depending whether or not $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$ and whether or not $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$. We will see that the composition factors of $S_{((n-m),(1^m))}$ arise as quotients of the images and kernels of the aforementioned homomorphisms in Proposition 6.2.

7.1 The action of $\mathscr{R}_n^\Lambda$ on $S_{((n-m),(1^m))}$ II

We let $\lambda = ((n-m),(1^m))$ throughout this section. In order to do determine irreducibility of $\mathscr{R}_n^\Lambda$-submodules of Specht modules labelled by hook bipartitions, namely of S_λ, we now establish results toward this end.

Each basis vector v_T of a Specht module labelled by a hook bipartition equals $\psi_{\omega_T} z_\lambda$ for a $\psi_{\omega_T} \in \mathscr{R}_n$, that depends on a reduced expression of $w_T \in \mathfrak{S}_n$. We wish to determine the non-trivial mappings between these basis vectors by the cyclotomic Khovanov–Lauda–Rouquier generators $\psi_1, \dots, \psi_{n-1}$. Appealing to Lemma 5.1 and Lemma 5.3, we explicitly describe the action of these generators on the basis vectors of S_λ, which act non-trivially only in a small number of cases.

Theorem 7.1. *Let $1 \leqslant l \leqslant n-1$, $T \in \operatorname{Std}(\lambda)$, and for $1 \leqslant r \leqslant m$, set $a_r := T(r,1,2)$. Then*

$$\psi_l v(a_1, \dots, a_m) = 0$$

for every T, except in the following cases.

(i) Suppose that $a_r = l$ for some $1 \leqslant r \leqslant m$, and that either $r = m$ or $a_{r+1} \geqslant l+2$. Then

$$\psi_l v(a_1, \dots, a_m) = v(a_1, \dots, a_{r-1}, l+1, a_{r+1}, \dots, a_m). \tag{7.1.1}$$

(ii) *Suppose that* $l \equiv \kappa_2 - \kappa_1 \pmod{e}$ *and* $l < n-1$.

- ◇ *Suppose* $a_r = l+1$ *and* $a_{r+1} = l+2$ *for some* $1 \leqslant r \leqslant n-1$, *and that either* $r = 1$ *or* $a_{r-1} \leqslant l-1$. *Then*

$$\psi_l v(a_1, \dots, a_m) = v(a_1, \dots, a_{r-1}, l, l+1, a_{r+2}, \dots, a_m). \tag{7.1.2}$$

- ◇ *Suppose* $a_r = l+2$ *for some* $1 \leqslant r \leqslant m$, *and that either* $r = 1$ *or* $a_{r-1} \leqslant l-1$. *Then*

$$\psi_l v(a_1, \dots, a_m) = -v(a_1, \dots, a_{r-1}, l, a_{r+1}, \dots, a_m). \tag{7.1.3}$$

(iii) *Suppose that* $l \equiv 2 + \kappa_2 - \kappa_1 \pmod{e}$.

- ◇ *Suppose* $a_r = l$ *and* $a_{r+1} = l+1$ *for some* $1 \leqslant r \leqslant m-1$, *and that either* $r = 1$ *or* $a_{r-1} \leqslant l-2$. *Then*

$$\psi_l v(a_1, \dots, a_m) = v(a_1, \dots, a_{r-1}, l-1, l, a_{r+2}, \dots, a_m). \tag{7.1.4}$$

- ◇ *Suppose* $a_r = l+1$ *for some* $1 \leqslant r \leqslant m$, *and that either* $r = 1$ *or* $a_{r-1} \leqslant l-2$. *Then*

$$\psi_l v(a_1, \dots, a_{r-1}, l+1, a_{r+1}, \dots, a_m) = -v(a_1, \dots, a_{r-1}, l-1, a_{r+1}, \dots, a_m). \tag{7.1.5}$$

(iv) *Suppose that* $l + \kappa_1 - \kappa_2 \not\equiv 0, 1, 2 \pmod{e}$, $a_r = l+1$ *for some* $1 \leqslant r \leqslant m$, *and either* $r = 1$ *or* $a_{r-1} \leqslant l-1$. *Then*

$$\psi_l v(a_1, \dots, a_m) = v(a_1, \dots, a_{r-1}, l, a_{r+1}, \dots, a_m). \tag{7.1.6}$$

Proof. We consider $\psi_l v(a_1, \dots, a_{r-1}, a_r, a_{r+1}, \dots, a_m)$, for all $a_r \geqslant l$.

1. We let $a_r = l$ and suppose $a_{r+1} \geqslant l+2$. Then

$$\begin{aligned}
\psi_l v(a_1, \dots, a_m) &= \psi_l \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{l-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{l}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= v(a_1, \dots, a_{r-1}, l+1, a_{r+1}, \dots, a_m),
\end{aligned}$$

which satisfies Equation (7.1.1).

2. Suppose $a_{r-1} + 1 \leqslant l \leqslant a_r - 3$. We have

$$\psi_l v(a_1, \dots, a_m) = \psi_l \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$

$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{l+2}{\downarrow}} \psi_l \Psi \overset{l+1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{l+2}{\downarrow}} \psi_l v(1,\dots,r-1,l+2,a_{r+1},\dots,a_m).$$

By part two of Lemma 5.1, $\psi_l v(1,\dots,r-1,l+2,a_{r+1},\dots,a_m)$ equals zero if $l \not\equiv \kappa_2-\kappa_1 \pmod e$. Suppose instead $l \equiv \kappa_2-\kappa_1 \pmod e$. Then $\psi_l v(a_1,\dots,a_m)$ becomes

$$\Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{l+2}{\downarrow}} \psi_l\psi_{l+1}\psi_l \Psi \overset{l-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{l+2}{\downarrow}} (\psi_{l+1}\psi_l\psi_{l+1}-1) \Psi \overset{l-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{l+3}{\downarrow}} \psi_{l+1} \Psi \overset{l+2}{\underset{r+1}{\downarrow}} \Psi \overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$- \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{l+3}{\downarrow}} \Psi \overset{l-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{l+4}{\downarrow}} \psi_{l+2} \Psi \overset{l+3}{\underset{r+1}{\downarrow}} \Psi \overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{l+3}{\downarrow}} \psi_{l+1} v(1,\dots,r,l+3,a_{r+2},\dots,a_m)$$
$$- \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{a_r-1}{\underset{l+3}{\downarrow}} \Psi \overset{l-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{l+4}{\downarrow}} \psi_{l+2} v(1,\dots,r,l+4,a_{r+2},\dots,a_m).$$

By part two of Lemma 5.1, $\psi_{l+1}v(1,\dots,r,l+3,a_{r+2},\dots,a_m)$ equals zero since $l \not\equiv \kappa_2-\kappa_1-1 \pmod e$ and $\psi_{l+2}v(1,\dots,r,l+4,a_{r+2},\dots,a_m)$ equals zero since $l \not\equiv \kappa_2-\kappa_1-2 \pmod e$.

3. Let $a_r = l+2$ and suppose that $a_{r-1} \leqslant l-1$.

   (i) Suppose $l \equiv \kappa_2-\kappa_1-1 \pmod e$. Firstly, let $l < n-2$. Then

$$\psi_l v(a_1,\dots,a_m)$$
$$= \psi_l \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{l+1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_l\psi_{l+1}\psi_l \Psi \overset{l-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{l+1}\psi_l\psi_{l+1} \Psi \overset{l-1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{l+1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{l+3}{\downarrow}} \psi_{l+1}\psi_{l+2}\psi_{l+1} \Psi \overset{l}{\underset{r+1}{\downarrow}} \Psi \overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{l+1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{l+3}{\downarrow}} (\psi_{l+2}\psi_{l+1}\psi_{l+2}-1) \Psi \overset{l}{\underset{r+1}{\downarrow}} \Psi \overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda.$$

   The first term becomes

$$\Psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{l+1}{\underset{r}{\downarrow}} \Psi \overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \Psi \overset{a_{r+2}-1}{\underset{l+4}{\downarrow}} \psi_{l+2} \Psi \overset{l+3}{\underset{r+2}{\downarrow}} \Psi \overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$

$$= \Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \Psi\underset{r}{\overset{l+1}{\downarrow}} \Psi\underset{r+1}{\overset{a_{r+1}-1}{\downarrow}} \Psi\underset{l+4}{\overset{a_{r+2}-1}{\downarrow}} \psi_{l+2}v(1,\dots,r+1,l+4,a_{r+3},\dots,a_m),$$

where $\psi_{l+2}v(1,\dots,r+1,l+4,a_{r+3},\dots,a_m)$ equals zero by part two of Lemma 5.1 since $l \not\equiv \kappa_2-\kappa_1-2 \pmod e$. If $a_{r+1} \geqslant l+4$ then the second term becomes

$$\begin{aligned} &- \Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \Psi\underset{r}{\overset{l+1}{\downarrow}} \Psi\underset{l+4}{\overset{a_{r+1}-1}{\downarrow}} \Psi\underset{r+1}{\overset{l}{\downarrow}} \Psi\underset{l+5}{\overset{a_{r+2}-1}{\downarrow}} \psi_{l+3}\Psi\underset{r+2}{\overset{l+4}{\downarrow}} \Psi\underset{r+3}{\overset{a_{r+3}-1}{\downarrow}} \dots \Psi\underset{m}{\overset{a_m-1}{\downarrow}} z_\lambda \\ &= -\Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \Psi\underset{r}{\overset{l+1}{\downarrow}} \Psi\underset{l+4}{\overset{a_{r+1}-1}{\downarrow}} \Psi\underset{r+1}{\overset{l}{\downarrow}} \Psi\underset{l+5}{\overset{a_{r+2}-1}{\downarrow}} \\ &\qquad \cdot \psi_{l+3}v(1,\dots,r+1,l+5,a_{r+3},\dots,a_m), \end{aligned}$$

where $\psi_{l+3}v(1,\dots,r+1,l+5,a_{r+3},\dots,a_m)$ equals zero since $l \not\equiv \kappa_2-\kappa_1-3$. Now suppose $a_{r+1}=l+3$. Then the second term becomes

$$\begin{aligned} &- \Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \Psi\underset{r}{\overset{l+1}{\downarrow}} \Psi\underset{r+1}{\overset{l}{\downarrow}} \Psi\underset{r+2}{\overset{a_{r+2}-1}{\downarrow}} \dots \Psi\underset{m}{\overset{a_m-1}{\downarrow}} z_\lambda \\ &= -\Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \psi_{l+1}\psi_l\Psi\underset{r}{\overset{l-1}{\downarrow}} \Psi\underset{r+1}{\overset{l}{\downarrow}} \Psi\underset{r+2}{\overset{a_{r+2}-1}{\downarrow}} \dots \Psi\underset{m}{\overset{a_m-1}{\downarrow}} z_\lambda \\ &= -\Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \psi_{l+1}\psi_l v(1,\dots,r-1,l,l+1,a_{r+2},\dots,a_m). \end{aligned}$$

If $e>3$ then $\psi_l v(1,\dots,r-1,l,l+1,a_{r+2},a_m)$ equals zero by the first part of Lemma 5.1 since $l \equiv \kappa_2-\kappa_1-1$. So suppose $e=3$. Then the second term becomes

$$\begin{aligned} &- \Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \psi_{l+1}\psi_l\psi_{l-1}\psi_l\Psi\underset{r}{\overset{l-2}{\downarrow}} \Psi\underset{r+1}{\overset{l-1}{\downarrow}} \Psi\underset{r+2}{\overset{a_{r+2}-1}{\downarrow}} \dots \Psi\underset{m}{\overset{a_m-1}{\downarrow}} z_\lambda \\ &= -\Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \psi_{l+1}(\psi_{l-1}\psi_l\psi_{l-1}+1)\Psi\underset{r}{\overset{l-2}{\downarrow}} \Psi\underset{r+1}{\overset{l-1}{\downarrow}} \Psi\underset{r+2}{\overset{a_{r+2}-1}{\downarrow}} \dots \Psi\underset{m}{\overset{a_m-1}{\downarrow}} z_\lambda. \end{aligned}$$

The first term becomes

$$\begin{aligned} &- \Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \psi_{l+1}\psi_{l-1}\psi_l\psi_{l-1}\Psi\underset{r}{\overset{l-2}{\downarrow}} \Psi\underset{r+1}{\overset{l-1}{\downarrow}} \Psi\underset{r+2}{\overset{a_{r+2}-1}{\downarrow}} \dots \Psi\underset{m}{\overset{a_m-1}{\downarrow}} z_\lambda \\ &= -\Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \psi_{l+1}\psi_{l-1}\psi_l\psi_{l-1}v(1,\dots,r-1,l-1,l,a_{r+2},\dots,a_m), \end{aligned}$$

where $\psi_{l-1}v(1,\dots,r-1,l-1,l,a_{r+2},\dots,a_m)$ equals zero by the first part of Lemma 5.1 since $l \not\equiv 0 \pmod 3$. Now the second term becomes

$$\begin{aligned} &- \Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \Psi\underset{r}{\overset{l-2}{\downarrow}} \Psi\underset{r+1}{\overset{l-1}{\downarrow}} \Psi\underset{l+3}{\overset{a_{r+2}-1}{\downarrow}} \psi_{l+1}\psi_{l+2}\psi_{l+1}\Psi\underset{r+2}{\overset{l}{\downarrow}} \Psi\underset{r+3}{\overset{a_{r+3}-1}{\downarrow}} \dots \Psi\underset{m}{\overset{a_m-1}{\downarrow}} z_\lambda \\ &= -\Psi\underset{1}{\overset{a_1-1}{\downarrow}} \dots \Psi\underset{r-1}{\overset{a_{r-1}-1}{\downarrow}} \Psi\underset{r}{\overset{l-2}{\downarrow}} \Psi\underset{r+1}{\overset{l-1}{\downarrow}} \Psi\underset{l+3}{\overset{a_{r+2}-1}{\downarrow}} \\ &\qquad \cdot (\psi_{l+2}\psi_{l+1}\psi_{l+2}-1)\Psi\underset{r+2}{\overset{l}{\downarrow}} \Psi\underset{r+3}{\overset{a_{r+3}-1}{\downarrow}} \dots \Psi\underset{m}{\overset{a_m-1}{\downarrow}} z_\lambda \end{aligned}$$

$$
\begin{aligned}
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l-2}{\underset{r}{\downarrow}} \Psi\overset{l-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{l+4}{\downarrow}} \psi_{l+2}\Psi\overset{l+3}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&\quad + \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l-2}{\underset{r}{\downarrow}} \Psi\overset{l-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{l+4}{\downarrow}} \Psi\overset{l}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{l+5}{\downarrow}} \psi_{l+3}\Psi\overset{l+4}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l-2}{\underset{r}{\downarrow}} \Psi\overset{l-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{l+4}{\downarrow}} \\
&\qquad \cdot \psi_{l+2} v(1,\dots,r+2,l+4,a_{r+4},\dots,a_m) \\
&\quad + \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l-2}{\underset{r}{\downarrow}} \Psi\overset{l-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{l+4}{\downarrow}} \Psi\overset{l}{\underset{r+2}{\downarrow}} \Psi\overset{a_{r+3}-1}{\underset{l+5}{\downarrow}} \\
&\qquad \cdot \psi_{l+3} v(1,\dots,r+2,l+5,a_{r+4},\dots,a_m).
\end{aligned}
$$

By part two of Lemma 5.1, $\psi_{l+2}v(1,\dots,r+2,l+4,a_{r+4},\dots,a_m)$ equals zero since $l \not\equiv \kappa_2 - \kappa - 2 \pmod{e}$ and $\psi_{l+3}v(1,\dots,r+2,l+5,a_{r+4},\dots,a_m)$ equals zero since $l \not\equiv \kappa_2 - \kappa_1 \pmod{e}$.

Secondly, let $l = n-2$. Then $r = m$ and $a_{m-1} \leqslant n-4$, so that

$$
\begin{aligned}
&\psi_{n-2} v(a_1,\dots,a_m) \\
&= \psi_{n-2}\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{m-2}-1}{\underset{m-2}{\downarrow}} \Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}} \Psi\overset{n-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{m-2}-1}{\underset{m-2}{\downarrow}} \Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}} \psi_{n-2}\psi_{n-1}\psi_{n-2}\Psi\overset{n-3}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{m-2}-1}{\underset{m-2}{\downarrow}} \Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}} \psi_{n-1}\psi_{n-2}\psi_{n-1}\Psi\overset{n-3}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{m-2}-1}{\underset{m-2}{\downarrow}} \Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}} \Psi\overset{n-1}{\underset{m}{\downarrow}} \psi_{n-1} z_\lambda \\
&= 0.
\end{aligned}
$$

(ii) Suppose $l \equiv \kappa_2 - \kappa_1 \pmod{e}$. Then

$$
\begin{aligned}
&\psi_l v(a_1,\dots,a_m) \\
&= \psi_l \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l+1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} (\psi_l\psi_{l+1}\psi_l)\Psi\overset{l-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} (\psi_{l+1}\psi_l\psi_{l+1} - 1)\Psi\overset{l-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda.
\end{aligned}
$$

The first term becomes

$$
\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{l+1}\psi_l\psi_{l+1}\Psi\overset{l-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{l+1}\psi_l\psi_{l+1} v(1,\dots,r-1,l,a_{r+1},\dots,a_m),
\end{aligned}
$$

where $\psi_{l+1}v(1,\dots,r-1,l,a_{r+1},\dots,a_m)$ equals zero by part two of Lemma 5.1 since $l \not\equiv \kappa_2-\kappa_1-1 \pmod e$. Now the second term becomes

$$\begin{aligned}&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda\\ &=-v(a_1,\dots,a_{r-1},l,a_{r+1},\dots,a_m).\end{aligned}$$

Hence Equation (7.1.3) is satisfied.

(iii) Suppose that $l+\kappa_1-\kappa_2 \not\equiv -1,0 \pmod e$. We have

$$\begin{aligned}&\psi_l v(a_1,\dots,a_m)\\
&=\psi_l\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \Psi\overset{l+1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_l\psi_{l+1}\psi_l\Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_{l+1}\psi_l\psi_{l+1}\Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \Psi\overset{l+1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+3}{\downarrow}}\ \psi_{l+1}\Psi\overset{l+2}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \Psi\overset{l+1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+3}{\downarrow}}\ \psi_{l+1}v(1,\dots,r,l+3,a_{r+2},\dots,a_m),\end{aligned}$$

where $\psi_{l+1}v(1,\dots,r,l+3,a_{r+2},\dots,a_m)$ equals zero by part two of Lemma 5.1 since $l \not\equiv \kappa_2-\kappa_1-1 \pmod e$.

4. (a) Let $a_r = l+1$ and suppose that $a_{r-1} \leqslant l-1$.

i. Suppose $l \equiv \kappa_2-\kappa_1 \pmod e$. Let $l < n-1$, and firstly suppose that $a_{r+1} \geqslant l+3$. Then we have

$$\begin{aligned}\psi_l v(a_1,\dots,a_m) &= \psi_l\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{l}{\underset{r}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\psi_l^2\Psi\overset{l-1}{\underset{r}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots(y_l-y_{l+1})\Psi\overset{l-1}{\underset{r}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda.\end{aligned}$$

The first term is

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots y_l\Psi\overset{l-1}{\underset{r}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots y_l v(1,\dots,r-1,l,a_{r+1}\dots a_m),$$

where $y_l v(1,\dots,r-1,l,a_{r+1}\dots a_m)$ equals zero by part three of Lemma 5.3 since $l \not\equiv 1,2 \pmod e$. The second term is

$$-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots y_{l+1}\Psi\overset{l-1}{\underset{r}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+2}{\downarrow}}\ y_{l+1}\psi_{l+1}\Psi\overset{l}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+2}{\downarrow}}\ (\psi_{l+1}y_{l+2}-1)\Psi\overset{l}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda.$$

Its first term becomes

$$-\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+2}{\downarrow}}\ \psi_{l+1}y_{l+2}\Psi\overset{l}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{l+3}{\downarrow}}\ y_{l+2}\Psi\overset{l}{\underset{r+1}{\downarrow}}\ \Psi\overset{l+2}{\underset{r+2}{\downarrow}}\ \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{l+3}{\downarrow}}\ y_{l+2}v(1,\dots,r,l+1,l+3,a_{r+3},\dots,a_m),$$

where $y_{l+2}v(1,\dots,r,l+1,l+3,a_{r+3},\dots,a_m)$ equals zero by part two of Lemma 5.3. Whereas, its second term becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+2}{\downarrow}}\ \Psi\overset{l}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+3}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{l+4}{\downarrow}}\ \psi_{l+2}\Psi\overset{l}{\underset{r+1}{\downarrow}}\ \Psi\overset{l+3}{\underset{r+2}{\downarrow}}\ \Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{a_{r+1}-1}{\underset{l+3}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{l+4}{\downarrow}}\ \psi_{l+2}v(1,\dots,r,l+1,l+4,a_{r+3},\dots,a_m),$$

where $\psi_{l+2}v(1,\dots,r,l+1,l+4,a_{r+3},\dots,a_m)$ equals zero by part two of Lemma 5.1 since $l \not\equiv -2 \pmod e$.
Secondly, suppose $a_{r+1} = l+2$. We have

$$\psi_l v(a_1,\dots,a_m)$$

$$= \psi_l\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \Psi\overset{l}{\underset{r}{\downarrow}}\ \Psi\overset{l+1}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ \psi_l^2\Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{l+1}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ (y_l - y_{l+1})\Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{l+1}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda.$$

The first term becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ y_l\Psi\overset{l-1}{\underset{r}{\downarrow}}\ \Psi\overset{l+1}{\underset{r+1}{\downarrow}}\ \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\ z_\lambda$$

$$= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\ y_l v(1,\dots,r-1,l,l+2,a_{r+2},\dots,a_m),$$

where $y_l v(1,\dots,r-1,l,l+2,a_{r+2},\dots,a_m)$ equals zero by part three of

Lemma 5.3. The second term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{l-1}{\underset{r}{\downarrow}}\,y_{l+1}\Psi\overset{l+1}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\,z_\lambda\\
&=-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{l-1}{\underset{r}{\downarrow}}\,(\psi_{l+1}y_{l+2}-1)\Psi\overset{l}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\,z_\lambda.
\end{aligned}$$

Its first term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{l-1}{\underset{r}{\downarrow}}\,\psi_{l+1}y_{l+2}\Psi\overset{l}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\,z_\lambda\\
&=-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{l-1}{\underset{r}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{l+3}{\downarrow}}\,\psi_{l+1}y_{l+2}\Psi\overset{l}{\underset{r+1}{\downarrow}}\Psi\overset{l+2}{\underset{r+2}{\downarrow}}\Psi\overset{a_{r+3}-1}{\underset{r+3}{\downarrow}}\dots\\
&\qquad\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\,z_\lambda\\
&=-\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{l-1}{\underset{r}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{l+3}{\downarrow}}\\
&\qquad\cdot\psi_{l+1}y_{l+2}v(1,\dots,r,l+1,l+3,a_{r+3},\dots,a_m)
\end{aligned}$$

where $y_{l+2}v(1,\dots,r,l+1,l+3,a_{r+3},\dots,a_m)$ equals zero by part two of Lemma 5.3. Whereas, its second term becomes

$$\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{l-1}{\underset{r}{\downarrow}}\Psi\overset{l}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\,z_\lambda\\
&=v(a_1,\dots,a_{r-1},l,l+1,a_{r+2},\dots,a_m),
\end{aligned}$$

which satisfies Equation (7.1.2).

Now let $l=n-1$. Then $r=m$ and $a_{m-1}\leqslant l-1$, so that

$$\begin{aligned}
&\psi_{n-1}v(a_1,\dots,a_m)\\
&=\psi_{n-1}\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{m-2}-1}{\underset{m-2}{\downarrow}}\Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}}\Psi\overset{n-1}{\underset{m}{\downarrow}}\,z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{m-2}-1}{\underset{m-2}{\downarrow}}\Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}}\,\psi_{n-1}^2\Psi\overset{n-2}{\underset{m}{\downarrow}}\,z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{m-2}-1}{\underset{m-2}{\downarrow}}\Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}}\,(y_{n-1}-y_n)\Psi\overset{n-2}{\underset{m}{\downarrow}}\,z_\lambda\\
&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{m-2}-1}{\underset{m-2}{\downarrow}}\Psi\overset{a_{m-1}-1}{\underset{m-1}{\downarrow}}\,y_{n-1}\Psi\overset{n-2}{\underset{m}{\downarrow}}\,z_\lambda,
\end{aligned}$$

which equals zero by part three of Lemma 5.3.

ii. Suppose $l\equiv 1+\kappa_2-\kappa_1 \pmod e$. Then

$$\psi_l v(a_1,\dots,a_m)=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\psi_l^2\Psi\overset{l-1}{\underset{r}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}\,z_\lambda=0.$$

iii. Suppose $l \equiv 2+\kappa_2-\kappa_1 \pmod e$. Then

$$\begin{aligned}
&\psi_l v(a_1,\dots,a_m)\\
&= \psi_l \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad \Psi\overset{l}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_\lambda\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad \psi_l^2\Psi\overset{l-1}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_\lambda\\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad (y_{l+1}-y_l)\Psi\overset{l-1}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_\lambda.
\end{aligned}$$

The first term equals zero by part three of Lemma 5.3. The second term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad y_l\Psi\overset{l-1}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_\lambda\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad (\psi_{l-1}y_{l-1}+1)\Psi\overset{l-2}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_\lambda.
\end{aligned}$$

Its first term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad \psi_{l-1}y_{l-1}\Psi\overset{l-2}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_\lambda\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad \psi_{l-1}y_{l-1}v(1,\dots,r-1,l-1,a_{r+1},\dots,a_m),
\end{aligned}$$

where $y_{l-1}v(1,\dots,r-1,l-1,a_{r+1},\dots,a_m)$ equals zero by part one of Lemma 5.3. If $a_{r-1} \leqslant l-2$ then its second term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad \Psi\overset{l-2}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_\lambda\\
&= -v(a_1,\dots,a_{r-1},l-1,a_{r+1},\dots,a_m),
\end{aligned}$$

which satisfies Equation (7.1.5). Whereas, if $r>1$ and $a_{r-1}=l-1$, then its second term becomes

$$\begin{aligned}
&-\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-2}-1}{\underset{r-2}{\downarrow}} \quad \Psi\overset{l-2}{\underset{r-1}{\downarrow}}\,\Psi\overset{l-2}{\underset{r}{\downarrow}}\,\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}}\, z_\lambda\\
&= -\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \quad \psi_{l-2}v(1,\dots,r-1,l-2,l-1,a_{r+1},\dots,a_m),
\end{aligned}$$

where $\psi_{l-2}v(1,\dots,r-1,l-2,l-1,a_{r+1},\dots,a_m)$ equals zero by part one of Lemma 5.1 since $l \not\equiv 4+\kappa_2-\kappa_1 \pmod e$.

iv. Suppose $l+\kappa_1-\kappa_2 \not\equiv 0,1,2 \pmod e$. We have

$$\psi_l v(a_1,\dots,a_m)$$

$$
\begin{aligned}
&= \psi_l \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_l^2 \Psi\overset{l-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l-1}{\underset{r}{\downarrow}} \Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= v(a_1, \dots, a_{r-1}, l, a_{r+1}, \dots, a_m),
\end{aligned}
$$

which satisfies Equation (7.1.6).

(b) Suppose $a_r = a_{r-1} + 1$. Firstly, suppose $l \not\equiv 2 + \kappa_2 - \kappa_1 \pmod e$. Then

$$
\begin{aligned}
&\psi_l v(a_1, \dots, a_m) \\
&= \psi_l \left(\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l-1}{\underset{r}{\downarrow}} \Psi\overset{l}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \right) \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_l v(1, \dots, r-1, l, l+1, a_{r+2}, \dots, a_m),
\end{aligned}
$$

where $\psi_l v(1, \dots, r-1, l, l+1, a_{r+2}, \dots, a_m)$ equals zero by part one of Lemma 5.1 since $l \not\equiv 2 + \kappa_2 - \kappa_1 \pmod e$.

Now suppose that $l \equiv 2 + \kappa_2 - \kappa_1 \pmod e$. Then

$$
\begin{aligned}
&\psi_l v(a_1, \dots, a_m) \\
&= \psi_l \left(\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l-1}{\underset{r}{\downarrow}} \Psi\overset{l}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \right) \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_l \psi_{l-1} \psi_l \Psi\overset{l-2}{\underset{r}{\downarrow}} \Psi\overset{l-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} (\psi_{l-1} \psi_l \psi_{l-1} + 1) \Psi\overset{l-2}{\underset{r}{\downarrow}} \Psi\overset{l-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda.
\end{aligned}
$$

The first term becomes

$$
\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{l-1} \psi_l \psi_{l-1} \Psi\overset{l-2}{\underset{r}{\downarrow}} \Psi\overset{l-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= \Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{l-1} \psi_l \psi_{l-1} v(1, \dots, r-1, l-1, l, a_{r+2}, \dots, a_m),
\end{aligned}
$$

where $\psi_{l-1} v(1, \dots, r-1, l-1, l, a_{r+2}, \dots, a_m)$ equals zero by part one of Lemma 5.1 since $l \not\equiv 3 + \kappa_2 - \kappa_1 \pmod e$. If $r = 1$ or $r > 1$ and $a_{r-1} \leqslant l - 2$ then the second term becomes

$$
\begin{aligned}
&\Psi\overset{a_1-1}{\underset{1}{\downarrow}} \dots \Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi\overset{l-2}{\underset{r}{\downarrow}} \Psi\overset{l-1}{\underset{r+1}{\downarrow}} \Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \Psi\overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda \\
&= v(a_1, \dots, a_{r-1}, l-1, l, a_{r+2}, \dots, a_m),
\end{aligned}
$$

which satisfies Equation (7.1.4). Whereas, if $r > 1$ and $a_{r-1} = l - 1$ then the second term becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-2}-1}{\underset{r-2}{\downarrow}}\Psi\overset{l-2}{\underset{r-1}{\downarrow}}\Psi\overset{l-2}{\underset{r}{\downarrow}}\Psi\overset{l-1}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_\lambda$$
$$=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-2}-1}{\underset{r-2}{\downarrow}}\psi_{l-2}v(1,\dots,r-2,l-2,l-1,l,a_{r+2},\dots,a_m),$$

where $\psi_{l-2}v(1,\dots,r-2,l-2,l-1,l,a_{r+2},\dots,a_m)$ equals zero by part one of Lemma 5.1 since $l\not\equiv 2+\kappa_2-\kappa_1 \pmod e$.

5. Suppose $a_r=l$ and $a_{r+1}\geqslant l+3$. Then

$$\begin{aligned}\psi_l v(a_1,\dots,a_m)&=\psi_l\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{l-1}{\underset{r}{\downarrow}}\Psi\overset{a_{r+1}-1}{\underset{r+1}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_\lambda\\&=v(a_1,\dots,a_{r-1},l,a_{r+1},\dots,a_m),\end{aligned}$$

which satisfies Equation (7.1.1). Now suppose $a_{r+1}=l+2$. Then

$$\begin{aligned}\psi_l v(a_1,\dots,a_m)&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\psi_l\Psi\overset{l-1}{\underset{r}{\downarrow}}\Psi\overset{l}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_\lambda\\&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\psi_l v(1,\dots,r-1,l,l+1,a_{r+2},\dots,a_m).\end{aligned}$$

By part one of Lemma 5.1, $\psi_l v(1,\dots,r-1,l,l+1,a_{r+2},\dots,a_m)$ equals zero if $l\not\equiv 2+\kappa_2-\kappa_1 \pmod e$. Suppose instead that $l\equiv 2+\kappa_2-\kappa_1 \pmod e$. Firstly, let $a_{r-1}\leqslant l-2$. Then

$$\begin{aligned}&\psi_l v(a_1,\dots,a_m)\\&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\psi_l\psi_{l-1}\psi_l\Psi\overset{l-2}{\underset{r}{\downarrow}}\Psi\overset{l-1}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_\lambda\\&=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}(\psi_{l-1}\psi_l\psi_{l-1}+1)\Psi\overset{l-2}{\underset{r}{\downarrow}}\Psi\overset{l-1}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_\lambda.\end{aligned}$$

The first term becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\psi_{l-1}\psi_l\psi_{l-1}\Psi\overset{l-2}{\underset{r}{\downarrow}}\Psi\overset{l-1}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_\lambda$$
$$=\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\psi_{l-1}\psi_l\psi_{l-1}v(1,\dots,r-1,l-1,l,a_{r+2},\dots,a_m),$$

where $\psi_{l-1}v(1,\dots,r-1,l-1,l,a_{r+2},\dots,a_m)$ equals zero by part one of Lemma 5.1 since $l\not\equiv 3+\kappa_2-\kappa_1 \pmod e$. Whereas, the second term becomes

$$\Psi\overset{a_1-1}{\underset{1}{\downarrow}}\dots\Psi\overset{a_{r-1}-1}{\underset{r-1}{\downarrow}}\Psi\overset{l-2}{\underset{r}{\downarrow}}\Psi\overset{l-1}{\underset{r+1}{\downarrow}}\Psi\overset{a_{r+2}-1}{\underset{r+2}{\downarrow}}\dots\Psi\overset{a_m-1}{\underset{m}{\downarrow}}z_\lambda$$
$$=v(a_1,\dots,a_{r-1},l-1,l,a_{r+2},a_m),$$

and hence Equation (7.1.4) is satisfied. Now let $a_{r-1} = l-1$. Then the second term becomes

$$\Psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \Psi \overset{l-2}{\underset{r}{\downarrow}} \Psi \overset{l-2}{\underset{r+1}{\downarrow}} \Psi \overset{l-1}{\underset{r+2}{\downarrow}} \Psi \overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \ldots \Psi \overset{a_m-1}{\underset{m}{\downarrow}} z_\lambda$$
$$= \Psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \Psi \overset{a_{r-1}-1}{\underset{r-1}{\downarrow}} \psi_{l-2} v(1,\ldots,r-1,l-2,l-1,l,a_{r+2},\ldots,a_m),$$

where $\psi_{l-2}v(1,\ldots,r-1,l-2,l-1,l,a_{r+2},\ldots,a_m)$ equals zero by part one of Lemma 5.1 since $l \not\equiv 4+\kappa_2-\kappa_1 \pmod e$.

□

Corollary 7.2. *The matrix of the action of ψ_l on $S_{((n-m),(1^m))}$ with respect to our chosen standard basis has at most one non-zero entry in each row and in each column.*

Ultimately, when S_λ is irreducible, we will show that we can map any basis vector of S_λ under the action of $\mathscr{R}_n^\Lambda$ to the standard generator z_λ. If S_λ is not irreducible, then our following results aid us to map every basis vector v_T of its composition factors to a basis vector v_S, where S is a more dominant λ-tableau than T.

For $S,T \in \mathrm{Std}(\lambda)$ and a reduced expression $\omega \in \mathfrak{S}_n$, we explicitly map each standard basis vector v_T of S_λ to another standard basis vector v_S by a $\psi_w \in \mathscr{R}_n^\Lambda$, where S is indeed more dominant than T.

Proposition 7.3. *Suppose that $a_i > i$. Then there exists an element $x \in \mathscr{R}_n^\Lambda$, equal to $\pm\psi_w$ for some $w \in \mathfrak{S}_n$ and some choice of reduced expression of w, such that*

$$xv(1,\ldots,i-1,a_i,a_{i+1},\ldots,a_m) = v(1,\ldots,i-1,i,a_{i+1},\ldots,a_m),$$

where x is given as follows.

1. *Let $i \equiv 1+\kappa_2-\kappa_1 \pmod e$.*
   (a) *If $a_i \equiv 1+\kappa_2-\kappa_1 \pmod e$ and $a_i \neq n$ when $i=m$, then*
   $$x = \begin{cases} -\psi_{a_i}\Psi \overset{a_i-1}{\underset{i+1}{\uparrow}} & \text{if } a_{i+1} = a_i+1, \\ \Psi \overset{a_i}{\underset{i+1}{\uparrow}} & \text{if } a_{i+1} \geqslant a_i+2. \end{cases}$$
   (b) *If $a_i \equiv 2+\kappa_2-\kappa_1 \pmod e$ and*
      i. *$a_i = i+1$, then $x = -\psi_{i+1}^2$.*
      ii. *$a_i > i+1$, then $x = \Psi \overset{a_i-2}{\underset{i+1}{\uparrow}}$,*
   (c) *If $a_i+\kappa_1-\kappa_2 \not\equiv 1,2 \pmod e$, then $x = -\Psi \overset{a_i-1}{\underset{i+1}{\uparrow}}$.*
2. *Let $i \equiv 2+\kappa_2-\kappa_1 \pmod e$ and suppose that $i \neq 1$ when $\kappa_2 \equiv \kappa_1 - 1 \pmod e$.*

(a) *If* $a_i \equiv 1+\kappa_2-\kappa_1 \pmod e$ *and* $a_i \neq n$ *when* $i=m$, *then*

$$x = \begin{cases} \psi_{a_i}\Psi\overset{i}{\underset{i-1}{\downarrow}}\ \Psi\overset{a_i-1}{\underset{i+1}{\uparrow}} & \text{if } a_{i+1}=a_i+1,\\ -\Psi\overset{i}{\underset{i-1}{\downarrow}}\ \Psi\overset{a_i}{\underset{i+1}{\uparrow}} & \text{if } a_{i+1}\geqslant a_i+2. \end{cases}$$

(b) *If* $a_i \equiv 2+\kappa_2-\kappa_1 \pmod e$, *then* $x = -\Psi\overset{i}{\underset{i-1}{\downarrow}}\ \Psi\overset{a_i-2}{\underset{i+1}{\uparrow}}$;

(c) *If* $a_i+\kappa_1-\kappa_2 \not\equiv 1,2 \pmod e$, *then* $x = \Psi\overset{i}{\underset{i-1}{\downarrow}}\ \Psi\overset{a_i-1}{\underset{i+1}{\uparrow}}$.

3. *Let* $i+\kappa_1-\kappa_2 \not\equiv 1,2 \pmod e$.

(a) *If* $a_i \equiv 1+\kappa_2-\kappa_1 \pmod e$ *and* $a_i \neq n$ *when* $i=m$, *then*

$$x = \begin{cases} \psi_{a_i}\Psi\overset{a_i-1}{\underset{i}{\uparrow}} & \text{if } a_{i+1}=a_i+1,\\ -\Psi\overset{a_i}{\underset{i}{\uparrow}} & \text{if } a_{i+1}\geqslant a_i+2. \end{cases}$$

(b) *If* $a_i \equiv 2+\kappa_2-\kappa_1 \pmod e$, *then* $x = -\Psi\overset{a_i-2}{\underset{i}{\uparrow}}$;

(c) *If* $a_i+\kappa_1-\kappa_2 \not\equiv 1,2 \pmod e$, *then* $x = \Psi\overset{a_i-1}{\underset{i}{\uparrow}}$.

Proof. Let $i \equiv \kappa_2-\kappa_1+1 \pmod e$.

1. Suppose that $a_i \equiv 1+\kappa_2-\kappa_1 \pmod e$ and $a_{i+1} \geqslant a_i+2$. Then $a_i = i+ke$ for some $k>0$. We proceed by induction on k. Set $i = 1+\kappa_2-\kappa_1$ and $a_i = 1+\kappa_2-\kappa_1+e$ for the base case, so that

$$\begin{aligned}
&\Psi\overset{1+\kappa_2-\kappa_1+e}{\underset{2+\kappa_2-\kappa_1}{\uparrow}} v(1,\dots,i-1,1+\kappa_2-\kappa_1+e,a_{i+1},\dots,a_m)\\
&= \Psi\overset{\kappa_2-\kappa_1+e}{\underset{2+\kappa_2-\kappa_1}{\uparrow}} v(1,\dots,i-1,2+\kappa_2-\kappa_1+e,a_{i+1},\dots,a_m) && \text{by Equation (7.1.1)}\\
&= -\Psi\overset{\kappa_2-\kappa_1+e-1}{\underset{2+\kappa_2-\kappa_1}{\uparrow}} v(1,\dots,i-1,\kappa_2-\kappa_1+e,a_{i+1},\dots,a_m) && \text{by Equation (7.1.3)}\\
&= -\psi_{2+\kappa_2-\kappa_1} v(1,\dots,i-1,3+\kappa_2-\kappa_1,a_{i+1},\dots,a_m) && \text{by Equation (7.1.6)}\\
&= v(1,\dots,i-1,1+\kappa_2-\kappa_1,a_{i+1},\dots,a_m) && \text{by Equation (7.1.5)}\ ,
\end{aligned}$$

as required. Now assume that $\Psi\overset{i+ke}{\underset{i+1}{\uparrow}} v(1,\dots,i-1,i+ke,a_{i+1},\dots,a_m) = v(1,\dots,i-1,i,a_{i+1},\dots,a_m)$ for some $k>0$. Observe

$$\begin{aligned}
&\Psi\overset{i+(k+1)e}{\underset{i+1}{\uparrow}} v(1,\dots,i-1,i+(k+1)e,a_{i+1},\dots,a_m)\\
&= \Psi\overset{i+(k+1)e-1}{\underset{i+1}{\uparrow}} v(1,\dots,i-1,i+(k+1)e+1,a_{i+1},\dots,a_m) && \text{by Equation (7.1.1)}
\end{aligned}$$

$$
\begin{aligned}
&= -\Psi \overset{i+(k+1)e-2}{\underset{i+1}{\uparrow}} v(1,\dots,i-1,i+(k+1)e-1,a_{i+1},\dots,a_m) && \text{by Equation (7.1.3)}\\
&= -\Psi \overset{i+ke+1}{\underset{i+1}{\uparrow}} v(1,\dots,i-1,i+ke+2,a_{i+1},\dots,a_m) && \text{by Equation (7.1.6)}\\
&= \Psi \overset{i+ke}{\underset{i+1}{\uparrow}} v(1,\dots,i-1,i+ke,a_{i+1},\dots,a_m) && \text{by Equation (7.1.5)}\\
&= v(1,\dots,i-1,i,a_{i+1},\dots,a_m),
\end{aligned}
$$

by the inductive hypothesis as required.

The other parts are similarly proven using induction on i. □

If $\kappa_2 \equiv \kappa_1 - 1 \pmod e$, then there does not exist a reduced expression ψ_w of w for which $\psi_w v(2,a_2,\dots,a_m) = v(1,a_2,\dots,a_m)$. Instead, we map every basis vector of S_λ to $v_T = \psi_1\psi_2\dots\psi_m z_\lambda$, where $2,3,\dots,m+1$ lie in the leg of T.

Lemma 7.4. *If $\kappa_2 \equiv \kappa_1 - 1 \pmod e$, then there exists an element $\psi_w \in \mathscr{R}_n^\Lambda$, for a reduced expression $w \in \mathfrak{S}_n$, such that $\psi_w v(a_1,a_2,\dots,a_m) = v(2,a_2,\dots,a_m)$ as given in the following cases.*

1. *If $a_1 \equiv 1 + \kappa_2 - \kappa_1 \pmod e$ then*

$$
\psi_w = \begin{cases} \psi_{a_i}\Psi \overset{a_1-1}{\underset{2}{\uparrow}} & (a_{1+1} = a_1+1),\\ \Psi\overset{a_1}{\underset{2}{\uparrow}} & (a_{1+1} > a_1+1).\end{cases}
$$

2. *If $a_1 \equiv 2 + \kappa_2 - \kappa_1 \pmod e$ then $\psi_w = \Psi \overset{a_1-2}{\underset{2}{\uparrow}}$.*

3. *If $a_1 - \kappa_1 + \kappa_2 \not\equiv 1,2 \pmod e$ then $\psi_w = \Psi \overset{a_1-1}{\underset{2}{\uparrow}}$.*

The next result will be a useful addition for determining irreducibility of certain $\mathscr{R}_n^\Lambda$-modules in the following chapter.

Corollary 7.5. *Let $1 \leqslant i \leqslant m-1$.*

1. *If $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$, then there exists an element $\psi_w \in \mathscr{R}_n^\Lambda$, for a reduced expression of a $w \in \mathfrak{S}_n$ such that*

$$\psi_w v(1,\dots,i-1,a_i,\dots,a_m) = v(1,\dots,i,a_{i+1},\dots,a_m).$$

2. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod e$.*

   (a) *If $a_i > i+1$, then there exists an element $\psi_w \in \mathscr{R}_n^\Lambda$, for a reduced expression of a $w \in \mathfrak{S}_n$ such that*

$$\psi_w v(2,\dots,i,a_i,\dots,a_{m-1},n) = v(2,\dots,i+1,a_{i+1},\dots,a_{m-1},n).$$

(b) *If $a_i > i > 1$, then there exists an element $\psi_w \in \mathscr{R}_n^\Lambda$, for a reduced expression of a $w \in \mathfrak{S}_n$ such that*

$$\psi_w v(1,\dots,i-1,a_i,\dots,a_m) = v(1,\dots,i,a_{i+1},\dots,a_m).$$

7.2 Linear combinations of basis vectors

To ascertain irreducibility of a non-zero submodule, say M, of a Specht module labelled by a hook bipartition, we need to ultimately show that this submodule is generated by any element in the basis of M. However, it is non-trivial that an arbitrary, non-zero submodule of S_λ even contains a single basis element of S_λ. To this end, we first introduce a result necessary for understanding the action of $\mathscr{R}_n^\Lambda$ on non-zero linear combinations of standard basis elements of S_λ.

Proposition 7.6. *Let $S, T \in \mathrm{Std}((n-m),(1^m))$ be distinct standard tableaux such that $\mathbf{i}_S = \mathbf{i}_T$. Then there exists $x \in \mathscr{R}_n^\Lambda$ of the form $x = \psi_w$ for some $w \in \mathfrak{S}_n$ or $x = e(\mathbf{i})$ for some $\mathbf{i} \in I^n$, such that exactly one of xv_S and xv_T is zero.*

Proof. Set $a_r := T(r,1,2)$ and $b_r := S(r,1,2)$ for $1 \leqslant r \leqslant m$ and first suppose that $a_m = b_m$.

If $a_r > b_r$ then observe that $T(r,1,2) = S(1,a_r-r,1) = a_r$ where $\mathrm{res}(r,1,2) = \kappa_2+1-r$, $\mathrm{res}(1,a_r-r,1) = \kappa_1+a_r-r-1$. Thus, $e_S \neq e_T$ if $a_r \not\equiv 2+\kappa_2-\kappa_1 \pmod e$, hence $e_T(v(a_1,\dots,a_m)+v(b_1,\dots,b_m)) = v(a_1,\dots,a_m)$ and $e_S(v(a_1,\dots,a_m)+v(b_1,\dots,b_m)) = v(b_1,\dots,b_m)$.

If $m=1$ then $a_1 = i > b_1$. We have $e_T \neq e_S$ if $i \not\equiv 2+\kappa_2-\kappa_1 \pmod e$. So suppose $i \equiv 2+\kappa_2-\kappa_1 \pmod e$. Then $\psi_i v(a_1) = v(i+1)$ by Equation (7.1.1), whereas $\psi_i v(b_1) = \psi_i \Psi \overset{b_1-1}{\underset{1}{\downarrow}} z_\lambda = \Psi \overset{b_1-1}{\underset{1}{\downarrow}} \psi_i z_\lambda = 0$.

Now suppose $m > 1$ and let r be maximal such that $a_r \neq b_r$ without loss of generality. Set $a_r = i > b_r$ and $a_{r+1} = b_{r+1} = j$.

1. Suppose $j \geqslant i+3$. Then $\psi_i v(b_1,\dots,b_m) = 0$, whereas, by Equation (7.1.1),

$$\psi_i v(a_1,\dots,a_m) = \psi_i v(a_1,\dots,a_{r-1},i,j,a_{r+2},\dots,a_m) = v(a_1,\dots,a_{r-1},i{+}1,j,a_{r+2}).$$

2. Suppose $j = i+2$.
   (a) Suppose $j \not\equiv 2+\kappa_2-\kappa_1 \pmod e$. Then $\psi_i v(b_1,\dots,b_m) = 0$, whereas $\psi_i v(a_1,\dots,a_m) = v(a_1,\dots,a_{r-1},i{+}1,i{+}2,a_{r+2},\dots,a_m)$, by Equation (7.1.1).
   (b) Suppose $j \equiv 2+\kappa_2-\kappa_1 \pmod e$. Then $e_S \neq e_T$ since $i \not\equiv 2+\kappa_2-\kappa_2 \pmod e$.
3. Suppose that $j = i+1$.

(a) Suppose $j \not\equiv 3+\kappa_2-\kappa_1 \pmod e$. Then $e_S \neq e_T$ since $i \not\equiv 2+\kappa_2-\kappa_1 \pmod e$.

(b) Suppose $j \equiv 3+\kappa_2-\kappa_1 \pmod e$. Firstly, let $r \neq 1$.

i. Suppose $a_{r-1} = i-1$. By Theorem 7.1, $\psi_{i-1}v(a_1,\dots,a_m) = 0$ and $\psi_i v(a_1,\dots,a_m) = 0$. Whereas, if $b_r = i-1$ then

$$\psi_{i-1}v(b_1,\dots,b_m) = v(b_1,\dots,b_{r-1},i,i+1,b_{r+2},\dots,b_m),$$

by Equation (7.1.1); if $b_r \leqslant i-2$ then

$$\psi_i v(b_1,\dots,b_m) = -v(b_1,\dots,b_{r-1},\leqslant i-2,i-1,b_{r+2},\dots,b_m),$$

by Equation (7.1.5).

ii. Suppose $a_{r-1} \leqslant i-2$.

A. Suppose $b_r = i-1$. Then

$$\psi_i v(a_1,\dots,a_m) = v(a_1,\dots,a_{r-2},\leqslant i-2,i-1,i,a_{r+2},\dots,a_m),$$

by Equation (7.1.4), whereas $\psi_i v(b_1,\dots,b_m) = 0$.

B. Suppose $b_r \leqslant i-2$. Then

$$\begin{aligned}\psi_i^2 v(a_1,\dots,a_m) =& \psi_i v(a_1,\dots,a_{r-1},\leqslant i-2,i-1,i,a_{r+2},\dots,a_m)\\ =& v(a_1,\dots,a_{r-1},\leqslant i-2,i-1,i+1,a_{r+2},\dots,a_m),\end{aligned}$$

by Equation (7.1.1) and Equation (7.1.4), whereas

$$\psi_i^2 v(b_1,\dots,b_m) = -\psi_i v(b_1,\dots,b_{r-1},\leqslant i-2,i-1,b_{r+2},\dots,b_m) = 0,$$

by Equation (7.1.5).

Now let $r = 1$. If $b_r = i-1$ then

$$\psi_i v(a_1,\dots,a_m) = v(i-1,i,a_3,\dots,a_m),$$

by Equation (7.1.4), whereas $\psi_i v(b_1,\dots,b_m) = 0$. If $b_r \leqslant i-2$ then

$$\psi_{i-1}\psi_i v(a_1,\dots,a_m) = \psi_{i-1}v(i-1,i,a_3,\dots,a_m) = 0,$$

by Equation (7.1.4), whereas

$$\begin{aligned}\psi_{i-1}\psi_i v(b_1,\dots,b_m) =& -\psi_{i-1}v(\leqslant i-2,i-1,a_3,\dots,a_m)\\ =& -v(\leqslant i-2,i,a_3,\dots,a_m),\end{aligned}$$

by Equation (7.1.1) and Equation (7.1.5).

Now suppose that $a_m \neq b_m$. It is sufficient to consider the following three cases:

- ◇ $a_{m-1} = b_{m-1} = i$, $a_m = j > b_m$,
- ◇ $a_{m-1} = i > b_{m-1}$, $a_m = j > b_m$,
- ◇ $a_{m-1} = i < b_{m-1}$, $a_m = j > b_m$.

Observe that $T(m,1,2) = S(1,j-m,1) = j$ where $\mathrm{res}(m,1,2) = \kappa_1 + 1 - m$ and $\mathrm{res}(1,j-m,1) = \kappa_1 + j - m - 1$. Thus, if $j \not\equiv 2 + \kappa_2 - \kappa_1 \pmod e$ then $e_T \neq e_S$. So suppose $j \equiv 2 + \kappa_2 - \kappa_1 \pmod e$. If $j < n$ then

$$\psi_j((v(a_1,\ldots,a_m) + v(b_1,\ldots,b_m)) = \psi_j(v(a_1,\ldots,a_m) = v(a_1,\ldots,a_{m-1},j+1),$$

by Equation (7.1.1). Now suppose $j = n$ and let $b_m \leqslant n-2$. We have $T(1,n-m,1) = S(1,n-m-1,1)$ where $\mathrm{res}(1,n-m,1) = \kappa_2 + 1 - m \neq \kappa_2 - m = \mathrm{res}(1,n-m-1,1)$, and thus $e_T \neq e_S$. Whereas, if $b_m = n-1$ then

$$\psi_{n-1}(v(a_1,\ldots,a_m) + v(b_1,\ldots,b_m)) = \psi_{n-1}v(b_1,\ldots,b_m) = (b_1,\ldots,b_{m-1},n),$$

by Equation (7.1.1). □

Lemma 7.7. *Any non-zero submodule of $S_{((n-m),(1^m))}$ contains a standard basis vector v_T, for some $T \in \mathrm{Std}((n-m),(1^m))$.*

Proof. Let $0 \neq M \subset S_{((n-m),(1^m))}$ and consider an arbitrary non-zero element v of M. Then v is an $\mathbb{F}$-linear combination of r distinct basis vectors of $S_{((n-m),(1^m))}$, for $r \geqslant 1$. We write

$$v = \sum_{i=1}^{r} c_i v_{T_i}, \quad 0 \neq c_i \in \mathbb{F},$$

where $v_{T_i} \neq v_{T_j}$ for all $1 \leqslant i \neq j \leqslant r$. We can instead replace v with $e(\mathbf{i})v$ such that $e(\mathbf{i})v = v$, and thus assume that $\mathbf{i}_{T_i} = \mathbf{i}_{T_j}$ for all $i,j \in \{1,\ldots,r\}$.

We choose v with $r \geqslant 1$ minimal. If $r = 1$, we are done, so we suppose that $r > 1$. Then, by Proposition 7.6, we can find an $x = \psi_w \in \mathscr{R}_n^\Lambda$ for $w \in \mathfrak{S}_n$ such that exactly one of xv_{T_1}, xv_{T_2} is zero. Without loss of generality, we let $xv_{T_1} = 0$. Then we have

$$M \ni xv = \sum_{i=2}^{r} \pm c_i v_{T_i},$$

where, for all $i \in \{2,\ldots,r\}$, $c_i x v_{T_i}$ equals zero or $\pm c_i v_{S_i}$ for some standard $((n-m),(1^m))$-tableau S_i. Moreover, by Corollary 7.2, we know that $i = j$ whenever $s_i = s_j$, for $i,j \in \{2,\ldots,r\}$. We have thus contradicted the minimality of r, and hence there must exist an $x \in \mathscr{R}_n^\Lambda$ such that $xv = \alpha v_T$, where $\alpha \in \mathbb{F}$ and $v_T \in S_{((n-m),(1^m))}$. □

7.3 Case I: composition series of $S_{((n-m),(1^m))}$ for $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv \kappa_2 - \kappa_1 + 1 \pmod e$

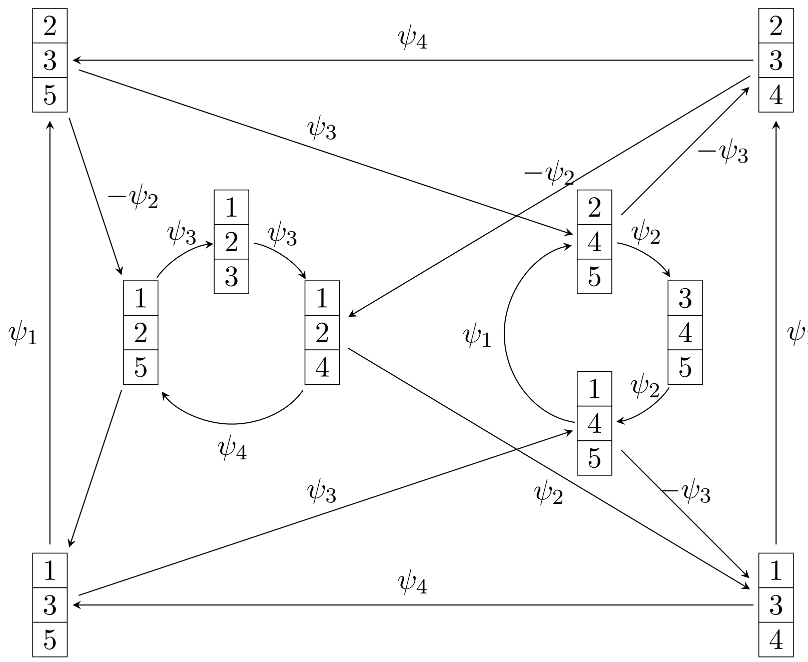
For this case, we claim that Specht modules labelled by hook bipartitions are irreducible, and thus are generated by any standard basis element.

Theorem 7.8. *If $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv \kappa_2 - \kappa_1 + 1 \pmod e$, then $S_{((n-m),(1^m))}$ is irreducible.*

Proof. We have $S_{((n-m),(1^m))} = \mathrm{span}\{v_T \mid T \in \mathrm{Std}\,((n-m),(1^m))\}$. Since we know from Lemma 7.7 that any non-zero submodule of $S_{((n-m),(1^m))}$ contains a standard basis element v_T, it suffices to show that any standard basis element $v(a_1,\ldots,a_m)$ generates $S_{((n-m),(1^m))}$. We know that $v(1,\ldots,m)$, which equals z_λ, generates $S_{((n-m),(1^m))}$, so we now let i be minimal such that $a_i > i$, for some $i \in \{1,\ldots,m\}$, and proceed by downwards induction on i. By Corollary 7.5, there exists an element $\psi_\omega \in \mathscr{R}_n^\Lambda$ such that $\psi_\omega v(1,\ldots,i-1,a_i,\ldots,a_m) = v(1,\ldots,i,a_{i+1},\ldots,a_m)$, for a reduced expression of $\omega \in \mathfrak{S}_n$. By induction, we know that $v(1,\ldots,i,a_{i+1},\ldots,a_m)$ generates $S_{((n-m),(1^m))}$, and thus, $v(a_1,\ldots,a_m)$ also generates $S_{((n-m),(1^m))}$.

□

Example 7.9. *Set $e = 3$ and $\kappa = (0,0)$. We know that $S_{((2),(1^3))}$ is an irreducible $\mathscr{R}_n^\Lambda$-module, so, for all $S,T \in \mathrm{Std}((2),(1^3))$, there exists an expression $\psi_\omega \in \mathscr{R}_n^\Lambda$ for which $\psi_\omega v_T = \pm v_S$, for a reduced expression $\omega \in \mathfrak{S}_n$. Recall that a standard $((2),(1^3))$-tableau is completely determined by the three entries in its leg. We represent the basis elements of $S_{((2),(1^3))}$ by the legs of the corresponding $((2),(1^3))$-tableaux, together with the only non-trivial relations between these elements. Observe that we can find a directed path from any standard $((2),(1^3))$-tableau to any other standard $((2),(1^3))$-tableau, as* *expected.*



7.4 Case II: composition series of $S_{((n-m),(1^m))}$ for $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$

Here, we claim that the images of the homomorphisms γ_m given in Lemma 6.5, for $0 \leqslant m \leqslant n-1$, appear as composition factors of Specht modules labelled by hook bipartitions. We first determine irreducibility of these $\mathscr{R}_n^\Lambda$-modules.

Proposition 7.10. *If* $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ *and* $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$, *then* $\operatorname{im}(\gamma_m)$ *is an irreducible* $\mathscr{R}_n^\Lambda$*-module. Further,* $\operatorname{im}(\gamma_m) = \langle v(1,\dots,m,n)\rangle \subseteq S_{((n-m-1),(1^{m+1}))}$.

Proof. By Lemma 6.5 part 1(a),

$$\operatorname{im}(\gamma_m) = \operatorname{span}\left\{v_T \mid T \in \operatorname{Std}\left((n-m-1),(1^{m+1})\right), T(m+1,1,2) = n\right\},$$

that is the image of γ_m is spanned by all standard v_T where n lies in the leg of T, where T is a standard $((n-m-1),(1^{m+1}))$-tableau. We can write a non-zero standard basis element v of $\operatorname{im}(\gamma_m)$ as $v(a_1,\dots,a_m,n)$.

Since we know from Lemma 7.7 that any non-zero submodule of $S_{((n-m),(1^m))}$ contains a standard basis element v_T, it suffices to show that any standard basis element $v(a_1,\dots,a_m,n)$ generates $\operatorname{im}(\gamma_m)$. We know that $v(1,\dots,m,n)$ generates $\operatorname{im}(\gamma_m)$, so we now let i be minimal such that $a_i > i$, for some $i \in \{1,\dots,m\}$, and proceed by downwards induction on i. By Corollary 7.5, there exists an element $\psi_\omega \in \mathscr{R}_n^\Lambda$ such that $\psi_\omega v(1,\dots,i-1,a_i,a_{i+1},\dots,a_m,n) = v(1,\dots,i-1,i,a_{i+1},\dots,a_m,n)$, for a reduced expression of $\omega \in \mathfrak{S}_n$. By induction, we know that $v(1, \ldots, i-1, i, a_{i+1}, \ldots, a_m, n)$ generates the image of γ_m, and hence, $v(a_1, \ldots, a_m, n)$ generates $\operatorname{im}(\gamma_m)$ too. □

An immediate consequence of this result, together with part one of Lemma 6.6, is the following theorem.

Theorem 7.11. *Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$, $n \equiv \kappa_2 - \kappa_1 + 1 \pmod e$ and $1 \leqslant m \leqslant n-1$. Then $S_{((n-m),(1^m))}$ has the composition series*

$$0 \subset \operatorname{im}(\gamma_{m-1}) \subset S_{((n-m),(1^m))},$$

where its composition factors are $\operatorname{im}(\gamma_{m-1})$ and $\operatorname{im}(\gamma_m)$ from bottom to top.

Example 7.12. *Let $e = 3$ and $\kappa = (0,1)$. Then $S_{((2),(1^3))}$ has the composition series*

$$0 \subset \operatorname{im}(\gamma_2) \subset S_{((2),(1^3))}$$

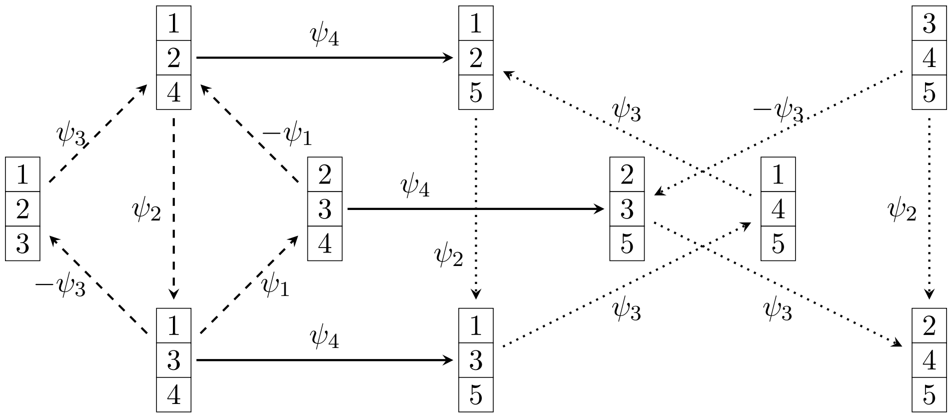
where $S_{((2),(1^3))}/\operatorname{im}(\gamma_2) \cong \operatorname{im}(\gamma_3)$. The basis elements of $\operatorname{im}(\gamma_2) = \langle z_{((2),(1^3))} \rangle$ correspond to the $((2),(1^3))$-tableaux

$$\begin{array}{cccc} \begin{array}{|c|c|}\hline 4&5\\\hline\end{array}, & \begin{array}{|c|c|}\hline 3&5\\\hline\end{array}, & \begin{array}{|c|c|}\hline 2&5\\\hline\end{array}, & \begin{array}{|c|c|}\hline 1&5\\\hline\end{array}, \\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} & \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array} \end{array}$$

and the basis elements of $\operatorname{im}(\gamma_3) = \left\langle \Psi\!\downarrow_{3}^{5} z_{((2),(1^3))} \right\rangle$ correspond to the $((2),(1^3))$-tableaux

$$\begin{array}{cccccc} \begin{array}{|c|c|}\hline 3&4\\\hline\end{array}, & \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}, & \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, & \begin{array}{|c|c|}\hline 1&4\\\hline\end{array}, & \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, & \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}. \\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 5\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 3\\\hline 5\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 4\\\hline 5\\\hline\end{array} & \begin{array}{|c|}\hline 2\\\hline 3\\\hline 5\\\hline\end{array} & \begin{array}{|c|}\hline 2\\\hline 4\\\hline 5\\\hline\end{array} & \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array} \end{array}$$

We know that $\operatorname{im}(\gamma_2)$ and $\operatorname{im}(\gamma_3)$ are irreducible, so for $R, S \in \operatorname{Std}((2),(1^3))$ such that $v_R, v_S \in \operatorname{im}(\gamma_2)$, there exists an element $\psi_{\omega_1} \in \mathscr{R}_n^\Lambda$ for which $\psi_{\omega_1} v_S = v_R$, for a reduced expression of $\omega_1 \in \mathfrak{S}_n$, and similarly, for $T, U \in \operatorname{Std}((2),(1^3))$ such that $v_T, v_U \in \operatorname{im}(\gamma_3)$, there exists an element $\psi_{\omega_2} \in \mathscr{R}_n^\Lambda$ for which $\psi_{\omega_2} v_U = v_T$, for a reduced expression of $\omega_2 \in \mathfrak{S}_n$. Recall that a standard $((2),(1^3))$-tableau is completely determined by the three entries in its leg. We represent the basis elements of $S_{((2),(1^3))}$ by the legs of the corresponding $((2),(1^3))$-tableaux, together with the only non-trivial relations between these elements. Observe that we can find a directed path from any standard $((2),(1^3))$-tableau in $\operatorname{im}(\gamma_2)$ to any other standard $((2),(1^3))$-tableau in $\operatorname{im}(\gamma_2)$, and similarly, we can find a directed path from any standard $((2),(1^3))$-tableau in $\operatorname{im}(\gamma_3)$ to any other standard $((2),(1^3))$-tableau in $\operatorname{im}(\gamma_3)$.



7.5 CASE III: COMPOSITION SERIES OF $S_{((n-m),(1^m))}$ FOR $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ AND $n \not\equiv \kappa_2 - \kappa_1 + 1 \pmod e$

We need only understand the homomorphism χ_m to determine the composition factors of Specht modules labelled by hook bipartitions in this case. Let us first confirm their irreducibility.

Proposition 7.13. *If $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv \kappa_2 - \kappa_1 + 1 \pmod e$, then $\operatorname{im}(\chi_m)$ and $S_{((n-m),(1^m))}/\operatorname{im}(\chi_m)$ are irreducible $\mathscr{R}_n^\Lambda$-modules. Moreover, $\operatorname{im}(\chi_m) = \langle v(2,\dots,m+1)\rangle$.*

Proof. We know that χ_m is injective and τ_m is surjective by the exact sequence given by Lemma 6.6 part two, so that $\operatorname{im}(\chi_m) \cong S_{((n-m,1^m),\varnothing)}$ and $\operatorname{im}(\tau_m) \cong S_{(\varnothing,(1^{n-m+1},1^{m-1}))}$. By appealing to the v-analogue of Peel's Theorem (see Proposition 4.2 or [CMT, Theorem 1(1)]), $S_{(n-m,1^m)}$ and $S_{(n-m+1,1^{m-1}))}$ are both irreducible, and hence, so are $S_{((n-m,1^m),\varnothing)}$ and $S_{(\varnothing,(1^{n-m+1},1^{m-1}))}$. Thus, $\operatorname{im}(\chi_m)$ and $S_{((n-m),(1^m))}/\operatorname{im}(\chi_m)$ are irreducible, as required. □

Hence, for $1 \leqslant m \leqslant n-1$, it is immediately obvious that $0 \subset \operatorname{im}(\chi_m) \subset S_{((n-m),(1^m))}$ is a composition series for $S_{((n-m),(1^m))}$ when $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv \kappa_2 - \kappa_1 + 1 \pmod e$.

Example 7.14. *Let $e = 3$ and $\kappa = (0,2)$. The composition series of $S_{((3),(1^2))}$ is*

$$0 \subset \operatorname{im}(\chi_2) \subset S_{((3),(1^2))}.$$

Elements of $\operatorname{im}(\chi_2)$ correspond to the $((3),(1^2))$-tableaux

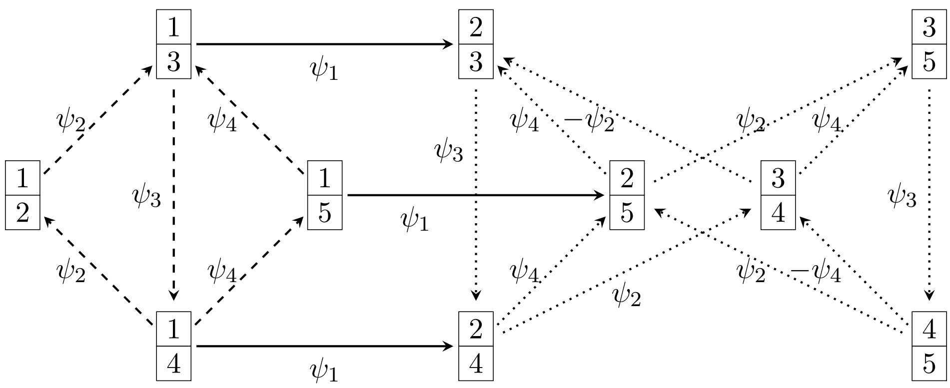
$$\begin{array}{|c|c|c|}\hline 3&4&5\\\hline\end{array},\quad \begin{array}{|c|c|c|}\hline 2&4&5\\\hline\end{array},\quad \begin{array}{|c|c|c|}\hline 2&3&5\\\hline\end{array},\quad \begin{array}{|c|c|c|}\hline 2&3&4\\\hline\end{array},$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array}$$

and elements of $S_{((3),(1^2))}/\operatorname{im}(\chi_2)$ *correspond to the* $((3),(1^2))$*-tableaux*

1 4 5 ,	1 3 5 ,	1 3 4 ,	1 2 5 ,	1 2 4 ,	1 2 3 .
2	2	2	3	3	4
3	4	5	4	5	5

We know that $\operatorname{im}(\chi_2)$ *and* $S_{((3),(1^2))}/\operatorname{im}(\chi_3)$ *are irreducible, so for* $R, S \in \operatorname{Std}((3),(1^2))$ *such that* $v_R, v_S \in \operatorname{im}(\chi_2)$*, there exists an element* $\psi_{\omega_1} \in \mathscr{R}_n^\Lambda$ *for which* $\psi_{\omega_1} v_S = v_R$*, for a reduced expression of* $\omega_1 \in \mathfrak{S}_n$*, and similarly, for* $T, U \in \operatorname{Std}((3),(1^2))$ *such that* $v_T, v_U \in S_{((3),(1^2))}/\operatorname{im}(\chi_3)$*, there exists an element* $\psi_{\omega_2} \in \mathscr{R}_n^\Lambda$ *for which* $\omega_2 v_U = v_T$*, for a reduced expression of* $\omega_2 \in \mathfrak{S}_n$*. Recall that a standard* $((3),(1^2))$*-tableau is completely determined by the two entries in its leg. We represent the basis elements of* $S_{((3),(1^2))}$ *by the legs of the corresponding* $((3),(1^2))$*-tableaux, together with the only non-trivial relations between these elements. Observe that we can find a directed path from any standard* $((3),(1^2))$*-tableau in* $\operatorname{im}(\chi_2)$ *to any other standard* $((3),(1^2))$*-tableau in* $\operatorname{im}(\chi_2)$*, and similarly, we can find a directed path from any standard* $((3),(1^2))$*-tableau in* $S_{((3),(1^2))}/\operatorname{im}(\chi_3)$ *to any other standard* $((3),(1^2))$*-tableau in* $S_{((3),(1^2))}/\operatorname{im}(\chi_3)$.



7.6 Case IV: composition series of $S_{((n-m),(1^m))}$ for $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \equiv 0 \pmod e$

The composition series of Specht modules labelled by hook bipartitions are the most complicated in this case, where each Specht module has up to four composition factors.

We determine the irreducibility of $\mathscr{R}_n^\Lambda$-modules $\operatorname{im}(\phi_m)$ and $\ker(\gamma_m)/\operatorname{im}(\phi_m)$.

Proposition 7.15. *If* $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ *and* $n \equiv 0 \pmod e$*, then* $\operatorname{im}(\phi_m)$ *and* $\ker(\gamma_m)/\operatorname{im}(\phi_m)$ *are irreducible* $\mathscr{R}_n^\Lambda$*-modules. Moreover,*

1. $\operatorname{im}(\phi_m) = \langle v(2, \dots, m, n) \rangle$
2. *and* $\ker(\gamma_m) = \langle v(1, \dots, m-1, n) \rangle$.

Proof. 1. By Lemma 6.5 part 3(c), we have

$$\operatorname{im}(\phi_m) = \operatorname{span}\{v_T \mid T \in \operatorname{Std}\left((n-m),(1^m)\right), T(1,1,1)=1, T(m,1,2)=n\},$$

that is the image of ϕ_m is spanned by all standard v_T where 1 lies in the arm of T and n lies in the leg of T, for a standard $((n-m),(1^m))$-tableau T.

Since we know from Lemma 7.7 that any non-zero submodule of $S_{((n-m),(1^m))}$ contains a standard basis element v_T, it suffices to show that any standard basis element $v(a_1,\dots,a_{m-1},n)$, with $a_i > i$ for all $i \in \{1,\dots,m-1\}$, generates $\operatorname{im}(\phi_m)$. We know that $v(2,\dots,m,n)$ generates $\operatorname{im}(\phi_m)$, so we now let i be minimal such that $a_i > i+1$ for all $i \in \{1,\dots,m-1\}$ and proceed by downwards induction on i. By Corollary 7.5, there exists an expression $\psi_\omega \in \mathscr{R}_n^\Lambda$ such that

$$\psi_\omega v(2,\dots,i,a_i,\dots,a_{m-1},n) = v(2,\dots,i+1,a_{i+1},\dots,a_{m-1},n),$$

for a reduced expression $\omega \in \mathfrak{S}_n$. By induction, we know that $v(2,\dots,i+1,a_{i+1},\dots,a_{m-1},n)$ generates $\operatorname{im}(\phi_m)$, and thus, so does $v(a_1,\dots,a_{m-1},n)$.

2. By Lemma 6.5 part 1(b), we have

$$\ker(\gamma_m) = \operatorname{span}\{v_T \mid T \in \operatorname{Std}\left((n-m),(1^m)\right), T(m,1,2)=n\},$$

that is $\ker(\gamma_m)$ is spanned by all standard v_T where n lies in the leg of T. Let $0 \neq v$ be an arbitrary element of $\ker(\gamma_m)$ that is not contained in $\operatorname{im}(\phi_m)$, so that v is equal, modulo $\operatorname{im}(\phi_m)$, to a linear combination of r distinct standard basis vectors, none of which is in $\operatorname{im}(\phi_m)$, for $r \geqslant 1$. That is, $v = \alpha_1 v_{T_1} + \cdots + \alpha_r v_{T_r}$ where $\alpha_1,\dots,\alpha_r \in \mathbb{F}$ and $v_{T_1},\dots,v_{T_r} \in \ker(\gamma_m)/\operatorname{im}(\phi_m)$. We proceed by induction on r to show that v generates $\ker(\gamma_m)/\operatorname{im}(\phi_m)$.

For $r=1$, we write $v = cv(a_1,\dots,a_{m-1},n)$ for some $0 \neq c \in \mathbb{F}$. Now let i be minimal such that $a_i > i$, for some $i \in \{2,\dots,m-1\}$, and proceed by downwards induction on i. By Corollary 7.5, there exists $\psi_\omega \in \mathscr{R}_n^\Lambda$ such that

$$\psi_\omega v(1,\dots,i-1,a_i,\dots,a_{m-1},n) = v(1,\dots,i,a_{i+1},\dots,a_{m-1},n),$$

for a reduced expression of $\omega \in \mathfrak{S}_n$. By induction, we know that $v(1,\dots,i,a_{i+1},\dots,a_{m-1},n)$ generates $\ker(\gamma_m)/\operatorname{im}(\phi_m)$, and thus, so does v.

Now suppose that $r>1$. Then by induction on r, there exists an $x \in \mathscr{R}_n^\Lambda$ such that

$$xv = \alpha v_T + \alpha_r x v_{T_r},$$

where $\alpha \in \mathbb{F}$, $v_T \in \ker(\gamma_m)/\operatorname{im}(\phi_m)$. If x kills $\alpha_r v_{T_r}$, then we are done. Otherwise, suppose that $\alpha_r x v_{T_r} = \beta v_S$, where $\beta \in \mathbb{F}$ and v_S is a standard basis vector lying in $\ker(\gamma_m)$ but not in $\operatorname{im}(\phi_m)$.

Take $v(1, a_2, \ldots, a_{m-1}, n) \in \ker(\gamma_m) \setminus \operatorname{im}(\phi_m)$ and observe that, by Theorem 7.1, $\psi_1 v(1, a_2, \ldots, a_{m-1}, n) = v(2, a_2, \ldots, a_{m-1}, n) \in \operatorname{im}(\phi_m)$, which is non-zero if and only if $2 < a_2 < \cdots < a_{m-1} < n$. Moreover, we also observe from Theorem 7.1 that there exists no $\psi_l \in \{\psi_1, \ldots, \psi_{n-1}\}$ such that

$$\psi_l v(2, a_2, \ldots, a_{m-1}, n) = v(1, b_2, \ldots, b_{m-1}, n) \in \ker(\gamma_m) \setminus \operatorname{im}(\phi_m),$$

where $1 < b_2 < \cdots < b_{m-1} < n$. We now consider the element $\psi_{w'} \psi_1 \psi_w \in \mathscr{R}_n^\Lambda$ such that ψ_w does not contain ψ_1, for $w, w' \in \mathfrak{S}_n$. If we suppose that $\psi_w v(1, a_2, \ldots, a_{m-1}, n) \neq 0$, then $\psi_1 \psi_w v(1, a_2, \ldots, a_{m-1}, n)$ is a non-zero element in $\operatorname{im}(\phi_m)$. Moreover, $\operatorname{im}(\phi_m)$ always contains $\psi_{w'} \psi_1 \psi_w v(1, a_2, \ldots, a_{m-1}, n)$.

Then, by appealing to Proposition 7.6 and Theorem 7.1, there exists an $x' = \psi_w \in \mathscr{R}_n^\Lambda$ that doesn't contain ψ_1, for $w \in \mathfrak{S}_n$, such that exactly one of $\alpha x' v_T$ and $\beta x' v_S$ is zero. Hence $x' x v = \beta x' v_S = \gamma v_R$, where $\gamma \in \mathbb{F}$ and v_R is a standard basis vector lying in $\ker(\gamma_m)$ but not in $\operatorname{im}(\phi_m)$, and thus v generates $\ker(\gamma_m) / \operatorname{im}(\phi_m)$, for $r > 1$.

□

Theorem 7.16. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$.*

1. *Then $S_{((n-1),(1))}$ has the composition series*

$$0 \subset \operatorname{im}(\phi_1) \subset \operatorname{im}(\chi_1) \subset S_{((n-1),(1))},$$

which has composition factors $S_{((n),\varnothing)}$, $\operatorname{im}(\phi_2)$ and $\operatorname{im}(\phi_3)$ from bottom to top.

2. *Let $2 \leqslant m \leqslant n - 2$. Then $S_{((n-m),(1^m))}$ has the composition series*

$$0 \subset \operatorname{im}(\phi_m) \subset \operatorname{im}(\chi_m) \subset \ker(\gamma_m) + \operatorname{im}(\chi_m) \subset S_{((n-m),(1^m))},$$

which has composition factors $\operatorname{im}(\phi_m)$, $\operatorname{im}(\phi_{m+1})$, $\ker(\gamma_m)/\operatorname{im}(\phi_m)$ and $\ker(\gamma_{m+1})/\operatorname{im}(\phi_{m+1})$ from bottom to top.

3. *Then $S_{((1),(1^{n-1}))}$ has the composition series*

$$0 \subset \operatorname{im}(\phi_{n-1}) \subset \operatorname{im}(\gamma_{n-2}) \subset S_{((1),(1^{n-1}))},$$

which has composition factors $\operatorname{im}(\phi_{n-1})$, $\operatorname{im}(\gamma_{n-2})/\operatorname{im}(\phi_{n-1})$ and $S_{(\varnothing,(1^n))}$ from bottom to top.

Proof. 1. From Lemma 6.8 we know that $S_{((n-1),(1))}$ has the filtration $\operatorname{im}(\phi_1) \subset \operatorname{im}(\chi_1) \subset S_{((n-1),(1))}$.

We know from Proposition 6.2 that $\gamma_1 \circ \phi_1 = 0$, so that by Lemma 6.8 the middle factor in the filtration of $S_{((n-1,1),(1))}$ is

$$\mathrm{im}(\chi_1)/\,\mathrm{im}(\phi_1) \cong \mathrm{im}(\gamma_1 \circ \chi_1)/\,\mathrm{im}(\gamma_1 \circ \phi_1) = \mathrm{im}(\gamma_1 \circ \chi_1) = \mathrm{im}(\phi_2).$$

By Lemma 6.5, we have

$$\ker(\tau_1) = \mathrm{span}\{T \in \mathrm{Std}\,((n-1),(1))\,, T(1,1,1) = 1.\}$$

Now, by using Lemma 6.4, the top factor in the filtration of $S_{((n-1),(1))}$ is given by

$$\begin{aligned}
&S_{((n-1),(1))}/\,\mathrm{im}(\chi_1)\\
&= S_{((n-1),(1))}/\ker(\tau_1)\\
&= \mathrm{span}\{v_T \mid T \in \mathrm{Std}\,((n-1),(1))\,, T(1,1,2) = 1\}\\
&\cong \mathrm{span}\{\gamma_3 \circ \gamma_2(v_T) \mid T \in \mathrm{Std}\,((n-1),(1))\,, T(1,1,2) = 1\}\\
&= \mathrm{span}\{v_T \mid T \in \mathrm{Std}\left((n-3),(1^3)\right), T(1,1,2) = 1, T(3,1,2) = n\}\\
&= \mathrm{im}(\phi_3).
\end{aligned}$$

2. Let $2 \leqslant m \leqslant n-2$. By Lemma 6.8, we know that $S_{((n-m),(1^m))}$ has the filtration

$$\mathrm{im}(\phi_m) \subset \mathrm{im}(\chi_m) \subset \ker(\gamma_m) + \mathrm{im}(\chi_m) \subset S_{((n-m),(1^m))}.$$

We know from Proposition 6.2 that $\gamma_m \circ \phi_m = 0$. Thus, together with Lemma 6.8, we know that the second from bottom factor in the filtration of $S_{((n-m,1^m))}$ is

$$\begin{aligned}
\mathrm{im}(\chi_m)/\,\mathrm{im}(\phi_m) \cong \mathrm{im}(\gamma_m \circ \chi_m)/\,\mathrm{im}(\gamma_m \circ \phi_m) &= \mathrm{im}(\gamma_m \circ \chi_m)\\
&= \mathrm{im}(\phi_{m+1}).
\end{aligned}$$

Using Lemma 6.8, we find that the second from top factor in the filtration of $S_{((n-m,1^m))}$ is

$$\begin{aligned}
(\ker(\gamma_m) + \mathrm{im}(\chi_m)/\,\mathrm{im}(\chi_m)) &= \ker(\gamma_m)/\,(\ker(\gamma_m) \cap \mathrm{im}(\chi_m))\\
&= \ker(\gamma_m)/\,(\mathrm{im}(\gamma_{m-1}) \cap \mathrm{im}(\chi_m))\\
&= \ker(\gamma_m)/\,\mathrm{im}(\phi_m).
\end{aligned}$$

Finally, using Lemma 6.8, the top factor in the filtration of $S_{((n-m,1^m))}$ is

$$\begin{aligned}
S_{((n-m,1^m))}/\,(\ker(\gamma_m) + \mathrm{im}(\chi_m)) &\cong (\mathrm{im}(\gamma_m) + \mathrm{im}(\chi_{m+1}))\,/\,\mathrm{im}(\chi_{m+1})\\
&= (\ker(\gamma_{m+1}) + \mathrm{im}(\chi_{m+1}))\,/\,\mathrm{im}(\chi_{m+1})\\
&= \ker(\gamma_{m+1})/\,(\ker(\gamma_{m+1}) \cap \mathrm{im}(\chi_{m+1}))\\
&= \ker(\gamma_{m+1})/\,\mathrm{im}(\phi_{m+1}).
\end{aligned}$$

3. It is clear from Lemma 6.8 that $\operatorname{im}(\phi_{n-1}) \subset \operatorname{im}(\gamma_{n-2}) \subset S_{((1),(1^{n-1}))}$ is a filtration of $S_{((1),(1^{n-1}))}$.

   Also by Lemma 6.8, the top factor of $S_{((1)(1^{n-1})}$ is

$$\begin{aligned} S_{((1),(1^{n-1})}/\operatorname{im}(\gamma_{n-2}) &= (\ker(\gamma_{n-1}+\operatorname{im}(\gamma_{n-1}))\,/\operatorname{im}(\gamma_{n-2}) \\ &= (\ker(\gamma_{n-1}+\operatorname{im}(\gamma_{n-1}))\,/\ker(\gamma_{n-1}) \\ &= \operatorname{im}(\gamma_{n-1})/\ker(\gamma_{n-1})\cap\operatorname{im}(\gamma_{n-1}) \\ &= \operatorname{im}(\gamma_{n-1}) \\ &\cong S_{(\varnothing,(1^n))}. \end{aligned}$$

□

Example 7.17. *Let $e = 3$ and $\kappa = (0,2)$. $S_{((3),(1^3))}$ has the composition series*

$$0 \subset \operatorname{im}(\phi_3) \subset \operatorname{im}(\chi_3) \subset \ker(\gamma_3)+\operatorname{im}(\chi_3) \subset S_{((3),(1^3))}$$

where $\operatorname{im}(\chi_3)/\operatorname{im}(\phi_3) \cong \operatorname{im}(\phi_4)$, $(\ker(\gamma_3)+\operatorname{im}(\chi_3))\,/\operatorname{im}(\chi_3) \cong \ker(\gamma_3)/\operatorname{im}(\phi_3)$ *and* $S_{((3),(1^3))}/\,(\ker(\gamma_3)+\operatorname{im}(\chi_3)) \cong \ker(\gamma_4)/\operatorname{im}(\phi_4)$. *The elements of* $\operatorname{im}(\phi_3) = \left\langle \psi_1\psi_2\Psi\overset{5}{\underset{3}{\downarrow}} z_{((3),(1^3))} \right\rangle$ *correspond to the* $((3),(1^3))$*-tableaux*

$$\begin{array}{|c|c|c|}\hline 1&4&5\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&3&5\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&5\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}.$$
$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|}\hline 2\\\hline 4\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|}\hline 3\\\hline 4\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|}\hline 2\\\hline 5\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|}\hline 3\\\hline 5\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|}\hline 4\\\hline 5\\\hline 6\\\hline\end{array}$$

$\operatorname{im}(\chi_3) = \langle \psi_1\psi_2\psi_3 z_{((3),(1^3))}\rangle$, *so the elements of* $\operatorname{im}(\phi_4)$ *correspond to the* $((3),(1^3))$*-tableaux*

$$\begin{array}{|c|c|c|}\hline 1&5&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&4&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&3&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&6\\\hline\end{array}.$$
$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\qquad \begin{array}{|c|}\hline 2\\\hline 3\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 2\\\hline 4\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$$

$\ker(\gamma_3)+\operatorname{im}(\chi_3) = \left\langle \Psi\overset{5}{\underset{3}{\downarrow}} z_{((3),(1^3))} \right\rangle$, *so the elements of* $\ker(\gamma_3)/\operatorname{im}(\phi_3)$ *correspond to the* $((3),(1^3))$*-tableaux*

$$\begin{array}{|c|c|c|}\hline 3&4&5\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 2&4&5\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 2&3&5\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 2&3&4\\\hline\end{array}.$$
$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 4\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 5\\\hline 6\\\hline\end{array}$$

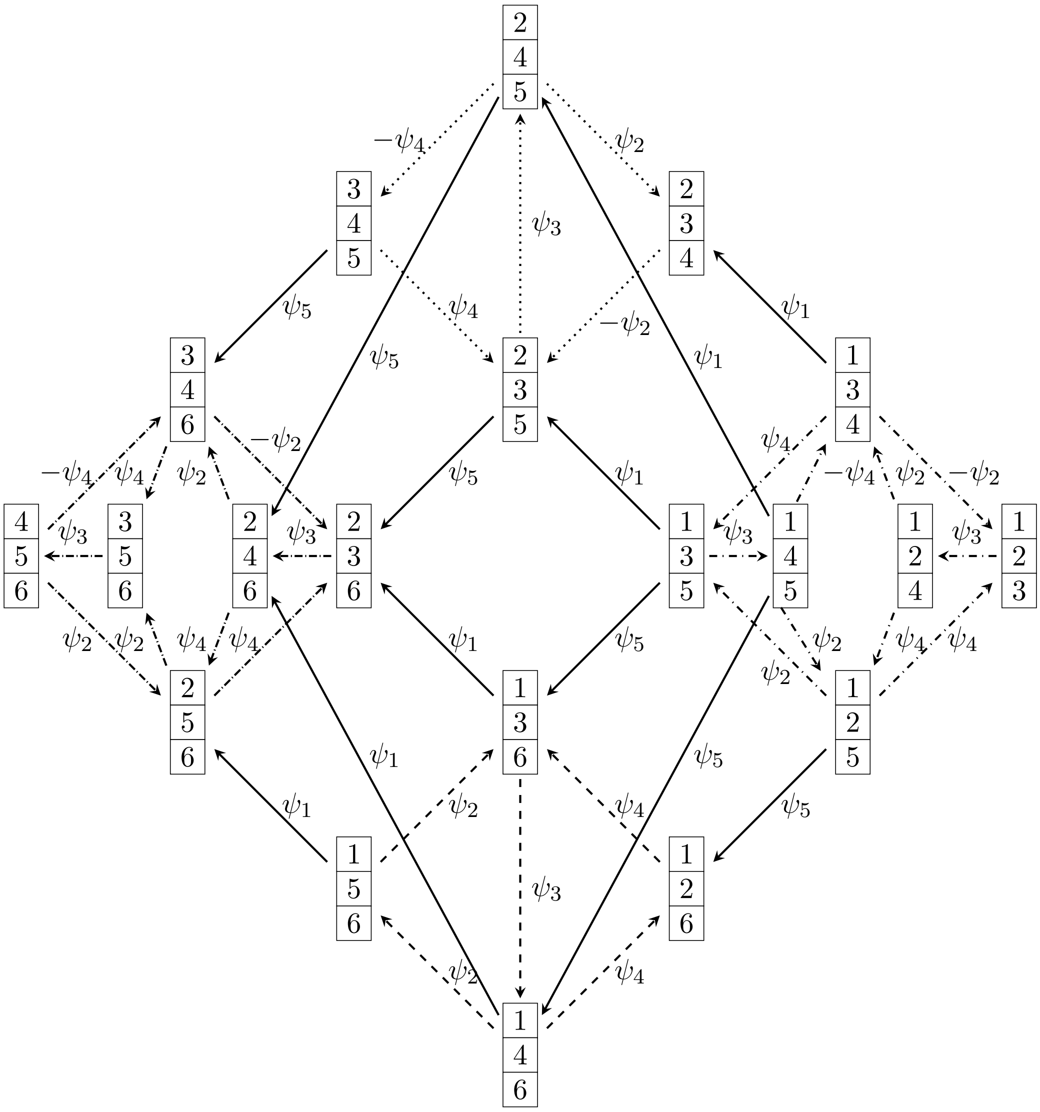
$S_{((3),(1^3))} = \langle z_{((3),(1^3))}\rangle$, *so the elements of* $\ker(\gamma_4)/\operatorname{im}(\phi_4)$ *correspond to the* $((3),(1^3))$-

tableaux

$$\begin{array}{|c|c|c|}\hline 4&5&6\\\hline\end{array},\quad \begin{array}{|c|c|c|}\hline 3&5&6\\\hline\end{array},\quad \begin{array}{|c|c|c|}\hline 3&4&6\\\hline\end{array},\quad \begin{array}{|c|c|c|}\hline 2&5&6\\\hline\end{array},\begin{array}{|c|c|c|}\hline 2&4&6\\\hline\end{array},\begin{array}{|c|c|c|}\hline 2&3&6\\\hline\end{array}.$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 4\\\hline 5\\\hline\end{array}$$

We know that $\mathrm{im}(\phi_3)$, $\mathrm{im}(\phi_4)$, $\ker(\gamma_3)/\mathrm{im}(\gamma_3)$ *and* $\ker(\gamma_4)/\mathrm{im}(\gamma_4)$ *are irreducible. Recall that a standard* $((3),(1^3))$*-tableau is completely determined by the three entries in its leg. We represent the basis elements of* $S_{((3),(1^3))}$ *by the legs of the corresponding* $((3),(1^3))$*-tableaux, together with the only non-trivial relations between these elements. Observe that for any* $v_R, v_S \in \mathrm{im}(\chi_3)$ *we can find a directed path from* R *to* S*. Similarly, for any* $v_T, v_U \in \mathrm{im}(\phi_4)$, $v_W, v_X \in \ker(\gamma_3)/\mathrm{im}(\gamma_3)$ *and* $v_Y, v_Z \in \ker(\gamma_4)/\mathrm{im}(\gamma_4)$ *we can find a directed path from* T *to* U*, from* W *to* X *and from* Y *to* Z*, respectively.*



Thus, in Cases II–IV, $S_{((n-m),(1^m))}$ has at least two composition factors for $1 \leqslant m \leqslant$

$n-1$. We need only determine if these factors, in each case, are non-isomorphic, to deduce part of the decomposition matrix for $\mathscr{R}_n^\Lambda$ comprising rows that correspond to hook bipartitions.

Chapter 8

Restriction and induction of $S_{((n-m),(1^m))}$

In this chapter we introduce specific restriction functors and their 'dual' induction functors over $\mathscr{R}_n^\Lambda$, which will assist us to obtain the labels of the composition factors of Specht modules labelled by hook bipartitions. Our general approach for determining these labels is given as follows.

We exploit Brundan and Kleshchev's i-restriction functors $e_i : \mathscr{R}_n^\Lambda \to \mathscr{R}_{n-1}^\Lambda$ and i-induction functors $f_i : \mathscr{R}_n^\Lambda \to \mathscr{R}_{n+1}^\Lambda$, given in Section 1.6, and apply them to Specht modules. Since the composition of exact functors is exact, we compose these restriction functors in a certain fashion, introducing several functors, including *arm* and *leg* functors, e_{arm} and e_{leg} respectively; we analogously introduce induction functors f_{arm} and f_{leg}. Appealing to Proposition 1.50, the $\mathscr{R}_n^\Lambda$-module M obtained by applying one of these restriction functors, say e_{arm}, to $S_{((n-m),(1^m))}$ is irreducible. Furthermore, if we then apply the analogous induction functor, f_{arm} here, we find one of the composition factors of $S_{((n-m),(1^m))}$. Since all of the other composition factors of $S_{((n-m),(1^m))}$ are killed by this induction functor, the surviving composition factor is, in fact, the socle of $f_{\mathrm{arm}}M$.

We define l to be the residue of $\kappa_2-\kappa_1$ modulo e throughout, so that $l \in \{0,\dots,e-1\}$.

8.1 Irreducible labels of one-dimensional Specht modules

We know $S_{((n),\varnothing)} = \left\{z_{((n),\varnothing)}\right\}$ and $S_{(\varnothing,(1^n))} = \left\{z_{(\varnothing,(1^n))}\right\}$ are both one-dimensional $\mathscr{R}_n^\Lambda$-modules, and hence are both irreducible. In fact, $S_{((n),\varnothing)} \cong D_{((n),\varnothing)}$. We now determine the bipartition $\mu \in \mathscr{RP}_n^2$ where $S_{(\varnothing,(1^n))} \cong D_\mu$.

For, $1 \leqslant r \leqslant n$, $S_{(\varnothing,(1^r))}$ only has one removable node, namely $(r,1,2)$ where $\operatorname{res}(r,1,2) \equiv \kappa_2+1-r \pmod e$. So, $e_{\kappa_2+1-r} : \mathscr{R}_r^\Lambda\text{-mod} \to \mathscr{R}_{r-1}^\Lambda\text{-mod}$ is the only restriction functor which acts non-trivially on $S_{(\varnothing,(1^r))}$ where $e_{\kappa_2+1-r}S_{(\varnothing,(1^r))} \cong S_{(\varnothing,(1^{r-1}))}$.

Define the sgn-*restriction functor* to be $e_{\text{sgn}} := e_{\kappa_2} \circ e_{\kappa_2-1} \circ \cdots \circ e_{\kappa_2+1-n}$, with the property that

$$e_{\text{sgn}} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_0^\Lambda\text{-mod}; \quad e_{\text{sgn}}S_{(\varnothing,(1^n))} \cong S_{(\varnothing,\varnothing)}.$$

We see that e_{sgn} is the only composition of restriction functors which acts non-trivially on $S_{(\varnothing,(1^n))}$. Define the sgn-*induction functor* to be $f_{\text{sgn}} = f_{\kappa_2+1-n} \circ f_{\kappa_2+2-n} \circ \cdots \circ f_{\kappa_2}$, where

$$f_{\text{sgn}} : \mathscr{R}_0^\Lambda\text{-mod} \to \mathscr{R}_n^\Lambda\text{-mod}.$$

The sgn-induction functor acts non-trivially on $S_{(\varnothing,\varnothing)}$; we determine the socle of $f_{\text{sgn}}S_{(\varnothing,\varnothing)}$.

Definition 8.1. *For $a \in \mathbb{N}^0$, we write*

$$\{a\} := \left\lfloor \frac{a+e-2}{e-1} \right\rfloor, \left\lfloor \frac{a+e-3}{e-1} \right\rfloor, \ldots, \left\lfloor \frac{a}{e-1} \right\rfloor,$$

that is, the weakly decreasing sequence of $e-1$ integers that sum to a and differ by at most 1.

We are now ready to give an explicit description of the e-regular bipartition μ where $S_{(\varnothing,(1^n))} \cong D_\mu$.

Lemma 8.2. *1. If $n < l$, then $S_{(\varnothing,(1^n))} \cong D_{(\varnothing,(1^n))}$.*

2. If $n \geqslant l$, then $S_{(\varnothing,(1^n))} \cong D_{((\{n-l\}),(1^l))}$.

Proof. Let $1 \leqslant r \leqslant n$ and $S_{(\varnothing,(1^n))} \cong D_\mu$ for some bipartition μ. By [BK2, Lemma 2.5],

$$\operatorname{res}^{\mathscr{R}_n^\Lambda}_{\mathscr{R}_0^\Lambda} S_{(\varnothing,(1^n))} \cong e_{\text{sgn}}S_{(\varnothing,(1^n))}.$$

For any $r > 1$, there is only one removable (κ_2+1-r)-node of $[(\varnothing,(1^r))]$, so that $\epsilon_{\kappa_2+1-r}S_{(\varnothing,(1^r))} = 1$. Thus, by Corollary 1.44,

$$e_{\text{sgn}}S_{(\varnothing,(1^n))} \cong S_{(\varnothing,(1^n))^{\triangledown e_{\text{sgn}}}} \cong S_{(\varnothing,(1^n))^{\triangledown \kappa_2+1-n \triangledown \kappa_2+2-n \ldots \triangledown \kappa_2}} \cong S_{(\varnothing,\varnothing)}.$$

Define $\uparrow^n_{\text{sgn}} (\varnothing,\varnothing) := \uparrow^1_{\kappa_2+1-n}\uparrow^1_{\kappa_2+2-n} \cdots \uparrow^1_{\kappa_2} (\varnothing,\varnothing)$. Since $S_{(\varnothing,(1^n))}$ is irreducible, then $S_{(\varnothing,(1^n))} \cong D_{\uparrow^n_{\text{sgn}}(\varnothing,\varnothing)}$ by Proposition 1.50. To calculate $\uparrow^n_{\text{sgn}} (\varnothing,\varnothing)$ we add n nodes to $[(\varnothing,\varnothing)]$ by successively adding the highest conormal node of e-residues $\kappa_2, \kappa_2-1, \ldots, \kappa_2+1-n$, respectively.

Firstly, we successively add the highest l conormal nodes of e-residues $\kappa_2, \kappa_2-1, \ldots, \kappa_2-l+1$, respectively. Since $\kappa_1 \equiv \kappa_2 - l \pmod e$, it is easy to see that $(\varnothing,(1^i))$ has (κ_2-i)-signature $+$, corresponding to node $(i+1,1,2)$, for each $i \in \{0,\ldots,l-1\}$. Hence

$$\uparrow^1_{\kappa_2-l+1}\uparrow^1_{\kappa_2-l+2} \cdots \uparrow^1_{\kappa_2} (\varnothing,\varnothing) = (\varnothing,(1^l)).$$

In particular, if $n \leqslant l$, then we are done. So suppose that $n > l$.

We now successively add the highest e conormal nodes to $(\varnothing, (1^l))$ of e-residue $\kappa_1, \kappa_1 - 1, \ldots, \kappa_1 + 1$, respectively. Notice that $((1^i), (1^l))$ has $(\kappa_1 - i)$-signature $+$, corresponding to node $(i+1, 1, 1)$, for $i \in \{0, \ldots, e-1\}$, except in the following cases:

- ◇ The κ_1-signature of $(\varnothing, (1^l))$ is $++$, corresponding to nodes $(1,1,1)$ and $(l+1,1,2)$, respectively. Thus $\uparrow_{\kappa_1} (\varnothing, (1^l)) = ((1), (1^l))$;
- ◇ The $(\kappa_1 + l + 1)$-signature of $((1^{e-1}), (1^l))$ is $++$, corresponding to $(e-l, 1, 1)$ and $(1, 2, 2)$, respectively;
- ◇ The (κ_1+1)-signature of $((1^{e-l-1}), (1^l))$ is $++-$, corresponding to $(1,2,1)$, $(e,1,1)$ and $(l,1,2)$, respectively. Moreover, if $l = 0$, then $(1,2,2)$ is also a conormal (κ_1+1)-node of $((1^{e-1}), (1^l))$.

It follows that

$$\uparrow_{\kappa_1+1}\uparrow_{\kappa_1+2} \cdots \uparrow_{\kappa_1} (\varnothing, (1^l)) = ((2, 1^{e-2}), (1^l)),$$

and so the first component of $\uparrow^n_{\mathrm{sgn}} (\varnothing, \varnothing)$ has $e-1$ rows.

We successively add the remaining nodes to the first component of $((2, 1^{e-2}), (1^l))$, down each column from left to right. There are $n - l - r + 1$ nodes in

$$[\uparrow^n_{\mathrm{sgn}} (\varnothing, \varnothing) \backslash ((1,1,2), \ldots, (l,1,2)) \cup ((1,1,1), \ldots, (r-1,1,1))],$$

for all $r \in \{1, \ldots, e-1\}$. Since there are $e-1$ rows in the first component of μ, there are $\left\lfloor \frac{n-l-r+e-1}{e-1} \right\rfloor$ nodes in the rth row of the first component of $\uparrow^n_{\mathrm{sgn}} (\varnothing, \varnothing)$. □

8.2 Leg and arm functors

We construct two particular restriction (resp. induction) functors e_{leg}, e_{arm} (resp. f_{leg}, f_{arm}) that are pivotal in determining the labels of the two distinct composition factors of $S_{((n-m),(1^m))}$ for $1 \leqslant m \leqslant n-1$. We define k to be the residue of $n - l - 1$ modulo e, so that $k \in \{1, \ldots, e-1\}$.

Let $a + b = r \leqslant n$ such that $1 \leqslant b \leqslant r - 1$. Then there are two removable nodes of any Young diagram $[((a), (1^b))]$, namely $(1, a, 1)$ and $(b, 1, 2)$. If $\mathrm{res}(1,a,1) \neq \mathrm{res}(b,1,2)$, then there are two distinct restriction functors

$$e_{\mathrm{res}(1,a,1)}, e_{\mathrm{res}(b,1,2)} : \mathscr{R}^\Lambda_r \to \mathscr{R}^\Lambda_{r-1},$$

corresponding to the two removable nodes of $[((a), (1^b))]$ of distinct e-residues. These are the only functors that act non-trivially on $S_{((a),(1^b))}$ where $e_{\mathrm{res}(1,a,1)} S_{((a),(1^b))} \cong S_{((a-1),(1^b))}$ and $e_{\mathrm{res}(b,1,2)} S_{((a),(1^b))} \cong S_{((a),(1^{b-1}))}$. Whereas, if $\mathrm{res}(1,a,1) = \mathrm{res}(b,1,2)$

then there is one (divided power) restriction functor

$$e^{(2)}_{\mathrm{res}(1,a,1)} : \mathscr{R}^\Lambda_r \to \mathscr{R}^\Lambda_{r-2},$$

corresponding to the two removable nodes of $[((a),(1^b))]$ of equal e-residue. This is the only functor that acts non-trivially on $S_{((a),(1^b))}$ where $e^{(2)}_{\mathrm{res}(1,a,1)} S_{((a),(1^b))} \cong S_{((a-1),(1^{b-1}))}$. Notice that we lie in the former case if $r \not\equiv l+2 \pmod e$.

We define

$$(e_{i_1} \circ \cdots \circ e_{i_r})^* := f_{i_r} \circ \cdots \circ f_{i_1} \quad \text{and} \quad \overline{e_{i_1} \circ \cdots \circ e_{i_r}} := e_{i_r} \circ \cdots \circ e_{i_1}.$$

For brevity, we write e_i instead of $e_{i+e\mathbb{Z}}$, for $i \in \mathbb{Z}$. We now introduce the restriction functors g_i, $\bar{g}_i$, h_i and $\bar{h}_i$, and their corresponding induction functors $\bar{g}_i^*$, g_i^* and $\bar{h}_i^*$ and h_i^*:

$$\begin{aligned} g_i &= e^{(2)}_{i-1} \circ e_{i-2} \circ \cdots \circ e_{i+2} \circ e_{i+1}, & \bar{g}_i^* &= f^{(2)}_{i-1} \circ f_{i-2} \circ \cdots \circ f_{i+2} \circ f_{i+1}, \\ \bar{g}_i &= e_{i+1} \circ e_{i+2} \circ \cdots \circ e_{i-2} \circ e^{(2)}_{i-1}, & g_i^* &= f_{i+1} \circ f_{i+2} \circ \cdots \circ f_{i-2} \circ f^{(2)}_{i-1}, \\ h_i &= e^{(2)}_{i+1} \circ e_{i+2} \circ \cdots \circ e_{i-2} \circ e_{i-1}, & \bar{h}_i^* &= f^{(2)}_{i+1} \circ f_{i+2} \circ \cdots \circ f_{i-2} \circ f_{i-1}, \\ \bar{h}_i &= e_{i-1} \circ e_{i-2} \circ \cdots \circ e_{i+2} \circ e^{(2)}_{i+1}, & h_i^* &= f_{i-1} \circ f_{i-2} \circ \cdots \circ f_{i+2} \circ f^{(2)}_{i+1}. \end{aligned}$$

One sees that each of the restriction functors g_i, $\bar{g}_i$, h_i and $\bar{h}_i$ restrict an $\mathscr{R}^\Lambda_n$-module $S_{((n-m),(1^m))}$ to an $\mathscr{R}^\Lambda_{(n-e)}$-module as follows. Firstly, suppose that $n-l$ has e-residue 0. Then g_i restricts along the leg of $[((n-m),(1^m))]$ when $i \equiv \kappa_2 - m \pmod e$ by removing $e-1$ nodes from its leg together with its hand node

$$g_{\kappa_2 - m} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-e-1}))},$$

whereas, h_i restricts along the arm of $[((n-m),(1^m))]$ when $i \equiv \kappa_2 - m \pmod e$ by removing $e-1$ nodes from its arm together with its foot node

$$h_{\kappa_2 - m} S_{((n-m),(1^m))} \cong S_{((n-m-e+1),(1^{m-1}))}.$$

Secondly, suppose that $n-l$ has e-residue 2. Then $\bar{h}_i$ restricts along the leg of $[((n-m),(1^m))]$ when $i \equiv \kappa_2 - m \pmod e$ by removing $e-1$ nodes from its leg together with its hand node

$$\bar{h}_{\kappa_2 - m} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-e+1}))},$$

whereas, $\bar{g}_i$ restricts along the arm of $[((n-m),(1^m))]$ when $i \equiv \kappa_2 - m + 2 \pmod e$ by removing $e-1$ nodes from its arm together with its foot node

$$\bar{g}_{\kappa_2 - m + 2} S_{((n-m),(1^m))} \cong S_{((n-m-e+1),(1^{m-1}))}.$$

Clearly, each of the induction functors $\bar{g}_i^*$, g_i^*, $\bar{h}_i^*$ and h_i^* induce an $\mathscr{R}_n^\Lambda$-module to an $\mathscr{R}_{(n+e)}^\Lambda$-module. However, obtaining the labels of an $\mathscr{R}_{(n+e)}^\Lambda$-module by induction under these functors is less straightforward.

We have defined g_i and $\bar{h}_i$ to restrict $S_{((n-m),(1^m))}$ by successively removing nodes from $[((n-m),(1^m))]$ that correspond to the consecutive e-residues in its arm. Thus, we notice that the restriction of $S_{((n-m),(1^m))}$ under g_i and $\bar{h}_i$ gives us S_λ, where the arm of λ is larger than its leg. Whereas, under h_i and $\bar{g}_i$ we obtain S_μ, where the leg of μ is larger than its arm. These four restriction functors, coupled with their corresponding induction functors, form the backbone of the two crucial functors required to determine the labels of the composition factors of Specht modules indexed by hook bipartitions, namely the *leg functor* and the *arm functor* of $S_{((n-m),(1^m))}$.

8.2.1 Leg functors

We require the following specific functors in order to define the leg functor and its variations. Restricting along the remaining nodes left in the leg of $[((n-m),(1^m))]$, when its leg is substantially longer than its arm, we obtain the restriction functor

$$\mathrm{remleg}_m := e_{\kappa_2} \circ e_{\kappa_2 - 1} \circ \cdots \circ e_{\kappa_2 - m + 1},$$

consisting only of i-restriction functors, where

$$\mathrm{remleg}_m : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{(n-m)}^\Lambda\text{-mod}, \quad \mathrm{remleg}_m S_{((n-m),(1^m))} \cong S_{((n-m),\varnothing)}.$$

We set $\mathrm{remleg}_0 = \mathrm{id}$. Its corresponding induction functor is $\mathrm{remleg}_m^* = f_{\kappa_2 - m+1} \circ \cdots \circ f_{\kappa_2 - 1} \circ f_{\kappa_2}$.

Further, restricting along the last $k-1$ nodes in the end of the leg of $[((n-m),(1^m))]$, we obtain the restriction functor

$$\mathrm{endleg} := e_{\kappa_2 - m + k - 1} \circ \cdots \circ e_{\kappa_2 + 2 - m} \circ e_{\kappa_2 + 1 - m},$$

consisting only of i-restriction functors, where

$$\mathrm{endleg} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{(n-k+1)}^\Lambda\text{-mod}, \quad \mathrm{endleg}\, S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-k+1}))}.$$

If $k < 2$, then we set $\mathrm{endleg} = \mathrm{id}$. Its corresponding induction functor is $\mathrm{endleg}^* = f_{\kappa_2 + 1 - m} \circ f_{\kappa_2 + 2 - m} \circ \cdots \circ f_{\kappa_2 - m + k - 1}$.

8.2.2 The main leg functor

We define the first main functor to be the composition of at most m restriction functors corresponding to the increasing e-residues of nodes in the leg of $[((n-m),(1^m))]$, from bottom to top. We call this functor the *leg restriction functor* of $S_{((n-m),(1^m))}$, written as e_{leg}. Informally, e_{leg} acts on $S_{((n-m),(1^m))}$ by successively evaluating the residue of each node up the leg of $((n-m),(1^m))$, starting at its foot node. For each node A in its leg with residue i, we either apply

- the corresponding functor e_i if the respective hand node does not have residue i,
- or the corresponding divided power $e_i^{(2)}$ if the respective hand node also has residue i.

The former functor only removes node A, whilst the latter functor removes the respective hand node too. We continue this algorithm until we have removed every node from the leg of $((n-m),(1^m))$.

We first consider a special case of the main leg functor, the *small leg functor*, denoted e_{smlleg}. For $m < k-1$, no node in the leg of $((n-m),(1^m))$ shares the same e-residue as its hand node. Thus, we define the small leg functor to be the composition of m i-restriction functors as follows

$$e_{\text{smlleg}} := e_{\kappa_2} \circ e_{\kappa_2 - 1} \circ \cdots e_{\kappa_2+1-m}.$$

For $m \geqslant k-1$, we introduce four main functors dependent on if $n \equiv l+1 \pmod e$ or not, and depending on the bounds of m. We will see by the construction of this functor that the boundaries for m are strictly distinct for the two cases $n \equiv l+1 \pmod e$ and $n \not\equiv l+1 \pmod e$, so we write these cases separately. We will observe that by applying the main arm functor to $S_{((n-m),(1^m))}$, for $n \equiv l+1 \pmod e$, we remove at most $\frac{n}{e}$ nodes from the arm of $[((n-m),(1^m))]$, and thus we define different functors when the leg of $((n-m),(1^m))$ contains at most $n - \frac{n}{e}$ nodes and when its leg contains at least $n - \frac{n}{e} + 1$ nodes. However, for $n \not\equiv l+1 \pmod e$, the main arm functor acts on $S_{((n-m),(1^m))}$ by removing at most $\frac{n}{e}+1$ nodes from the arm of $[((n-m),(1^m))]$, and hence we introduce different functors when the leg of $((n-m),(1^m))$ contains at most $n - \frac{n}{e} - 1$ nodes and when its leg contains at least $n - \frac{n}{e}$ nodes.

8.2.2.1 The main leg functor when $n \equiv l+1 \pmod e$ and $1 \leqslant m \leqslant n - \frac{n}{e}$

In this case, we define the main leg functor to have the property that

$$e_{\text{leg}} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{n-m-\lfloor (m-1)/(e-1) \rfloor}^\Lambda\text{-mod},$$

where

$$e_{\text{leg}} S_{((n-m),(1^m))} \cong S_{((n-m-\lfloor (m-1)/(e-1) \rfloor),\varnothing)}.$$

We construct this leg restriction functor of $S_{((n-m),(1^m))}$ as follows. By removing the last node in the leg of $[((n-m),(1^m))]$, we obtain the restriction functor e_{κ_2+1-m} : $\mathscr{R}_n^\Lambda$-mod $\to \mathscr{R}_{n-1}^\Lambda$-mod with the property that

$$e_{\kappa_2+1-m} S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-1}))}.$$

Suppose we successively remove nodes corresponding to the e-residues of nodes in the leg of $[((n-m),(1^{m-1}))]$, from bottom to top. Observe that every $(e-1)$th node we remove from its leg shares the same e-residue of the removable node in its arm. Thus, removing nodes successively up the leg corresponds to restricting under the functor

$$g_{\kappa_2-m+1-i} : \mathscr{R}^{\Lambda}_{(n+(1-i)e-1)}\text{-mod} \to \mathscr{R}^{\Lambda}_{(n-ie-1)}\text{-mod},$$

with the property that

$$g_{\kappa_2-m+1-i}S_{((n-m-i+1),(1^{m+(i-1)(1-e)-1})} \cong S_{((n-m-i),(1^{m+i(1-e)-1}))},$$

for $0 \leqslant i \leqslant \lfloor (m-e)/(e-1) \rfloor$. Finally, we obtain the following restriction functor by removing the remaining $m+(1-e)\lfloor(m-1)/(e-1)\rfloor - 1$ nodes in its leg

$$\text{remleg}_{m-e-(e-1)\lfloor(m-e)/(e-1)\rfloor} : \mathscr{R}^{\Lambda}_{n-1-e\lfloor(m-1)/(e-1)\rfloor}\text{-mod} \to \mathscr{R}^{\Lambda}_{n-m-\lfloor(m-1)/(e-1)\rfloor}\text{-mod},$$

where

$$\begin{aligned}&\text{remleg}_{(m-e-(e-1)\lfloor(m-e)/(e-1)\rfloor)}\, S_{\left(\left(n-m-\left\lfloor\frac{m-1}{e-1}\right\rfloor\right),\left(1^{m-1+(1-e)\left\lfloor\frac{m-1}{e-1}\right\rfloor}\right)\right)}\\ &\cong S_{\left(\left(n-m-\left\lfloor\frac{m-1}{e-1}\right\rfloor\right),\varnothing\right)}.\end{aligned}$$

Thus, by taking a composition of these functors we obtain the leg functor

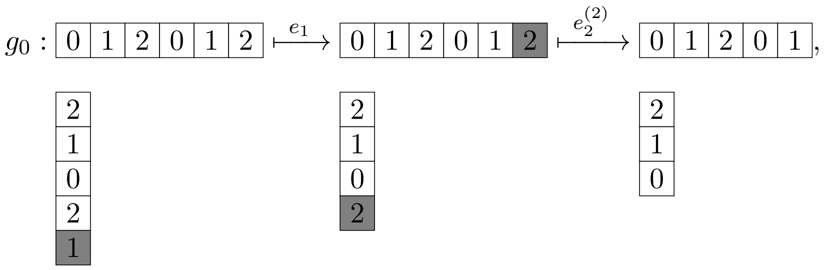
$$\begin{aligned}e_{\text{leg}} := \text{remleg}_{m-e-(e-1)\lfloor(m-e)/(e-1)\rfloor} \circ g_{\kappa_2-m+1-\lfloor(m-e)/(e-1)\rfloor} \circ \cdots\\ \cdots \circ g_{\kappa_2-m} \circ g_{\kappa_2-m+1} \circ e_{\kappa_2-m+1}.\end{aligned}$$

Example 8.3. *Let* $e=3$, $\kappa=(0,2)$. *We evaluate the restriction functor* $e_{\text{leg}} = \text{rem}_1 \circ g_2 \circ g_0 \circ e_0 = e_2 \circ e_1{}^{(2)} \circ e_0 \circ e_2{}^{(2)} \circ e_1 \circ e_0$, *where* $e_{\text{leg}} : \mathscr{R}^{\Lambda}_{12}\text{-mod} \to \mathscr{R}^{\Lambda}_{4}\text{-mod}$ *such that* $e_{\text{leg}}S_{((6),(1^6))} \cong S_{((4),\varnothing)}$. *Each restriction functor corresponds to removing at most two removable nodes from our Young diagram; these removable nodes are shaded in the following diagrams. Observing the e-residues of* $[((6),(1^6))]$, *node* $(6,1,2)$ *at the end of its leg has e-residue* 0. *So, restricting* $S_{((6),(1^6))}$ *by* e_0 *corresponds to removing node* $(6,1,2)$ *from* $[((6),(1^6))]$

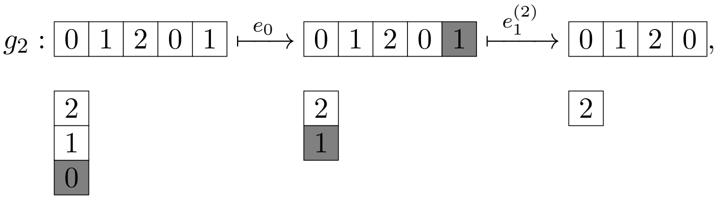
0 | 1 | 2 | 0 | 1 | 2 $\overset{e_0}{\longmapsto}$ 0 | 1 | 2 | 0 | 1 | 2,

2   2
1   1
0   0
2   2
1   1
0 (shaded)

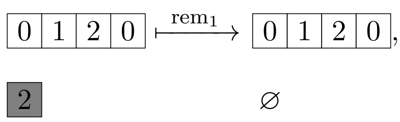
that is, $e_0 S_{((6),(1^6))} \cong S_{((6),(1^5))}$*. By the definition of the functor* g_0*, we remove* $e-1$ *nodes in the leg of* $[((6),(1^5))]$ *together with the node at the end of its arm*



that is, $g_0 S_{((6),(1^5))} \cong S_{((5),(1^3))}$*. Similarly, restricting* $S_{((5),(1^3))}$ *under* g_2 *corresponds to removing two nodes from the leg of* $[((6),(1^5))]$ *and one node from its arm*



that is, $g_2 S_{((5),(1^3))} \cong S_{((4),(1))}$*. One sees there is a single node remaining in the leg of* $[S_{((4),(1))}]$*, so restricting* $S_{((4),(1))}$ *under* rem_1 *we have*



that is, $\mathrm{rem}_1\, S_{((4),(1))} \cong S_{((4),\varnothing)}$*, as we wanted.*

Using the induction functor g_i^*, we now introduce the *leg induction functor* of $S_{((n-m),(1^m))}$, written f_{leg}, to be

$$f_{\mathrm{leg}} = f_{\kappa_2-m+1} \circ g^*_{\kappa_2-m+1} \circ g^*_{\kappa_2-m} \circ \cdots \\ \cdots \circ g^*_{\kappa_2-m+1-\lfloor (m-e)/(e-1) \rfloor} \circ \mathrm{remleg}^*_{m-e-(e-1)\lfloor (m-1)/(e-1) \rfloor}.$$

By the definition of the leg restriction functor above, one deduces that

$$\mathrm{remleg}^*_{m-e-(e-1)\lfloor (m-e)/(e-1) \rfloor} S_{\left(\left(n-m-\left\lfloor \frac{m-1}{e-1} \right\rfloor\right),\varnothing\right)} \cong S_{\left(\left(n-m-\left\lfloor \frac{m-1}{e-1} \right\rfloor\right),\left(1^{m-e-(e-1)\left\lfloor \frac{m-e}{e-1} \right\rfloor}\right)\right)}.$$

8.2.2.2 The main leg functor when $n \equiv l+1 \pmod e$ and $n - \frac{n}{e} < m \leqslant n-1$

We similarly define the leg restriction functor of $S_{((n-m),(1^m))}$ to be

$$e_{\mathrm{leg}} := \mathrm{remleg}_{em+n(1-e)-1} \circ g_{\kappa_2-n+2} \circ \cdots \circ g_{\kappa_2-m} \circ g_{\kappa_2-m+1} \circ e_{\kappa_2-m+1},$$

with the property that

$$e_{\text{leg}} : \mathscr{R}_n^{\Lambda}\text{-mod} \to \mathscr{R}_0^{\Lambda}\text{-mod}, \quad e_{\text{leg}} S_{((n-m),(1^m))} \cong S_{(\varnothing,\varnothing)}.$$

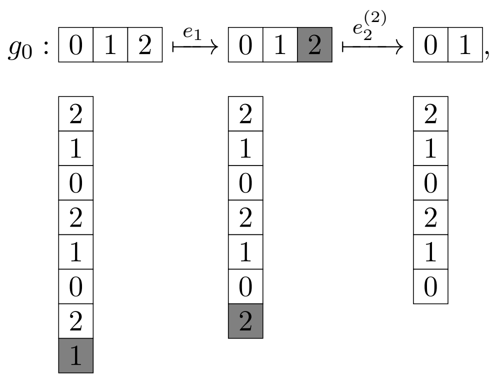
One notices there are precisely $n-m$ divided powers of two functors in this definition, and hence, as we restrict up the leg of $[((n-m),(1^m))]$, we remove each node in its arm. We induce $S_{(\varnothing,\varnothing)}$ to an $\mathscr{R}_n^{\Lambda}$-module under the leg induction functor

$$f_{\text{leg}} := f_{\kappa_2-m+1} \circ g^*_{\kappa_2-m+1} \circ g^*_{\kappa_2-m} \circ \cdots \circ g^*_{\kappa_2-n+2} \circ \text{remleg}^*_{em+n(1-e)-1}.$$

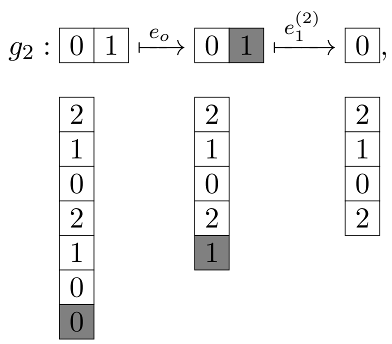
Example 8.4. *Set $e = 3$, $\kappa = (0,2)$ and $n = 12$, as in the previous example, and evaluate $e_{\text{leg}} S_{((3),(1^9))}$. We can check that*

$$e_{\text{leg}} = \text{remleg}_2 \circ g_1 \circ g_2 \circ g_0 \circ e_0 = e_2 \circ e_1 \left(e_0^{(2)} \circ e_2\right) \circ \left(e_1^{(2)} \circ e_0\right) \circ \left(e_2^{(2)} \circ e_1\right) \circ e_0.$$

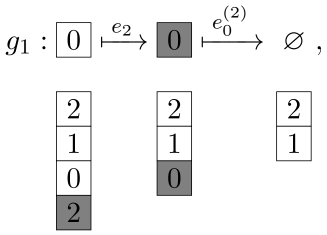
We claim that e_{leg} restricts $S_{((3),(1^9))}$ to the $\mathscr{R}_0^{\Lambda}$-module $S_{(\varnothing,\varnothing)}$. Clearly, $e_0 S_{((3),(1^9))} \cong S_{((3),(1^8))}$. Applying g_0 to $S_{((3),(1^8))}$ we have



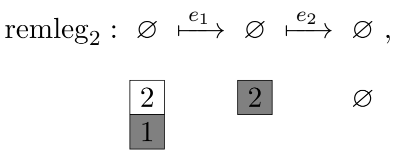
so $g_0 S_{((3),(1^8))} \cong S_{((2),(1^6))}$. Now applying g_2 we have



so $g_2 S_{((2),(1^6))} \cong S_{((1),(1^4))}$. *Now applying* g_1 *we have*



so $g_1 S_{((1),(1^4))} \cong S_{(\varnothing,(1^2))}$. *Finally, applying* remleg_2 *we have*



so $\mathrm{remleg}_2\, S_{(\varnothing,(1^2))} \cong S_{(\varnothing,\varnothing)}$, *and hence* $e_{\mathrm{leg}} S_{((3),(1^9))} \cong S_{(\varnothing,\varnothing)}$ *as we wanted.*

8.2.2.3 The main leg functor when $n \not\equiv l+1 \pmod e$ and $1 \leqslant m < n - \frac{n}{e}$

In this case, we define the main leg functor to have the property that

$$e_{\mathrm{leg}} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}^\Lambda_{n-m-\lfloor (m+e-k-1)/(e-1)\rfloor}\text{-mod},$$

where

$$e_{\mathrm{leg}} S_{((n-m),(1^m))} \cong S_{\left(\left(n-m-\left\lfloor \frac{m+e-k-1}{e-1}\right\rfloor\right),\varnothing\right)}.$$

We construct this leg restriction functor as follows.

Firstly, under endleg, $S_{((n-m),(1^m))}$ is restricted to the $\mathscr{R}^\Lambda_{(n-k+1)}$-module $S_{((n-m),(1^{m-k+1}))}$. Now removing nodes successively up the leg corresponds to restricting under the functor

$$\bar{h}_{\kappa_2-m+k-i} : \mathscr{R}^\Lambda_{(n-k+1-(i-1)e)}\text{-mod} \to \mathscr{R}^\Lambda_{(n-k+1-ie)}\text{-mod},$$

where

$$\bar{h}_{\kappa_2-m+k-i} S_{((n-m+1-i),(1^{m-k-(i-1)(e-1)+1}))} \cong S_{((n-m-i),(1^{m-k-i(e-1)+1}))},$$

for $1 \leqslant i \leqslant \lfloor (m-k)/(e-1)\rfloor$. Observe that the hand node and foot of $[((n-m-\lfloor (m-k)/(e-1)\rfloor),(1^{m-k+1-(e-1)\lfloor (m-k)/(e-1)\rfloor}))]$ share e-residue $\kappa_2-m+k+(e-1)\lfloor (m-k)/(e-1)\rfloor$, so under $e_{\kappa_2-m+k+(e-1)\lfloor (m-k)/(e-1)\rfloor}$ we restrict to the $\mathscr{R}^\Lambda_{(n-k-e\lfloor (m-k)/(e-1)\rfloor)}$-module

$$S_{\left(\left(n-m-\left\lfloor \frac{m-k}{e-1}\right\rfloor\right),\left(1^{m-k-(e-1)\left\lfloor \frac{m-k}{e-1}\right\rfloor}\right)\right)}.$$

Finally, by restricting along the remaining nodes in the leg by $\mathrm{remleg}_{(m-k+1-(e-1)\lfloor (m-k)/(e-1)\rfloor)}$

we restrict this Specht module to the $\mathscr{R}^{\Lambda}_{n-m-\lfloor (m+e-k-1)/(e-1)\rfloor}$-module

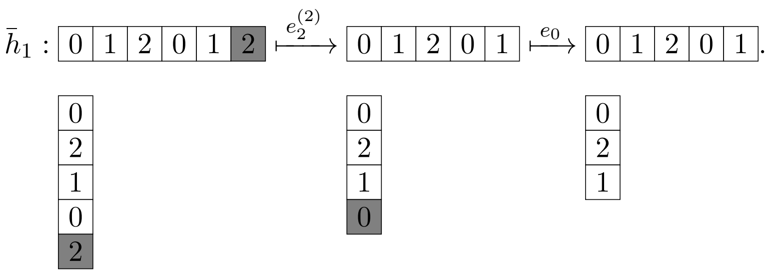
$$S_{\left(\left(n-m-\left\lfloor\frac{m+e-k-1}{e-1}\right\rfloor\right),\varnothing\right)},$$

as expected. Thus, by taking a composition of these restriction functors we define

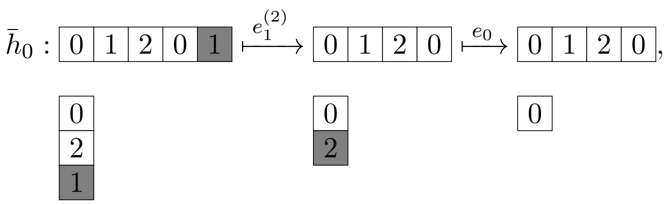
$$e_{\mathrm{leg}} := \mathrm{remleg}_{j+1-(e-1)\lfloor j/(e-1)\rfloor} \circ e_{\kappa_2-j+(e-1)\lfloor j/(e-1)\rfloor} \circ h'_{\kappa_2-j-\lfloor j/(e-1)\rfloor} \circ \cdots \cdots \circ h'_{\kappa_2-j-2} \circ h'_{\kappa_2-j-1} \circ \mathrm{endleg},$$

where $j = m - k$.

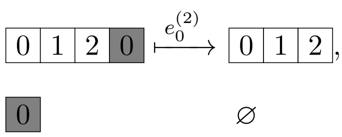
Example 8.5. *Let $e = 3$ and $\kappa = (0,0)$. We apply the restriction functor $e_{\mathrm{leg}} = \mathrm{remleg}_1 \circ e_0 \circ \bar{h}_0 \circ \bar{h}_1 \circ \mathrm{endleg}$ to $S_{((6),(1^5))}$. One can check that $e_{\mathrm{leg}} = e_0^{(2)} \circ e_2 \circ e_1^{(2)} \circ e_0 \circ e_2^{(2)}$. We claim that $e_{\mathrm{leg}} S_{((6),(1^5))} \cong S_{((3),\varnothing)}$. We first observe that $\bar{h}_1 S_{((6),(1^5))} \cong S_{((5),(1^3))}$*



Applying $\bar{h}_0$ to $S_{((5),(1^3))}$, we have



so $\bar{h}_0 S_{((5),(1^3))} \cong S_{((4),(1))}$. Finally, if we apply $e_0^{(2)}$ to $S_{((4),(1))}$, then we have the mapping



so that $e_0^{(2)} S_{((4),(1))} \cong S_{((3),\varnothing)}$, as we wanted.

8.2.2.4 The main leg functor when $n \not\equiv l+1 \pmod{e}$ and $n - \frac{n}{e} \leqslant m \leqslant n-1$

Similarly to the previous case, we define the leg restriction functor to be

$$e_{\mathrm{leg}} := \mathrm{remleg}_{e(m-n)+n-k+1} \circ \bar{h}_{\kappa_1-1} \circ \cdots \circ \bar{h}_{\kappa_2-m-2+k} \circ \bar{h}_{\kappa_2-m-1+k} \circ \mathrm{endleg},$$

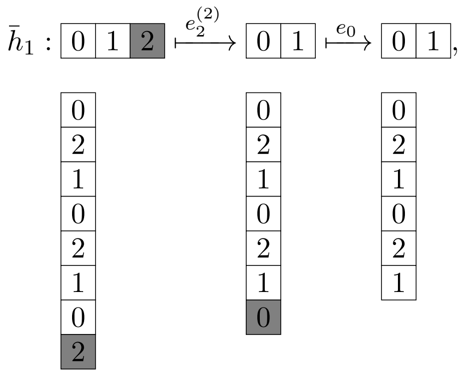
which has the property that

$$e_{\mathrm{leg}} := \mathscr{R}_n\text{-mod}^\Lambda \to \mathscr{R}_0\text{-mod}^\Lambda, \quad e_{\mathrm{leg}} S_{((n-m),(1^m))} \cong S_{(\varnothing,\varnothing)}.$$

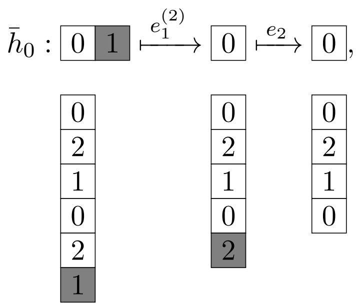
Example 8.6. *We set* $e = 3$ $\kappa = (0,0)$ *and* $n = 11$*, as in the previous example. We evaluate* $e_{\mathrm{leg}} S_{((3),(1^8))}$*. One can check that*

$$e_{\mathrm{leg}} = \mathrm{remleg}_2 \circ \bar{h}_2 \circ \bar{h}_0 \circ \bar{h}_1 \circ \mathrm{endleg} = e_0 \circ e_2 \circ \left(e_1 \circ e_o^{(2)}\right) \circ \left(e_2 \circ e_1^{(2)}\right) \circ \left(e_0 \circ e_2^{(2)}\right).$$

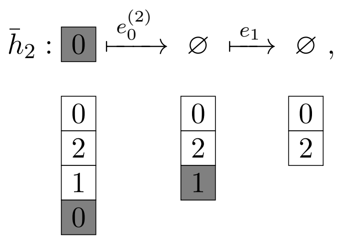
We claim that e_{leg} *restricts* $S_{((3),(1^8))}$ *to the* $\mathscr{R}_0^\Lambda$*-module* $S_{(\varnothing,\varnothing)}$*. Applying* $\bar{h}_1$ *to* $S_{((3),(1^8))}$ *we have*



so $\bar{h}_1 S_{((3),(1^8))} \cong S_{((2),(1^6))}$*. Now applying* $\bar{h}_0$ *to* $S_{((2),(1^6))}$ *we have*

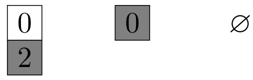


so $\bar{h}_0 S_{((2),(1^6))} \cong S_{((1),(1^4))}$*. Now applying* $\bar{h}_2$ *to* $S_{((1),(1^6))}$ *we have*



so $\bar{h}_2 S_{((1),(1^4))} \cong S_{(\varnothing,(1^2))}$. *Finally, applying* remleg_2 *to* $S_{(\varnothing,(1^2))}$ *we have*

$$\mathrm{remleg}_2 : \varnothing \xmapsto{e_2} \varnothing \xmapsto{e_0} \varnothing \;,$$



so $\mathrm{remleg}_2\, S_{(\varnothing,(1^2))} \cong S_{(\varnothing,\varnothing)}$, *and hence* $e_{\mathrm{leg}} S_{((3),(1^8))} \cong S_{(\varnothing,\varnothing)}$ *as we wanted.*

8.2.3 The first variation of the leg functor

The leg restriction functor corresponds to the increasing e-residues in the leg of $[((n-m),(1^m))]$, from bottom to top. Restricting $S_{((n-m),(1^m))}$ under this leg functor, certain nodes are also removed from the first component of $[((n-m),(1^m))]$. Supposing that we remove one less node from the first component of $[((n-m),(1^m))]$, restricting $S_{((n-m),(1^m))}$ in the same fashion as e_{leg}, brings us to the first variation of the leg restriction functor, written as $\widehat{e_{\mathrm{leg}}}$.

8.2.3.1 The first variation of the leg functor when $n \equiv l+1 \pmod e$

The first variation of the leg functor is only defined in this case for $n - \frac{n}{e} < m \leqslant n-1$. We define this functor to be

$$\widehat{e_{\mathrm{leg}}} := \mathrm{remleg}_{m-(e-1)(n-m-1)-1} \circ g_{\kappa_2-n+3} \circ \cdots \circ g_{\kappa_2-m} \circ g_{\kappa_2-m+1} \circ e_{\kappa_2-m+1},$$

where

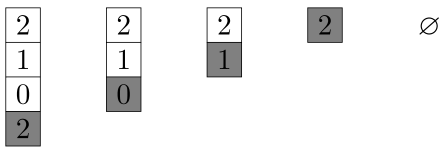
$$\widehat{e_{\mathrm{leg}}} : \mathscr{R}_n^{\Lambda}\text{-mod} \to \mathscr{R}_1^{\Lambda}\text{-mod}, \quad \widehat{e_{\mathrm{leg}}} S_{((n-m),(1^m))} \cong S_{((1),\varnothing)}.$$

Example 8.7. *As in Example 8.4, we set* $e=3$, $\kappa=(0,2)$, *and evaluate* $\widehat{e_{\mathrm{leg}}} S_{((3),(1^9))}$. *One can check that*

$$\widehat{e_{\mathrm{leg}}} = \mathrm{remleg}_4 \circ g_2 \circ g_0 \circ e_0 = e_2 \circ e_1 \circ e_0 \circ e_2 \circ \left(e_1^{(2)} \circ e_0\right) \circ \left(e_2^{(2)} \circ e_1\right) \circ e_0.$$

We claim that $\widehat{e_{\mathrm{leg}}} S_{((3),(1^9))} \cong S_{((1),\varnothing)}$. *By Example 8.4, we have* $g_2 \circ g_0 \circ e_0 S_{((3),(1^9))} \cong S_{((1),(1^4))}$. *Applying* remleg_4 *to* $S_{((1),(1^4))}$ *we have*

$$\mathrm{remleg}_4 : \boxed{0} \xmapsto{e_2} \boxed{0} \xmapsto{e_0} \boxed{0} \xmapsto{e_1} \boxed{0} \xmapsto{e_2} \boxed{0},$$



so $\mathrm{remleg}_4\, S_{((1),(1^4))} \cong S_{((1),\varnothing)}$, *as we claimed.*

8.2.3.2 The first variation of the leg functor when $n \not\equiv l+1 \pmod e$

In this case, the first variation of the leg functor is only defined for $n - \frac{n}{e} \leqslant m \leqslant n-1$. We define this functor to be

$$\widehat{e_{\mathrm{leg}}} := \mathrm{remleg}_{e(m-n+1)+n-k} \circ \bar{h}_{\kappa_1} \circ \bar{h}_{\kappa_1+1} \circ \cdots \circ \bar{h}_{\kappa_2-m-1+k} \circ \mathrm{endleg},$$

where

$$\widehat{e_{\mathrm{leg}}} : \mathscr{R}_n^{\Lambda}\text{-mod} \to \mathscr{R}_1^{\Lambda}\text{-mod}, \quad \widehat{e_{\mathrm{leg}}} S_{((n-m),(1^m))} \cong S_{((1),\varnothing)}.$$

Example 8.8. *We set $e = 3$ and $\kappa = (0,0)$ as in Example 8.6, and we evaluate $\widehat{e_{\mathrm{leg}}} S_{((3),(1^8))}$. One can check that*

$$\widehat{e_{\mathrm{leg}}} = \mathrm{remleg}_4 \circ \bar{h}_0 \circ \bar{h}_1 \circ \mathrm{endleg} = e_0 \circ e_2 \circ e_1 \circ e_0 \circ \left(e_2 \circ e_1^{(2)}\right) \circ \left(e_0 \circ e_2^{(2)}\right).$$

We claim that $\widehat{e_{\mathrm{leg}}} S_{((3),(1^8))} \cong S_{((1),\varnothing)}$. By Example 8.6, we have $\bar{h}_0 \circ \bar{h}_1 \circ \mathrm{endleg} S_{((3),(1^8))} \cong S_{((1),(1^4))}$. Applying remleg_4 to $S_{((1),(1^4))}$ we have

$$\mathrm{remleg}_4 : \boxed{0}\ \begin{array}{c}0\\2\\1\\\mathbf{0}\end{array} \xmapsto{e_0} \boxed{0}\ \begin{array}{c}0\\2\\\mathbf{1}\end{array} \xmapsto{e_1} \boxed{0}\ \begin{array}{c}0\\\mathbf{2}\end{array} \xmapsto{e_2} \boxed{0}\ \begin{array}{c}\mathbf{0}\end{array} \xmapsto{e_0} \boxed{0}\ \varnothing,$$

so $\mathrm{remleg}_4\, S_{((3),(1^8))} \cong S_{((1),\varnothing)}$ as we wanted.

8.2.4 The second variation of the leg functor

Suppose that $n \equiv l+1 \pmod e$ and $\kappa_2 \equiv \kappa_1 - 1 \pmod e$. Then the second variation of the leg restriction functor of $S_{((n-m),(1^m))}$ is formed by first removing node $(1, n-m, 1)$, then restricting in the same fashion as e_{leg} without removing node $(1,1,1)$.

8.2.4.1 The second variation of the leg functor for $\frac{n}{e} \leqslant m < n - \frac{n}{e}$

We define the second variation of the leg functor to be

$$\widetilde{e_{\mathrm{leg}}} := \mathrm{remleg}_{m-(e-1)\lfloor m/(e-1)\rfloor} \circ g_{\kappa_2-m+1-\lfloor m/(e-1)\rfloor} \circ \cdots \circ g_{\kappa_2-m-1} \circ g_{\kappa_2-m} \circ e_{\kappa_2-m},$$

where

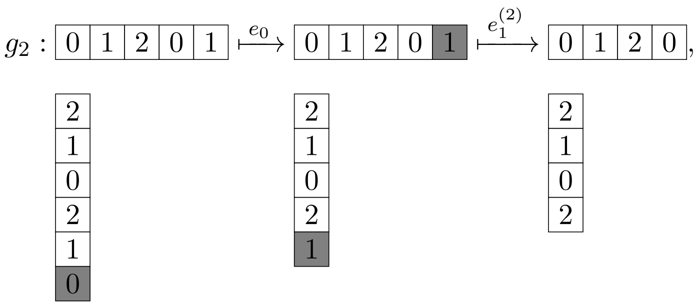
$$\widetilde{e_{\mathrm{leg}}} : \mathscr{R}_n^{\Lambda}\text{-mod} \to \mathscr{R}^{\Lambda}_{(n-m-1-\lfloor m/(e-1)\rfloor)}\text{-mod},$$
$$e_{\mathrm{leg}} \mapsto S_{((n-m),(1^m))} \cong S_{((n-m-1-\lfloor m/(e-1)\rfloor),\varnothing)}.$$

Example 8.9. *We set $e = 3$, $\kappa = (0,2)$ as in Example 8.3, and we evaluate $\widetilde{e_{\mathrm{leg}}} S_{((6),(1^6))}$.*

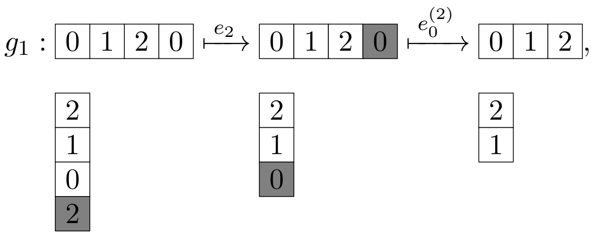
One can check that

$$\widetilde{e_{\text{leg}}} = \text{remleg}_0 \circ g_0 \circ g_1 \circ g_2 \circ e_2 = \left(e_2^{(2)} \circ e_1\right) \circ \left(e_0^{(2)} \circ e_2\right) \circ \left(e_1^{(2)} \circ e_0\right) \circ e_2,$$

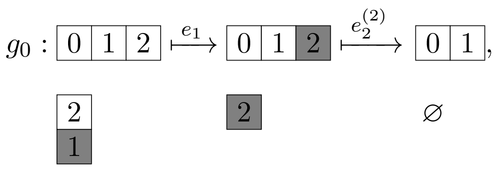
and claim that $\widetilde{e_{\text{leg}}} S_{((6),(1^6))} \cong S_{((2),\varnothing)}$. *Clearly, by removing the hand node of* $((6),(1^6))$ *we have* $e_2 S_{((6),(1^6))} \cong S_{((5),(1^6))}$. *Applying* g_2 *to* $S_{((5),(1^6))}$ *we have*



so $g_2 S_{((5),(1^6))} \cong S_{((4),(1^4))}$. *Now applying* g_1 *we have*



so $g_1 S_{((4),(1^4))} \cong S_{((3),(1^2))}$. *Now applying* g_0 *we have*



so $g_0 S_{((3),(1^2))} \cong S_{((2),\varnothing)}$ *as we wanted.*

8.2.4.2 The second variation of the leg functor for $n - \frac{n}{e} \leqslant m \leqslant n-2$

We define the functor $\widetilde{e_{\text{leg}}}$ by

$$\widetilde{e_{\text{leg}}} := \text{remleg}_{(n-2)(1-e)+em} \circ g_{\kappa_2 - n+3} \circ \cdots \circ g_{\kappa_2 - m-1} \circ g_{\kappa_2 - m} \circ e_{\kappa_2 - m},$$

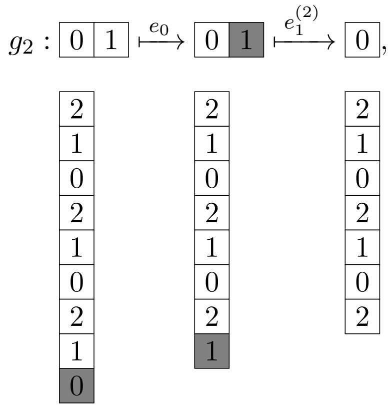
where

$$\widetilde{e_{\text{leg}}} : \mathscr{R}_n^{\Lambda}\text{-mod} \to \mathscr{R}_1^{\Lambda}\text{-mod}, \quad \widetilde{e_{\text{leg}}} S_{((n-m),(1^m))} \cong S_{((1),\varnothing)}.$$

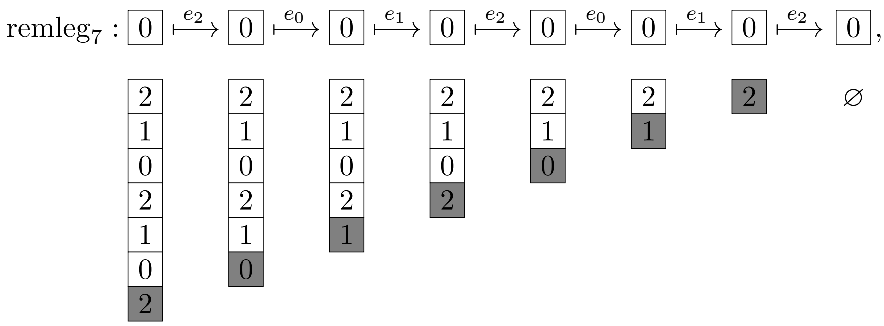
Example 8.10. *We set* $e = 3$, $\kappa = (0,2)$ *as in Example 8.4, and evaluate* $\widetilde{e_{\text{leg}}} S_{((3),(1^9))}$. *One checks that*

$$\widetilde{e_{\text{leg}}} = \text{remleg}_1 \circ g_2 \circ e_2 = e_2 \circ e_1 \circ e_0 \circ e_2 \circ e_1 \circ e_0 \circ e_2 \circ \left(e_1^{(2)} \circ e_0\right) \circ e_2,$$

and claim that $\widetilde{e_{\mathrm{leg}}} S_{((3),(1^9))} \cong S_{((1),\varnothing)}$. *By removing the hand node of* $((3),(1^9))$ *we have* $e_2 S_{((3),(1^9))} \cong S_{((2),(1^9))}$. *Now applying* g_2 *we have*



so $g_2 S_{((2),(1^9))} \cong S_{((1),(1^7))}$. *Finally, applying* remleg_7 *we have*



so $\mathrm{remleg}_7\, S_{((1),(1^7))} \cong S_{((1),\varnothing)}$ *as we wanted.*

8.2.5 Arm functors

The arm functors of a Specht module labelled by a hook-bipartition $((n-m),(1^m))$ are a composition of functors, where each functor corresponds to a node in the arm of $((n-m),(1^m))$. We first introduce specific functors necessary to define the main arm functor and its variations.

When the arm of a hook bipartition $((n-m),(1^m))$ is substantially larger than its leg, we define the restriction functor

$$\mathrm{remarm}_m = e_{\kappa_1} \circ e_{\kappa_1+1} \circ \cdots \circ e_{\kappa_1+m-2} \circ e_{\kappa_1+m-1},$$

where

$$\mathrm{remarm}_m : \mathscr{R}_n\text{-mod} \to \mathscr{R}_{(n-m)}\text{-mod}, \quad \mathrm{remarm}_m\, S_{((m),(1^{n-m}))} \cong S_{(\varnothing,(1^{n-m}))}.$$

We set $\mathrm{remarm}_0 = \mathrm{id}$.

We define its corresponding induction functor to be $\mathrm{remarm}_m^* = f_{\kappa_1+m-1} \circ f_{\kappa_1+m-2} \circ$

$\cdots \circ f_{\kappa_1+1} \circ f_{\kappa_1}$.

By restricting along the last $k-1$ nodes in the end of the arm of $[((n-m),(1^m))]$, we define the restriction functor

$$\text{endarm} = e_{\kappa_2-m+2} \circ \cdots \circ e_{\kappa_2-m+k-1} \circ e_{\kappa_2-m+k},$$

where

$$\text{endarm} : \mathscr{R}_n\text{-mod} \to \mathscr{R}_{(n-k+1)}\text{-mod}, \quad \text{endarm}\, S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-k+1}))}.$$

If $k < 2$, then we set endarm $=$ id.

We define its corresponding induction functor to be $\text{endarm}^* = f_{\kappa_2-m+k} \circ f_{\kappa_2-m+k-1} \circ \cdots \circ f_{\kappa_2-m+2}$.

8.2.6 The main arm functor

We define the second main functor to be the composition of at most $n-m$ restriction functors corresponding to the decreasing e-residues in the arm of $[((n-m),(1^m))]$, from right to left. We call this functor the *arm restriction functor* of $S_{((n-m),(1^m))}$, written as e_{arm}. Informally, we can think of e_{arm} as acting as the of 'dual' functor to e_{leg} on $S_{((n-m),(1^m))}$. That is, e_{arm} acts by successively evaluating the residue of each node along the arm of $((n-m),(1^m))$ from its hand node. For each node A in its arm with residue i, we either apply

- ◇ the corresponding functor e_i if the respective foot node does not have residue i,
- ◇ or the corresponding divided power $e_i^{(2)}$ if the respective foot node also has residue i.

The former functor only removes node A, whilst the latter functor removes the respective foot node too. We continue this algorithm until we have removed every node from the arm of $((n-m),(1^m))$.

Similar to the small leg functor, we introduce the *small arm functor*, a special case of the main arm functor. For $m > n-k+1$, no node in the arm of $((n-m),(1^m))$ shares the same e-residue as its foot node. So, we define the small leg functor, denoted by e_{smlarm}, to be the composition of $n-m$ i-restriction functors as follows

$$e_{\text{smlarm}} := e_{\kappa_1} \circ e_{\kappa_1+1} \circ \cdots \circ e_{\kappa_1+n-m-1}.$$

Now suppose that $m \leqslant n-k+1$. Then similarly to the main leg functor, we introduce four main arm functors dependent on if $n \equiv l+1 \pmod{e}$ or not, as well as depending on the bounds of m.

8.2.6.1 The main arm functor when $n \equiv l+1 \pmod e$ and $1 \leqslant m < \frac{n}{e}$

We define the main arm functor in this case to have the property that

$$e_{\text{arm}} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_0^\Lambda\text{-mod}, \quad e_{\text{arm}} S_{((n-m),(1^m))} \cong S_{(\varnothing,\varnothing)}$$

where this arm restriction functor of $S_{((n-m),(1^m))}$ is constructed as follows. By removing the last node in the arm of $[((n-m),(1^m))]$, we obtain the restriction functor

$$e_{\kappa_2 - m} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{(n-1)}^\Lambda\text{-mod}, \quad e_{\kappa_2-m} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^m))}.$$

Suppose we successively remove nodes corresponding to the e-residues of nodes in the arm of $[((n-m-1),(1^m))]$, from right to left. Observe that every $(e-1)$th node we remove from its arm shares the same e-residue of the removable node in its leg. Thus, removing nodes successively along the arm corresponds to restricting under the functor

$$h_{\kappa_2-m-1+i} : \mathscr{R}_{(n+(1-i)e-1)}^\Lambda\text{-mod} \to \mathscr{R}_{(n-ie-1)}^\Lambda\text{-mod}$$

where

$$h_{\kappa_2-m-1+i} S_{((n-m+(i-1)(1-e)-1),(1^{m-i+1}))} \cong S_{((n-m+i(1-e)-1),(1^{m-i}))}$$

for all $1 \leqslant i \leqslant m$. Finally, by removing the remaining $n-em-1$ nodes in its arm we obtain the restriction functor

$$\text{remarm}_{n-em-1}\, \mathscr{R}_{(n-em-1)}^\Lambda\text{-mod} \to \mathscr{R}_0^\Lambda\text{-mod}, \quad \text{remarm}_{n-em-1}\, S_{((n-em-1),\varnothing)} \cong S_{(\varnothing,\varnothing)}.$$

Thus, by taking a composition of these functors we obtain the arm functor

$$e_{\text{arm}} := \text{remarm}_{n-em-1} \circ h_{\kappa_2-1} \circ \cdots \circ h_{\kappa_2-m+1} \circ h_{\kappa_2-m} \circ e_{\kappa_2-m}.$$

8.2.6.2 The main arm functor when $n \equiv l+1 \pmod e$ and $\frac{n}{e} \leqslant m \leqslant n-1$

In this case, we construct the arm restriction functor of $S_{((n-m),(1^m))}$ to have the property that

$$e_{\text{arm}} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_{m-\lfloor (n-m-1)/(e-1)\rfloor}^\Lambda\text{-mod},$$

where

$$e_{\text{arm}} S_{((n-m),(1^m))} \cong S_{\left(\varnothing,\left(1^{m-\left\lfloor\frac{n-m-1}{e-1}\right\rfloor}\right)\right)}.$$

Firstly, by removing the last node in the arm of $[((n-m),(1^m))]$, we have

$$e_{\kappa_2-m} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^m))}.$$

Again, removing nodes successively along the arm corresponds to restricting under the functor h_{κ_2-m-i}, which restricts $S_{((n-m+(i-1(1-e)-1),(1^{m-i+1}))}$ to the $\mathscr{R}^{\Lambda}_{(n-ie-1)}$-module $S_{((n-m+i(1-e)-1),(1^{m-1}))}$ for $1 \leqslant i \leqslant \lfloor (n-m-1)/(e-1) \rfloor$. Finally, if we let $l = n-m-1$, then by removing the remaining $l + (1-e)\lfloor l/(e-1) \rfloor$ nodes in its arm we obtain the restriction functor

$$\mathrm{remarm}_{l-(e-1)\lfloor l/(e-1)\rfloor} : \mathscr{R}^{\Lambda}_{(n-1-e\lfloor l/(e-1)\rfloor)}\text{-mod} \to \mathscr{R}^{\Lambda}_{(m-\lfloor l/(e-1)\rfloor)}\text{-mod}$$

where

$$\mathrm{remarm}_{l-(e-1)\lfloor l/(e-1)\rfloor} S_{\left((l+(1-e)\lfloor l/(e-1)\rfloor),\left(1^{m-\lfloor l(e-1)\rfloor}\right)\right)} \cong S_{\left(\varnothing,\left(1^{m-\lfloor l/(e-1)\rfloor}\right)\right)}.$$

Thus, by taking a composition of these functors we obtain the arm functor

$$e_{\mathrm{arm}} := \mathrm{remarm}_{l-(e-1)\lfloor l/(e-1)\rfloor} \circ h_{\kappa_2-m-1+\lfloor l/(e-1)\rfloor} \circ \cdots \circ h_{\kappa_2-m+1} \circ h_{\kappa_2-m} \circ e_{\kappa_2-m}.$$

8.2.6.3 The main arm functor when $n \not\equiv l+1 \pmod e$ and $1 \leqslant m \leqslant \frac{n}{e}$

For this case, we define the main arm functor by

$$e_{\mathrm{arm}} := \mathrm{remarm}_{(n-k+1-em)} \circ \bar{g}_{\kappa_2+1} \circ \cdots \circ \bar{g}_{\kappa_2-m+3} \circ \bar{g}_{\kappa_2-m+2} \circ \mathrm{endarm},$$

with the property that

$$e_{\mathrm{arm}} : \mathscr{R}^{\Lambda}_n\text{-mod} \to \mathscr{R}^{\Lambda}_0\text{-mod}, \quad e_{\mathrm{arm}} S_{((n-m),(1^m))} \cong S_{(\varnothing,\varnothing)}.$$

We leave the details of the construction of this functor to the reader.

8.2.6.4 The main arm functor when $n \not\equiv l+1 \pmod e$ and $\frac{n}{e} < m \leqslant n-1$

In the fourth case, the main arm functor is defined to be

$$\begin{aligned} e_{\mathrm{arm}} := \mathrm{remarm}_{j+2-(e-1)\lfloor j/(e-1)\rfloor} \circ e_{\kappa_1+j+1-(e-1)\lfloor j/(e-1)\rfloor} \circ \bar{g}_{\kappa_2-m+1+\lfloor j/(e-1)\rfloor} \circ \cdots \\ \cdots \circ \bar{g}_{\kappa_2-m+3} \circ \bar{g}_{\kappa_2-m+2} \circ \mathrm{endarm}, \end{aligned}$$

where $j = n-m-k-1$, satisfying

$$e_{\mathrm{arm}} : \mathscr{R}^{\Lambda}_n\text{-mod} \to \mathscr{R}^{\Lambda}_{(m-1-\lfloor j/(e-1)\rfloor)}\text{-mod}, \quad e_{\mathrm{arm}} S_{((n-m),(1^m))} \cong S_{(\varnothing,(1^{m-1-\lfloor j/(e-1)\rfloor}))}.$$

Again, we leave the reader to understand its construction.

8.2.7 The first variation of the arm functor

We can similarly think of the first variation of the arm functor e_{arm} as the 'dual' functor of the first variation of the leg functor in the following sense.

The arm restriction functor corresponds to the decreasing e-residues along the arm of $[((n-m),(1^m))]$ from right to left. Restricting $S_{((n-m),(1^m))}$ under this arm functor, certain nodes are also removed from the second component of $[((n-m),(1^m))]$. Supposing that we remove one less node from the second component of $[((n-m),(1^m))]$, restricting $S_{((n-m),(1^m))}$ in the same fashion as e_{arm}, brings us to the first variation of the arm restriction functor, written as $\widehat{e_{\text{arm}}}$.

8.2.7.1 The first variation of the arm functor when $n \equiv l+1 \pmod e$

The following variation of the arm functor is only defined for $1 \leqslant m < \frac{n}{e}$

$$\widehat{e_{\text{arm}}} := \text{remarm}_{n+e(1-m)-2} \circ h_{\kappa_2-2} \circ \cdots \circ h_{\kappa_2-m+1} \circ h_{\kappa_2-m} \circ e_{\kappa_2-m},$$

where

$$\widehat{e_{\text{arm}}} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_1^\Lambda\text{-mod}, \quad \widehat{e_{\text{arm}}} S_{((n-m),(1^m))} \cong S_{(\varnothing,(1))}.$$

8.2.7.2 The first variation of the arm functor when $n \not\equiv l+1 \pmod e$

In this case, for $1 \leqslant m \leqslant \frac{n}{e}$, we define the first variation of the arm functor to be

$$\widehat{e_{\text{arm}}} := \text{remarm}_{n-k+e(1-m)} \circ \bar{g}_{\kappa_2} \circ \cdots \circ \bar{g}_{\kappa_2-m+3} \circ \bar{g}_{\kappa_2-m+2} \circ \text{endarm},$$

where

$$\widehat{e_{\text{arm}}} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_1^\Lambda\text{-mod}, \quad \widehat{e_{\text{arm}}} S_{((n-m),(1^m))} \cong S_{(\varnothing,(1))}.$$

8.2.8 The second variation of the arm functor

Let $n \equiv l+1 \pmod e$ and $\kappa_2 \equiv \kappa_1 - 1 \pmod e$. Again, we view this second variation of the arm functor $\widetilde{e_{\text{arm}}}$ as the 'dual' to the second variation of the leg functor $\widetilde{e_{\text{leg}}}$ in the following sense.

We form the second variation of the arm restriction functor of $S_{((n-m),(1^m))}$ by first removing its foot node $(m,1,2)$, and then restricting in the same fashion as e_{arm}, without removing node $(1,1,2)$.

8.2.8.1 The second variation of the arm functor when $2 \leqslant m \leqslant \frac{n}{e}$

We define the second variation of the arm functor in this case to be

$$\widetilde{e_{\text{arm}}} := \text{remarm}_{n+e(2-m)-2} \circ h_{\kappa_2-2} \circ \cdots \circ h_{\kappa_2+2-m} \circ h_{\kappa_2+1-m} \circ e_{\kappa_2+1-m},$$

where

$$\widetilde{e_{\text{arm}}} : \mathscr{R}_n^\Lambda\text{-mod} \to \mathscr{R}_1^\Lambda\text{-mod}, \quad \widetilde{e_{\text{arm}}} S_{((n-m),(1^m))} \cong S_{((\varnothing,(1))}.$$

8.2.8.2 The second variation of the arm functor when $\frac{n}{e} < m \leqslant n - \frac{n}{e}$

In this case, the second variation of the arm functor is defined to be

$$\begin{aligned}\widetilde{e_{\text{arm}}} := \text{remarm}_{n-m-(e-1)\lfloor (n-m)(e-1)\rfloor} \circ h_{\kappa_2 - m + \lfloor (n-m)(e-1)\rfloor} \circ \cdots \\ \cdots \circ h_{\kappa_2+2-m} \circ h_{\kappa_2+1-m} \circ e_{\kappa_2+1-m},\end{aligned}$$

where

$$\begin{aligned}\widetilde{e_{\text{arm}}} : \mathscr{R}^{\Lambda}_{n}\text{-mod} &\to \mathscr{R}^{\Lambda}_{m-1-\lfloor (n-m)/(e-1)\rfloor}\text{-mod} \\ e_{\text{arm}} &\mapsto S_{((n-m),(1^m))} \cong S_{(\varnothing,(1^{m-1-\lfloor (n-m)/(e-1)\rfloor}))}.\end{aligned}$$

Chapter 9

Labelling the composition factors of $S_{((n-m),(1^m))}$

We draw on the arm and leg functors defined in Section 8.2 to find all of the irreducible labels $\lambda \in \mathscr{RP}_n^2$ of $\mathscr{R}_n^\Lambda$-modules that factor Specht modules labelled by hook bipartitions. Recall that l is the residue of $\kappa_2 - \kappa_1$ modulo e.

9.1 Labelling the composition factors of $S_{((n-m),(1^m))}$ for $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$

Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ throughout this section.

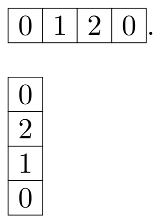
For $n \not\equiv l+1 \pmod{e}$, we recall from Theorem 7.8 that $S_{((n-m),(1^m))}$ is an irreducible $\mathscr{R}_n^\Lambda$-module, that is, $S_{((n-m),(1^m))} \cong D_\lambda$ for some bipartition $\lambda \in \mathscr{RP}_n^2$.

Definition 9.1. *Let $n \not\equiv l+1 \pmod{e}$. For $1 \leqslant m \leqslant n-1$, we define*

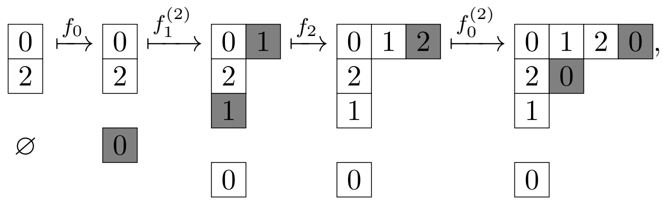
$$\mu_{n,m} = \begin{cases} ((n-m),(1^m)) & \text{if } 1 \leqslant m < l+1 \leqslant e-1, \\ ((n-m,\{m-l-1\}),(1^{l+1})) & \text{if } l+1 \leqslant m < n - \frac{n}{e}, \\ ((\{m-l\},n-m-1),(1^{l+1})) & \text{if } n - \frac{n}{e} \leqslant m \leqslant n-1. \end{cases}$$

We claim that $\lambda = \mu_{n,m}$, for $1 \leqslant m \leqslant n-1$. Now observe that the irreducible label λ can be obtained either by applying the restriction functor e_{leg} from Sections 8.2.2.3 and 8.2.2.4, or the restriction functor e_{arm} from Sections 8.2.6.3 and 8.2.6.4, to $S_{((n-m),(1^m))}$, together with their respective induction functors.

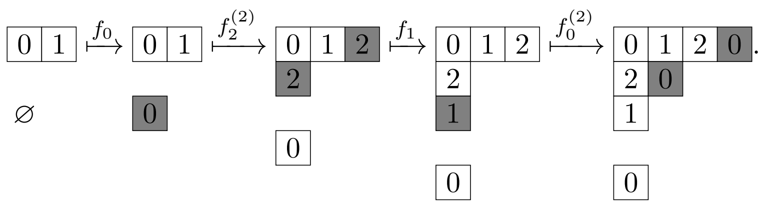
Example 9.2. *Let $e = 3$, $\kappa = (0,0)$. Observe that $S_{((4),(1^4))}$ has the following 3-residues*



We have $e_{\text{arm}} S_{((4),(1^4))} \cong S_{(\varnothing,(1^2))} \cong D_{((1^2),\varnothing)}$, by Lemma 8.2, where $e_{\text{arm}} = e_0 \circ e_1^{(2)} \circ e_2 \circ e_0^{(2)}$ and $e_{\text{leg}} S_{((4),(1^4))} \cong S_{((2),\varnothing)} \cong D_{((2),\varnothing)}$ where $e_{\text{leg}} = e_0 \circ e_2^{(2)} \circ e_1 \circ e_0^{(2)}$. Notice that as the arm and leg are of the same length, the arm and leg functors of $S_{((4),(1^4))}$ are of the same length, hence neither are simpler to apply to $S_{((4),(1^4))}$. Firstly, applying f_{arm} to $D_{((1^2),\varnothing)}$, we have



so that $S_{((4),(1^4))} \cong D_{((4,2,1),\varnothing)} \cong D_{\mu_{8,4}}$. Now, applying f_{leg} to $D_{((2),\varnothing)}$, we check that we obtain the same label, $\mu_{8,4}$



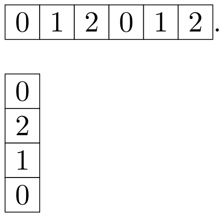
For $n \equiv l+1 \pmod e$ and $1 \leqslant m \leqslant n-1$, we recall from Theorem 7.11 that $S_{((n-m),(1^m))}$ has two composition factors, namely $\operatorname{im}(\gamma_{m-1})$ and $\operatorname{im}(\gamma_m)$. Thus, $\operatorname{im}(\gamma_{m-1}) \cong D_\lambda$ and $\operatorname{im}(\gamma_m) \cong D_\mu$ for some bipartitions $\lambda, \mu \in \mathscr{RP}_n^2$.

Definition 9.3. *Let $n \equiv l+1 \pmod e$. For $0 \leqslant m \leqslant n-1$, we define*

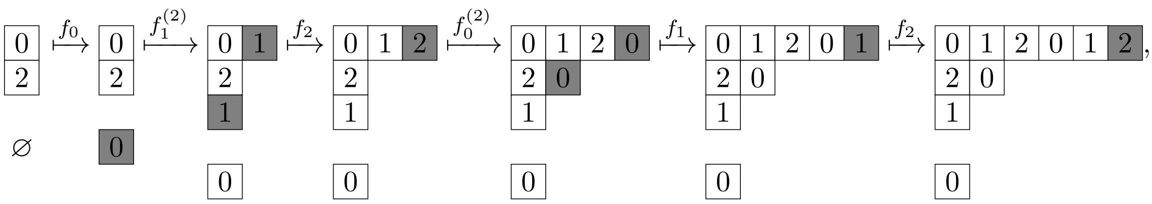
$$\mu_{n,m} = \begin{cases} ((n-m),(1^m)) & \text{if } 0 \leqslant m < l+1, \\ \left((n-m,\{m-l-1\}),(1^{l+1})\right) & \text{if } l+1 \leqslant m < n - \frac{n}{e}, \\ \left((\{m-l+1\},n-m-2),(1^{l+1})\right) & \text{if } n-\frac{n}{e} \leqslant m \leqslant n-2, \\ \left((\{n-l\}),(1^l)\right) & \text{if } m = n-1. \end{cases}$$

Notice that $\mu_{n,m-1}$ and $\mu_{n,m}$ are distinct. We claim that the two labels λ, μ of the composition factors of $S_{((n-m),(1^m))}$ are, in fact, $\mu_{n,m-1}$ and $\mu_{n,m}$, respectively, and hence that these factors are non-isomorphic. We also claim that these labels are obtained by independently applying the arm functor e_{arm} from Sections 8.2.6.1 and 8.2.6.2, and the leg functor e_{leg} from Sections 8.2.6.1 and 8.2.6.2, to $S_{((n-m),(1^m))}$, together with their respective induction functors.

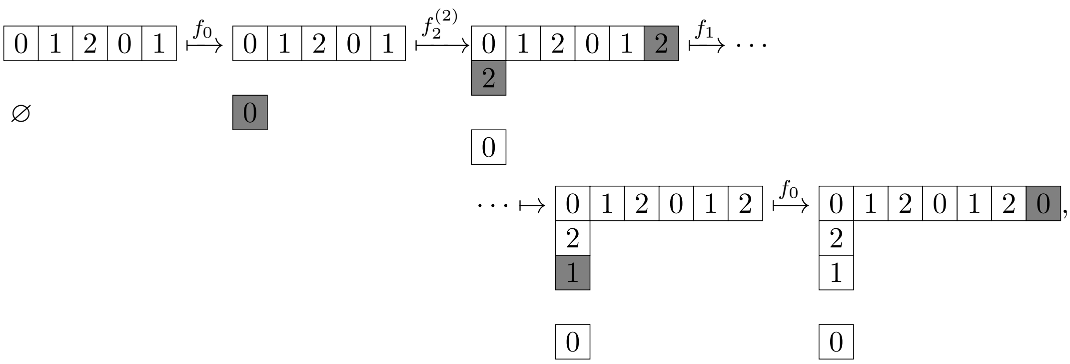
Example 9.4. *Let $e = 3$, $\kappa = (0, 0)$. Observe that $S_{((6),(1^4))}$ has the following 3-residues*



We have $e_{\text{arm}} S_{((6),(1^4))} \cong S_{(\varnothing,(1^2))} \cong S_{((1^2),\varnothing)}$ where $e_{\text{arm}} = e_0 \circ e_1^{(2)} \circ e_2 \circ e_0^{(2)} \circ e_1 \circ e_2$ and $e_{\text{leg}} S_{((6),(1^4))} \cong S_{((5),\varnothing)} \cong D_{((5),\varnothing)}$ where $e_{\text{leg}} = e_0 \circ e_2^{(2)} \circ e_1 \circ e_0$. Applying f_{arm} to $D_{((1^2),\varnothing)}$, we have



so $D_{((6,2,1),(1))} = D_{\mu_{10,4}}$ is a composition factor of $S_{((6),(1^4))}$. Now, applying f_{leg} to $S_{((6),(1^4))}$, we have



so $D_{((7,1^2),(1))} = D_{\mu_{10,3}}$ is the second composition factor of $S_{((6),(1^4))}$.

We require the following combinatorial result in order to confirm our claims.

Lemma 9.5. *1. If $n \equiv l \pmod{e}$, then*

$$\mu_{n,m} \uparrow_{\kappa_2 - m} = \mu_{n+1,m}. \tag{9.1.1}$$

2. If $n \not\equiv l \pmod{e}$, then

$$\mu_{n,m} \uparrow_{\kappa_2 - m} = \mu_{n+1,m+1}. \tag{9.1.2}$$

Proof. (i) Let $1 \leqslant m < l + 1 \leqslant e - 1$. Observe that $[((n - m), (1^m))]$ has addable $(\kappa_2 - m)$-node $(m + 1, 1, 2)$, as well as $(1, n - m + 1, 1)$ if $n \equiv l \pmod{e}$, and has removable $(\kappa_2 - m)$-node $(1, n-m, 1)$ if $n \equiv l+1 \pmod e$. We note that addable nodes $(2,1,1)$ and $(1,2,2)$ cannot have residue $\kappa_2 - m$ as $m < l+1$. So, if $n \equiv l \pmod e$ then $((n-m),(1^m))$ has $(\kappa_2 - m)$-signature $++$, corresponding to conormal nodes $(1, n-m+1, 1)$ and $(m+1,1,2)$. Adding the higher of these conormal nodes, we have

$$\mu_{n,m} \uparrow_{\kappa_2 - m} = ((n-m),(1^m)) \uparrow_{\kappa_2 - m} = ((n-m+1),(1^m)) = \mu_{n+1,m}.$$

However, if $n \equiv l+1 \pmod e$ then $((n-m),(1^m))$ has $(\kappa_2 - m)$-signature $-+$, and if $n - \kappa_2 + \kappa_1 \not\equiv 0, 1 \pmod e$ then $((n-m),(1^m))$ has $(\kappa_2 - m)$-signature $+$. The conormal node in each sequence is $(m+1,1,2)$, whereby adding this node gives

$$\mu_{n,m} \uparrow_{\kappa_2 - m} = ((n-m),(1^m)) \uparrow_{\kappa_2 - m} = ((n-m),(1^{m+1})) = \mu_{n+1,m+1}.$$

(ii) Let $l+1 \leqslant m < n - \lfloor \frac{n}{e} \rfloor$. Observe that $((n-m, \{m-l-1\}),(1^{l+1}))$ has the following addable/removable $(\kappa_2 - m)$-nodes

- ◇ addable node $(1, n-m+1, 1)$ if $n \equiv l \pmod e$,
- ◇ removable node $(1, n-m, 1)$ if $n \equiv l+1 \pmod e$,
- ◇ addable node at the end of the $\lfloor (m+e-l-2)/(e-1) \rfloor$th column in the first component,
- ◇ addable node $(e+1,1,1)$ and removable node $(l+1,1,2)$ if $m \equiv l \pmod e$,
- ◇ addable node $(1,2,2)$ if $m \equiv -1 \pmod e$,
- ◇ addable node $(l+2,1,2)$ if $m \equiv l+1 \pmod e$.

Suppose that $n \equiv l \pmod e$. Then $((n-m, \{m-l-1\}),(1^{l+1}))$ has $(\kappa_2 - m)$-signature

- ◇ $+++-$ if $m \equiv l \pmod e$,
- ◇ $++++$ if $m \equiv -1 \pmod e$ and $l = e-2$,
- ◇ $+++$ if $m \equiv -1 \pmod e$ and $l \neq e-2$ or $m \not\equiv -1 \pmod e$ and $m \equiv l+1 \pmod e$,
- ◇ $++$ for all other cases.

Adding the highest conormal $(\kappa_2 - m)$-node in these sequences, $(1, n-m+1, 1)$, we have

$$\begin{aligned}\mu_{n,m} \uparrow_{\kappa_2 - m} &= ((n-m, \{m-l-1\}),(1^{l+1})) \uparrow_{\kappa_2 - m} \\ &= ((n-m+1, \{m-l-1\}),(1^{l+1})) \\ &= \mu_{n+1,m}.\end{aligned}$$

Now, suppose that $n \equiv l+1 \pmod{e}$. Then $((n-m, \{m-l-1\}), (1^{l+1}))$ has $(\kappa_2 - m)$-signature

- ⋄ $-++-$ if $m \equiv l \pmod{e}$,
- ⋄ $-+++$ if $m \equiv -1 \pmod{e}$ and $l = e-2$,
- ⋄ $-++$ if $m \equiv -1 \pmod{e}$ and $l \neq e-2$ or $m \not\equiv -1 \pmod{e}$ and $m \equiv l+1 \pmod{e}$,
- ⋄ $-+$ for all other cases.

And, when $n-l \not\equiv 0, 1 \pmod{e}$, $((n-m, \{m-l-1\}), (1^{l+1}))$ has $(\kappa_2 - m)$-signature

- ⋄ $++-$ if $m \equiv l \pmod{e}$,
- ⋄ $+++$ if $m \equiv -1 \pmod{e}$ and $l = e-2$,
- ⋄ $++$ if $m \equiv -1 \pmod{e}$ and $l \neq e-2$ or $m \not\equiv -1 \pmod{e}$ and $m \equiv l+1 \pmod{e}$,
- ⋄ $+$ for all other cases.

So, for $n \not\equiv l \pmod{e}$, we observe that the highest conormal $(\kappa_2 - m)$-node in each signature is the addable node lying at the bottom of the $\lfloor (m+e-l-2)/(e-1) \rfloor$th column in the first component. Adding this node, we have

$$\begin{aligned}\mu_{n,m} \uparrow_{\kappa_2 - m} &= ((n-m, \{m-l-1\}), (1^{l+1})) \uparrow_{\kappa_2 - m} \\ &= ((n-m, \{m-l\}), (1^{l+1})) \\ &= \mu_{n+1,m+1}.\end{aligned}$$

(iii) Let $m \geqslant n - \lfloor \frac{n}{e} \rfloor$. Firstly, suppose that $n \not\equiv l+1 \pmod{e}$. Observe that $((\{m-l\}, n-m-1), (1^{l+1}))$ has the following addable/removable $(\kappa_2 - m)$-nodes

- ⋄ addable node at the bottom of the $\lfloor (m+e-l-2)/(e-1) \rfloor$th column in the first component,
- ⋄ addable node $(e, n-m, 1)$ if $n \equiv l \pmod{e}$,
- ⋄ addable node $(e+1, 1, 1)$ and removable node $(l+1, 1, 2)$ if $m \equiv l \pmod{e}$,
- ⋄ addable node $(1, 2, 2)$ if $m \equiv -1 \pmod{e}$,
- ⋄ addable node $(l+2, 1, 2)$ if $m \equiv l+1 \pmod{e}$.

For $n \equiv l \pmod{e}$, it follows that $((\{m-l\}, n-m-1), (1^{l+1}))$ has $(\kappa_2 - m)$-signature

- ⋄ $+++-$ if $m \equiv l \pmod{e}$,
- ⋄ $++++$ if $m \equiv -1 \pmod{e}$ and $l = e-2$,
- ⋄ $+++$ if $m \equiv -1 \pmod{e}$ and $l \neq e-2$ or $m \not\equiv -1 \pmod{e}$ and $m \equiv l+1 \pmod{e}$,

- $++$ for all other cases.

For $n-l \not\equiv 0, 1 \pmod e$, $(e, n-m, 1)$ is no longer an addable $(\kappa_2 - m)$-node, so $((\{m-l\}, n-m-1), (1^{l+1}))$ has $(\kappa_2 - m)$-signatures $++-$, $+++$, $++$ and $+$. So, for $n \not\equiv l+1 \pmod e$, the highest conormal $(\kappa_2 - m)$-node in each $(\kappa_2 - m)$-signature of $((\{m-l\}, n-m-1), (1^{l+1}))$ is the addable node at the bottom of the $\lfloor (m+e-l-2)/(e-1) \rfloor$th column in the first component. Hence

$$\begin{aligned}\mu_{n,m} \uparrow_{\kappa_2 - m} &= ((\{m-l\}, n-m-1), (1^{l+1})) \uparrow_{\kappa_2 - m} \\ &= ((\{m-l+1\}, n-m-1), (1^{l+1})) \\ &= \begin{cases} \mu_{n+1,m} & \text{if } n \equiv l \pmod e, \\ \mu_{n+1,m+1} & \text{if } n-l \not\equiv 0, 1 \pmod e. \end{cases}\end{aligned}$$

Secondly, suppose that $n \equiv l+1 \pmod e$. Observe that $((\{m-l+1\}, n-m-2), (1^{l+1}))$ has the following addable or removable $(\kappa_2 - m)$-nodes

- addable node $(e, n-m-1, 1)$,
- addable node $(e+1, 1, 1)$ and removable node $(l+1, 1, 2)$ if $m \equiv l \pmod e$,
- addable node $(1, 2, 2)$ if $m \equiv -1 \pmod e$,
- addable node $(l+2, 1, 2)$ if $m \equiv l+1 \pmod e$,
- the removable $(\kappa_2 - m)$-node at the bottom of the $(\lfloor (m+e-l-1)/(e-1) \rfloor)$th column in the first component.

Hence, $((\{m-l+1\}, n-m-2), (1^{l+1}))$ has $(\kappa_2 - m)$-signature

- $-++-$ if $m \equiv l \pmod e$,
- $-+++$ if $m \equiv -1 \pmod e$ and $l = e-2$,
- $-++$ if $m \equiv -1 \pmod e$ and $l \neq e-2$ or $m \not\equiv -1 \pmod e$ and $m \equiv l+1 \pmod e$,
- $-+$ for all other cases.

The highest conormal $(\kappa_2 - m)$-node in each sequence is $(e, n-m-1, 1)$, and adding this node we have

$$\begin{aligned}\mu_{n,m} \uparrow_{\kappa_2 - m} &= ((\{m-l+1\}, n-m-2), (1^{l+1})) \uparrow_{\kappa_2 - m} \\ &= ((\{m-l+1\}, n-m-1), (1^{l+1})) \\ &= \mu_{n+1,m+1}.\end{aligned}$$

□

Theorem 9.6. *Let* $n \not\equiv l+1 \pmod e$ *and* $1 \leqslant m \leqslant n-1$. *Then*

$$S_{((n-m),(1^m))} \cong D_{\mu_{n,m}}.$$

Proof. We proceed by induction on n.

1. Suppose that $n - l \not\equiv 2 \pmod e$. We obtain the irreducible label of $S_{((n-m),(1^m))}$ by first restricting it to an $\mathscr{R}^\Lambda_{n-1}$-module by removing its foot node of residue $\kappa_2 + 1 - m$ modulo e, and then inducing up to an $\mathscr{R}^\Lambda_n$-module by adding its highest conormal $(\kappa_2 + 1 - m)$-node. We have $e_{\text{leg}} S_{((n-1),(1))} = e_{\kappa_2} S_{((n-1),(1))} \cong S_{((n-1),\varnothing)}$. By Equation (9.1.2), $((n-1),\varnothing) \uparrow_{\kappa_2} = ((n-1),(1))$, and hence $S_{((n-1),(1))} \cong D_{((n-1),(1))}$ by Proposition 1.50.

   Assuming that $S_{((n-m),(1^{m-1}))} \cong D_{\mu_{n-1,m-1}}$ for $m > 1$, then

   $$e_{\kappa_2+1-m} S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-1}))} \cong D_{\mu_{n-1,m-1}}.$$

   By Proposition 1.50, and by $S_{((n-m),(1^m))}$ being an irreducible $\mathscr{R}^\Lambda_n$-module,

   $$S_{((n-m),(1^m))} \cong D_{\mu_{n-1,m-1} \uparrow_{\kappa_2 - m+1}} = D_{\mu_{n,m}},$$

   by Equation (9.1.2).

2. Suppose that $n - l \equiv 2 \pmod e$. We obtain the irreducible label of $S_{((n-m),(1^m))}$ by first restricting it to an $\mathscr{R}^\Lambda_{n-2}$-module by removing its hand node and its foot node of residue $\kappa_2 + 1 - m$ modulo e, and then inducing up to an $\mathscr{R}^\Lambda_n$-module by adding its two highest conormal $(\kappa_2 + 1 - m)$-nodes. We have $e_{\text{leg}} S_{((n-1),(1))} = e^{(2)}_{\kappa_2} S_{((n-1),(1))} \cong S_{((n-2),\varnothing)}$. By Equation (9.1.1) and Equation (9.1.2)), $((n-2),\varnothing) \uparrow^2_{\kappa_2} = ((n-1),\varnothing) \uparrow_{\kappa_2} = ((n-1),(1))$. Hence $S_{((n-1),(1))} \cong D_{((n-1),(1))}$ by Proposition 1.50.

   Assuming that $S_{((n-m-1),(1^{m-1}))} \cong D_{\mu_{n-2,m-1}}$ for $m > 1$, then

   $$e^{(2)}_{\kappa_2 - m+1} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-1}))} \cong D_{\mu_{n-2,m-1}}.$$

   By Proposition 1.50, and by $S_{((n-m),(1^m))}$ being an irreducible $\mathscr{R}^\Lambda_n$-module,

   $$\begin{aligned} S_{((n-m),(1^m))} \cong D_{\mu_{n-2,m-1} \uparrow^2_{\kappa_2 - m+1}} &= D_{\mu_{n-1,m-1} \uparrow_{\kappa_2 - m+1}} && \text{(Equation (9.1.1))} \\ &= D_{\mu_{n,m}} && \text{(Equation (9.1.2)).} \end{aligned}$$

□

Theorem 9.7. *Let $1 \leqslant m \leqslant n-1$. If $n \equiv l + 1 \pmod e$, then the composition factors of $S_{((n-m),(1^m))}$ are*

$$D_{\mu_{n,m-1}} \text{ and } D_{\mu_{n,m}}.$$

Moreover, $D_{\mu_{n,m}} \cong \operatorname{im}(\gamma_m)$.

Proof. We obtain the label of each of the two composition factors of a Specht module indexed by a hook bipartition $((n-m),(1^m))$ by first restricting it to an $\mathscr{R}^\Lambda_{n-1}$-module by either (1) removing its foot node of residue $\kappa_2 + 1 - m$, or (2) by removing its hand node of residue $\kappa_2 - m$, and then inducing up to an $\mathscr{R}_n^\Lambda$-module by adding the highest conormal node of residue $\kappa_2 + 1 - m$ or $\kappa_2 - m$, respectively.

1. By removing the foot node of $[((n-m),(1^m))]$, we have

$$e_{\kappa_2-m+1}S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-1}))},$$

and by Theorem 9.6, this is isomorphic to $D_{\mu_{n-1,m-1}}$. Equation (9.1.1), $\mu_{n-1,m-1}\!\uparrow_{\kappa_2+1-m} = \mu_{n,m-1}$. Then, by Proposition 1.50, $D_{\mu_{n,m-1}}$ is a composition factor of $S_{((n-m),(1^m))}$.

2. By removing the hand node of $[((n-m),(1^m))]$, we have

$$e_{\kappa_2-m}S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^m))},$$

and by Theorem 9.6, this is isomorphic to $D_{\mu_{n-1,m}}$. By Equation (9.1.1), $\mu_{n-1,m}\uparrow_{\kappa_2-m} = \mu_{n,m}$. Then, by Proposition 1.50, $D_{\mu_{n,m}}$ is a composition factor of $S_{((n-m),(1^m))}$.

Furthermore, from Theorem 7.11, we know that $\operatorname{im}(\gamma_{m-1})$ and $\operatorname{im}(\gamma_m)$ must somehow correspond to $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$, for $1 \leqslant m \leqslant n-1$. Moreover, $\operatorname{im}(\gamma_m)$ is a composition factor of both $S_{((n-m),(1^m))}$ and $S_{((n-m-1),(1^{m+1}))}$, and hence must be isomorphic to $D_{\mu_{n,m}}$, as required. □

9.2 Labelling the composition factors of $S_{((n-m),(1^m))}$ for $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ throughout this section.

For $n \not\equiv 0 \pmod{e}$ and $1 \leqslant m \leqslant n-1$, we recall from Proposition 7.13 that $S_{((n-m),(1^m))}$ has two composition factors, namely $\operatorname{im}(\chi_m)$ and $S_{((n-m),(1^m))}/\operatorname{im}(\chi_m)$. Thus $\operatorname{im}(\chi_m) \cong D_\lambda$ and $S_{((n-m),(1^m))}/\operatorname{im}(\chi_m) \cong D_\mu$ for some bipartitions $\lambda, \mu \in \mathscr{RP}_n^2$.

Definition 9.8. *Suppose that $n \not\equiv 0 \pmod{e}$. For $1 \leqslant m \leqslant n-1$, define*

$$\mu_{n,2m} = \begin{cases} ((n-m,\{m\}),\varnothing) & \text{if } 1 \leqslant m < n - \frac{n}{e}, \\ ((\{m+1\}, n-1-m),\varnothing) & \text{if } n - \frac{n}{e} \leqslant m \leqslant n-1, \end{cases}$$

and

$$\mu_{n,2m+1} = \begin{cases} ((n-m),(1^m)) & \text{if } 1 \leqslant m < e, \\ ((n-m,\{m-e\}),(2,1^{e-2})) & \text{if } e \leqslant m < n - \frac{n}{e}, \\ ((\{m-e+1\}, n-1-m),(2,1^{e-2})) & \text{if } n - \frac{n}{e} \leqslant m \leqslant n-1. \end{cases}$$

Notice that $\mu_{n,2m}$ and $\mu_{n,2m+1}$ are distinct. We claim that the labels λ, μ of the two composition factors of $S_{((n-m),(1^m))}$ are $\mu_{n,2m}$ and $\mu_{n,2m+1}$, respectively, and hence that these factors are non-isomorphic. We also claim that these labels are obtained by independently applying certain restriction functors to $S_{((n-m),(1^m))}$, together with their respective induction functors. In particular,

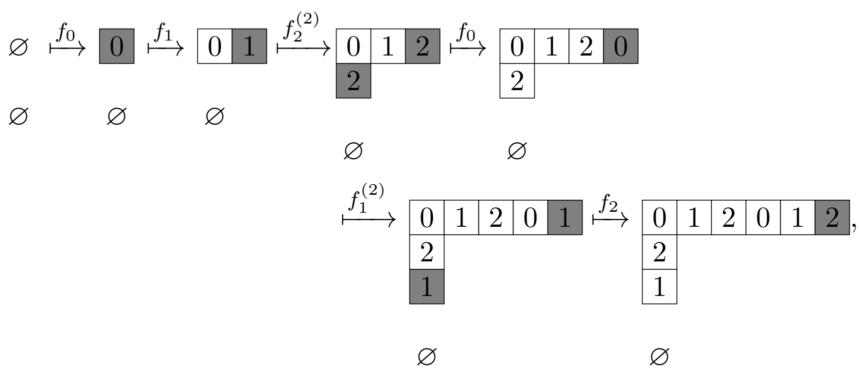
- if $1 \leqslant m \leqslant \frac{n}{e}$, we apply e_{arm} from Section 8.2.6.3 (or e_{leg} from Section 8.2.2.3) and $\widehat{e_{\text{arm}}}$ from Section 8.2.7.2,
- if $\frac{n}{e} < m < n - \frac{n}{e}$, we apply e_{arm} from Section 8.2.6.4 and e_{leg} from Section 8.2.2.3,
- if $n - \frac{n}{e} \leqslant m \leqslant n - 1$, we apply e_{arm} from Section 8.2.6.4 (or e_{leg} from Section 8.2.3.2) and $\widehat{e_{\text{leg}}}$ from Section 8.2.2.4.

Example 9.9. *Let $e = 3$, $\kappa = (0, 2)$. Observe that $S_{((6),(1^2))}$ has the following 3-residues*

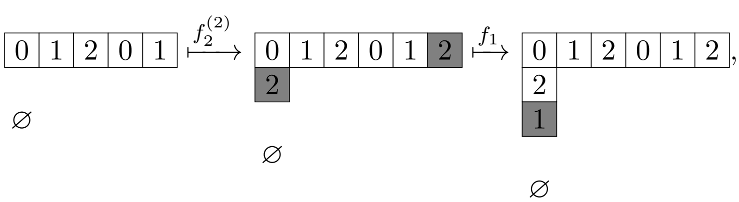
$$\begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&0&1&2\\\hline\end{array}.\qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$$

Now, $e_{\text{arm}}S_{((6),(1^2))} \cong S_{(\varnothing,\varnothing)} \cong D_{(\varnothing,\varnothing)}$ where $e_{\text{arm}} = e_0 \circ e_1 \circ e_2^{(2)} \circ e_0 \circ e_1^{(2)} \circ e_2$, $e_{\text{leg}}S_{((6),(1^2))} \cong S_{((5),\varnothing)} \cong D_{((5),\varnothing)}$, and $\widehat{e_{\text{arm}}}S_{((6),(1^2))} \cong S_{(\varnothing,(1))} \cong D_{(\varnothing,(1))}$ where $\widehat{e_{\text{arm}}} = e_0 \circ e_1 \circ e_2 \circ e_0 \circ e_1^{(2)} \circ e_2$.

Applying f_{arm} to $D_{(\varnothing,\varnothing)}$, we have

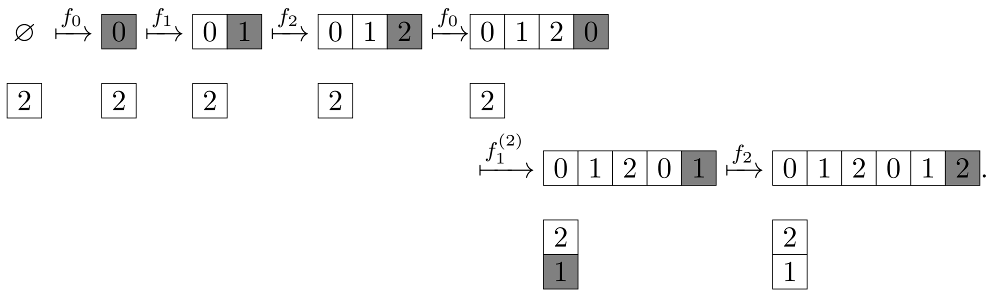


so $D_{((5,1^2),\varnothing)} = D_{\mu_{8,4}}$ is a composition factor of $S_{((6),(1^2))}$. Now, applying f_{leg} to $D_{((5),\varnothing)}$, we have



so either $S_{((6),(1^2))}$ has two composition factors isomorphic to $D_{\mu_{8,4}}$ or the same factor is obtained by applying the arm and leg functor independently.

Now, when we apply $\widehat{f_{\text{arm}}}$ to $D_{(\varnothing,(1))}$, we have



Thus, $D_{((6),(1^2))} = D_{\mu_{8,5}}$ is a composition factor of $S_{((6),(1^2))}$, and hence $S_{((6),(1^2))}$ has two non-isomorphic composition factors.

For $n \equiv 0 \pmod e$, we recall from Theorem 7.16 that $S_{((n-m),(1^m))}$ has four composition factors $\operatorname{im}(\phi_m)$, $\operatorname{im}(\phi_{m+1})$, $\ker(\gamma_m)/\operatorname{im}(\phi_m)$ and $\ker(\gamma_{m+1})/\operatorname{im}(\phi_{m+1})$ if $2 \leqslant m \leqslant n-2$, and that $S_{((n-1),(1))}$ and $S_{((1),(1^{n-1}))}$ both three composition factors. Thus $\operatorname{im}(\phi_m) \cong D_\lambda$, $\operatorname{im}(\phi_{m+1}) \cong D_\mu$, $\ker(\gamma_m)/\operatorname{im}(\phi_m) \cong D_\nu$ and $\ker(\gamma_{m+1})/\operatorname{im}(\phi_{m+1}) \cong D_\eta$, for some bipartitions $\lambda, \mu, \nu, \eta \in \mathscr{RP}^2_n$.

Definition 9.10. *Let $n \equiv 0 \pmod e$. For $2 \leqslant m \leqslant n-1$, define*

$$\mu_{n,2m} = \begin{cases} ((n-m+1,\{m-1\}),\varnothing) & \text{if } 2 \leqslant m \leqslant n - \frac{n}{e}, \\ ((\{m+1\}, n-m-1),\varnothing) & \text{if } n - \frac{n}{e} < m \leqslant n-1, \end{cases}$$

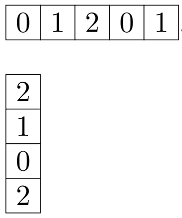
and

$$\mu_{n,2m+1} = \begin{cases} ((n-m+1),(1^{m-1})) & \text{if } 2 \leqslant m \leqslant e, \\ ((n-m+1,\{m-e-1\}),(2,1^{e-2})) & \text{if } e < m \leqslant n - \frac{n}{e}, \\ ((\{m-e+1\}, n-m-1),(2,1^{e-2})) & \text{if } n - \frac{n}{e} < m \leqslant n-1. \end{cases}$$

We notice that $\mu_{n,2m}$, $\mu_{n,2m+1}$, $\mu_{n,2m+2}$ and $\mu_{n,2m+2}$ are distinct bipartitions. We claim that the labels λ, μ, ν, η of the four composition factors of $S_{((n-m),(1^m))}$ are $\mu_{n,2m}$, $\mu_{n,2m+2}$, $\mu_{n,2m+1}$ and $\mu_{n,2m+3}$ for $2 \leqslant m \leqslant n-2$, and hence these factors are non-isomorphic. We also claim that these labels are obtained by independently applying four distinct restriction functors to $S_{((n-m),(1^m))}$, together with their respective induction functors. In particular,

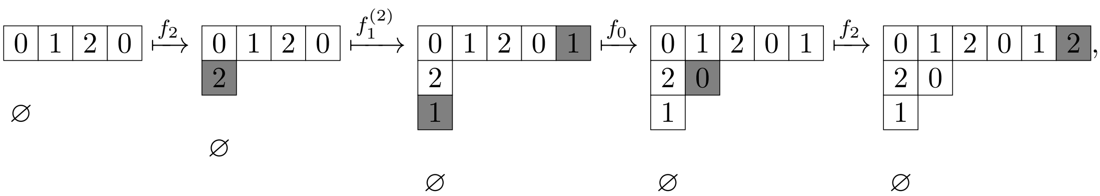
- ◇ if $2 \leqslant n < \frac{n}{e}$, we apply e_{leg} from Section 8.2.2.1, e_{arm} from Section 8.2.6.1, $\widetilde{e_{\text{arm}}}$ from Section 8.2.8.1 and $\widehat{e_{\text{arm}}}$ from Section 8.2.7.1,
- ◇ if $\frac{n}{e} < m \leqslant n - \frac{n}{e}$, we apply e_{leg} from Section 8.2.2.1, $\widetilde{e_{\text{leg}}}$ from Section 8.2.4.1, e_{arm} from Section 8.2.6.2 and $\widetilde{e_{\text{arm}}}$ from Section 8.2.8.2,
- ◇ if $n - \frac{n}{e} < m \leqslant n-2$, we apply e_{arm} from Section 8.2.6.2, e_{leg} from Section 8.2.2.2, $\widetilde{e_{\text{leg}}}$ from Section 8.2.4.2 and $\widehat{e_{\text{leg}}}$ from Section 8.2.3.1.

Example 9.11. *Let $e = 3$, $\kappa = (0, 2)$. Observe that $S_{((5),(1^4))}$ has the following 3-residues*



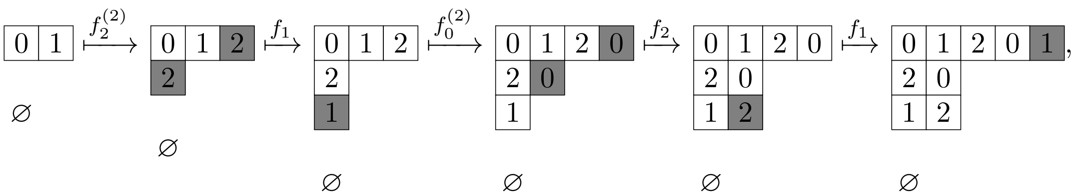
We have $e_{\text{leg}} S_{((5),(1^4))} \cong S_{((4),\varnothing)} \cong D_{((4),\varnothing)}$ where $e_{\text{leg}} = e_2 \circ e_1^{(2)} \circ e_0 \circ e_2$, $\widetilde{e_{\text{leg}}} S_{((5),(1^4))} \cong S_{((2),\varnothing)} \cong D_{((2),\varnothing)}$ where $\widetilde{e_{\text{leg}}} = e_2^{(2)} \circ e_1 \circ e_0^{(2)} \circ e_2 \circ e_1$, $e_{\text{arm}} S_{((5),(1^4))} \cong S_{(\varnothing,(1^2))} \cong D_{(\varnothing,(1^2))}$ where $e_{\text{arm}} = e_0^{(2)} \circ e_1 \circ e_2^{(2)} \circ e_0 \circ e_1$, and $\widetilde{e_{\text{arm}}} S_{((5),(1^4))} \cong S_{(\varnothing,(1))} \cong D_{(\varnothing,(1))}$ where $\widetilde{e_{\text{arm}}} = e_0 \circ e_1^{(2)} \circ e_2 \circ e_0^{(2)} \circ e_1 \circ e_2$.

Applying f_{leg} to $D_{((4),\varnothing)}$, we have



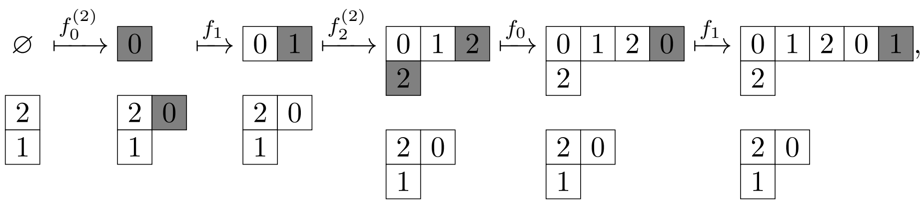
so that $D_{((6,2,1),\varnothing)} = D_{\mu_{9,8}}$ is a composition factor of $S_{((5),(1^4))}$.

Applying $\widetilde{f_{\text{leg}}}$ to $D_{((2),\varnothing)}$, we have



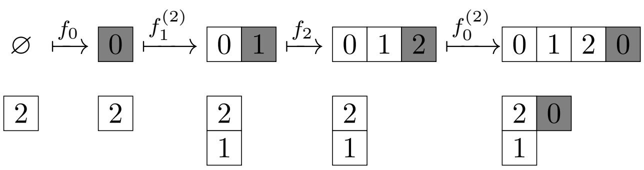
so $D_{((5,2^2),\varnothing)} = D_{\mu_{9,10}}$ is a composition factor of $S_{((5),(1^4))}$.

Applying f_{arm} to $D_{(\varnothing,(1^2))}$, we have



so $D_{((5,1),(2,1))} = D_{\mu_{9,11}}$ is a composition factor of $S_{((5),(1^4))}$.

Applying $\widetilde{f_{\text{arm}}}$ to $D_{(\varnothing,(1))}$, we have



$$\overset{f_1}{\longmapsto} \begin{array}{c} \begin{array}{|c|c|c|c|c|}\hline 0&1&2&0&\mathbf{1}\\\hline\end{array} \\ \\ \begin{array}{|c|c|}\hline 2&0\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \end{array} \overset{f_2}{\longmapsto} \begin{array}{c} \begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&0&1&\mathbf{2}\\\hline\end{array} \\ \\ \begin{array}{|c|c|}\hline 2&0\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \end{array},$$

so $D_{((6),(2,1))} = D_{\mu_{9,9}}$ *is a composition factor of* $S_{((5),(1^4))}$.

To confirm our claims, we require the following combinatorial result.

Lemma 9.12. *1. If* $n \not\equiv 0 \pmod{e}$*, then*

$$\mu_{n,2m} \uparrow_{\kappa_2 - m} = \mu_{n+1,2m+2}, \tag{9.2.1a}$$

$$\mu_{n,2m+1} \uparrow_{\kappa_2 - m} = \mu_{n+1,2m+3}. \tag{9.2.1b}$$

2. If $n \equiv 0 \pmod{e}$*, then*

$$\mu_{n,2m} \uparrow_{\kappa_2+1-m} = \mu_{n+1,2m}, \quad \text{for } 1 \leqslant m \leqslant n-1, \tag{9.2.2a}$$

$$\mu_{n,2m+1} \uparrow_{\kappa_2+1-m} = \mu_{n+1,2m+1}, \ \text{for } 2 \leqslant m \leqslant n-1. \tag{9.2.2b}$$

Proof. 1. Suppose that $n \not\equiv 0 \pmod{e}$ and let $i = \kappa_2 - m$.

(i) Let $1 \leqslant m < e$. Then $((n-m, \{m\}), \varnothing)$ has the following addable/removable i-nodes

- ◇ addable node $(1, n-m+1, 1)$ if $n \equiv l \pmod{e}$,
- ◇ removable node $(1, n-m, 1)$ if $n \equiv 0 \pmod{e}$,
- ◇ addable node at the bottom of the $\lfloor (m+e-2)/(e-1) \rfloor$th column in the first component,
- ◇ addable node $(e+1, 1, 1)$ if $m \equiv -1 \pmod{e}$,
- ◇ addable node $(1, 1, 2)$ if $m \equiv 0 \pmod{e}$.

So, when $n \equiv l \pmod{e}$, the i-signature of $((n-m, \{m\}), \varnothing)$ is either $++$ if $m \not\equiv -1, 0 \pmod{e}$, or is $+++$ if $m \equiv -1, 0 \pmod{e}$. The highest conormal node in each sequence is $(1, n-m+1, 1)$, and adding this node gives

$$\mu_{n,2m} \uparrow_i = ((n-m, \{m\}), \varnothing) \uparrow_i = ((n-m+1, \{m\}), \varnothing) = \mu_{n+1,2m+2}.$$

Now, if $n - l \not\equiv 0, 1 \pmod{e}$, then the i-signature of $((n-m, \{m\}), \varnothing)$ is either $+$ if $m \not\equiv -1, 0 \pmod{e}$ or is $++$ if $m \equiv 0, 1 \pmod{e}$. The highest conormal node in each sequence is the addable node at the end of the $\lfloor (m+e-2)/(e-1) \rfloor$th column in the first component, and hence

$$\mu_{n,2m} \uparrow_i = ((n-m, \{m\}), \varnothing) \uparrow_i = ((n-m, \{m+1\}), \varnothing) = \mu_{n+1,2m+2}.$$

We now observe that $((n-m), (1^m))$ has the following addable/removable i-nodes

- ⋄ addable node $(1, n-m+1, 1)$ if $n \equiv l \pmod e$,
- ⋄ removable node $(1, n-m, 1)$ if $n \equiv 0 \pmod e$,
- ⋄ addable node $(1, 2, 2)$ if $m \equiv -1 \pmod e$,
- ⋄ addable node $(m+1, 1, 2)$.

We note that $(2, 1, 1)$ cannot have residue i modulo e as $m < e$.

Suppose that $n \equiv l \pmod e$. Then $((n-m), (1^m))$ has i-signature $++$ if $m < e-1$, with highest conormal i-node $(1, n-m+1, 1)$. Hence

$$\mu_{n,2m+1} \uparrow_i = ((n-m), (1^m)) \uparrow_i = ((n-m+1), (1^m)) = \mu_{n+1,2m+3}.$$

However, if $m = e-1$, then $((n-e+1), (1^{e-1}))$ has i-signature $+++$, with highest conormal $(\kappa_2 - m)$-node $(1, n-e+2, 1)$. Thus $\mu_{n,2e-1} \uparrow_i = \mu_{n+1,2e+1}$.

Instead, suppose that $n - l \not\equiv 0, 1 \pmod e$. Then $((n-m), (1^m))$ has i-signature $+$ if $m < e-1$, corresponding to cornomal node $(m+1, 1, 2)$. Hence

$$\mu_{n,2m+1} \uparrow_i = ((n-m), (1^m)) \uparrow_i = ((n-m), (1^{m+1})) = \mu_{n+1,2m+3}.$$

However, if $m = e-1$, then $((n-m), (1^m))$ has i-signature $++$, with highest conormal node $(1, 2, 2)$. Hence

$$\mu_{n,2e-1} \uparrow_i = ((n-e+1), (1^{e-1})) \uparrow_i = ((n-e+1), (2, 1^{e-2})) = \mu_{n+1,2e+1}.$$

(ii) Let $e \leqslant m < n - \lfloor \frac{n}{e} \rfloor$. By the first part, it follows that $\mu_{n,2m} \uparrow_i = \mu_{n+1,2m+2}$ if $n \not\equiv 0 \pmod e$.

We now observe that $((n-m, \{m-e\}), (2, 1^{e-2}))$ has the following addable and removable i-nodes

- ⋄ addable node $(1, n-m+1, 1)$ if $n \equiv l \pmod e$,
- ⋄ removable node $(1, n-m, 1)$ if $n \equiv 0 \pmod e$,
- ⋄ addable nodes $(e+1, 1, 1)$, $(e, 1, 2)$ and removable node $(1, 2, 2)$ if $m \equiv -1 \pmod e$,
- ⋄ addable node $(2, 2, 2)$ if $m \equiv 0 \pmod e$,
- ⋄ addable node $(1, 3, 2)$ and removable node $(e-1, 1, 2)$ if $m \equiv -2 \pmod e$.

Suppose that $n \equiv l \pmod e$. Then $((n-m, \{m-e\}), (2, 1^{e-2}))$ has i-signature $+++$ if $m \equiv 0 \pmod e$, $+++-+$ if $m \equiv -1 \pmod e$, $+++-$ if $m \equiv -2 \pmod e$, and $++$ otherwise. The highest conormal node in each of these sequences is $(1, n-m+1, 1)$, and hence

$$\begin{aligned}\mu_{n,2m+1} \uparrow_i &= ((n-m, \{m-e\}), (2, 1^{e-2})) \uparrow_i \\ &= ((n-m+1, \{m-e\}), (2, 1^{e-2}))\end{aligned}$$

$$= \mu_{n+1,2m+3}.$$

Now, suppose that $n - l \not\equiv 0, 1 \pmod e$. Then $((n-m, \{m-e\}), (2, 1^{e-2}))$ has i-signature $++$ if $m \equiv 0 \pmod e$, $++-+$ if $m \equiv -1 \pmod e$, $++-$ if $m \equiv -2 \pmod e$, and $+$ otherwise. The highest conormal node in each of these sequences is the addable node at the bottom of the $\lfloor (m-e)/(e-1) \rfloor$th column in the first component. Hence

$$\begin{aligned}
\mu_{n,2m+1} \uparrow_i &= ((n-m, \{m-e\}), (2, 1^{e-2})) \uparrow_i \\
&= ((n-m, \{m-e+1\}), (2, 1^{e-2})) \\
&= \mu_{n+1,2m+3}.
\end{aligned}$$

(iii) Let $\lfloor \frac{n}{e} \rfloor \leqslant m \leqslant n-1$. We observe that $((\{m+1\}, n-m-1), \varnothing)$ has the following addable/removable i-nodes

- ◇ addable node at the bottom of the $\lfloor (m+e-1)/(e-1) \rfloor$th column in the first component,
- ◇ addable node $(e, n-m, 1)$ if $n \equiv l \pmod e$,
- ◇ removable node $(e, n-m-1, 1)$ if $n \equiv 0 \pmod e$,
- ◇ addable node $(e+1, 1, 1)$ if $m \equiv -1 \pmod e$,
- ◇ addable node $(1, 1, 2)$ if $m \equiv 0 \pmod e$.

So, suppose that $n \equiv l \pmod e$. Then $((\{m+1\}, n-m-1), \varnothing)$ has i-signature $++$ if $m \not\equiv -1, 0 \pmod e$, and $+++$ if $m \equiv -1, 0 \pmod e$. Whereas, when $n-l \not\equiv 0, 1 \pmod e$, $((\{m+1\}, n-m-1), \varnothing)$ has i-signature $+$ if $m \not\equiv -1, 0 \pmod e$, and $++$ if $m \equiv -1, 0 \pmod e$. The highest conormal node in each of these sequences is the addable node at the bottom of the $\lfloor (m+e-1)/(e-1) \rfloor$th column in the first component. Hence

$$\begin{aligned}
\mu_{n,2m} \uparrow_i &= ((\{m+1\}, n-m-1), \varnothing) \uparrow_i \\
&= ((\{m+2\}, n-m-1), \varnothing) \\
&= \mu_{n+1,2m+2}.
\end{aligned}$$

We now observe that $((\{m-e+1\}, n-m-1), (2, 1^{e-2}))$ has the following addable/removable i-nodes

- ◇ addable node in the $\lfloor (m-1)/(e-1) \rfloor$th column of the first component,
- ◇ addable node $(e, n-m, 1)$ if $n \equiv l \pmod e$,
- ◇ removable node $(e, n-m-1, 1)$ if $n \equiv 0 \pmod e$,
- ◇ addable node $(2, 2, 2)$ if $m \equiv 0 \pmod e$,
- ◇ addable node $(e+1, 1, 1)$, $(e, 1, 2)$ and removable node $(1, 2, 2)$ if $m \equiv -1 \pmod e$,

◇ addable node $(1,3,2)$ and removable node $(e-1,1,2)$ if $m \equiv -2 \pmod e$.

Suppose that $n \equiv l \pmod e$. Then $((\{m-e+1\}, n-m-1), (2, 1^{e-2}))$ has i-signature $+++$ if $m \equiv 0 \pmod e$, $+++-+$ if $m \equiv -1 \pmod e$, $+++-$ if $m \equiv -2 \pmod e$, $++$ otherwise. Instead, suppose that $n - l \not\equiv 0, 1 \pmod e$. Then $((\{m-e+1\}, n-m-1), (2, 1^{e-2}))$ has i-signature $++$ if $m \equiv 0 \pmod e$, $++-+$ if $m \equiv -1 \pmod e$, $++-$ if $m \equiv -2 \pmod e$, $+$ otherwise. In each of these sequences, the highest conormal i-node is the addable node in the $\lfloor (m-1)/(e-1) \rfloor$th column of the first component, and hence

$$\begin{aligned}\mu_{n,2m+1} \uparrow_i &= ((\{m-e+1\}, n-m-1), (2, 1^{e-2})) \uparrow_i \\ &= ((\{m-e+2\}, n-m-1), (2, 1^{e-2})) \\ &= \mu_{n+1,2m+3}.\end{aligned}$$

2. Suppose that $n \equiv 0 \pmod e$ and let $i = \kappa_2 + 1 - m$.

   (a) Let $1 \leqslant m \leqslant e$. We observe that $((n-m+1, \{m-1\}), \varnothing)$ has the following removable/addable i-nodes

   ◇ removable node $(1, n-m+1, 1)$,

   ◇ addable node at the bottom of the $\lfloor (m+e-3)/(e-1) \rfloor$th column in the first component,

   ◇ addable node $(e+1, 1, 1)$ if $m \equiv 0 \pmod e$,

   ◇ addable node $(1, 1, 2)$ if $m \equiv 1 \pmod e$.

   The i-signature of $((n-m+1, \{m-1\}), \varnothing)$ is $-+$ if $m \not\equiv 0, 1 \pmod e$, or is $-++$ if $m \equiv 0, 1 \pmod e$. The highest conormal i-node in each sequence is the addable node at the bottom of the $\lfloor (m+e-3)/(e-1) \rfloor$th column in the first component, and adding this node we have

$$\begin{aligned}\mu_{n,2m} \uparrow_{\kappa_2+1-m} &= ((n-m+1, \{m-1\}), \varnothing) \uparrow_{\kappa_2+1-m} \\ &= ((n-m+1, \{m\}), \varnothing) \\ &= \mu_{n+1,2m}.\end{aligned}$$

   For $m > 1$, we now observe that $((n-m+1), (1^{m-1}))$ has the following removable/addable i-nodes

   ◇ removable node $(1, n-m+1, 1)$,

   ◇ addable node $(1, 2, 2)$ if $m = e$,

   ◇ addable node $(m, 1, 2)$.

   The i-signature of $((n-m+1), (1^{m-1}))$ is $-+$ if $m \neq e$, and is $-++$ if $m = e$. The highest conormal i-node in each sequence is $(m, 1, 2)$, and adding this node we have

$$\begin{aligned}\mu_{n,2m+1} \uparrow_{\kappa_2+1-m} &= ((n-m+1),(1^{m-1})) \uparrow_{\kappa_2+1-m} \\ &= ((n-m+1),(1^m)) \\ &= \mu_{n+1,2m+1}.\end{aligned}$$

(b) Let $e < m \leqslant n - \frac{n}{e}$. By the previous part, $\mu_{n,2m} \uparrow_{\kappa_2+1-m} = \mu_{n+1,2m}$. We observe that $((n-m+1,\{m-e-1\}),(2,1^{e-2}))$ has the following addable and removable i-nodes

- ◇ removable node $(1, n-m+1, 1)$,
- ◇ addable node at the bottom of the $\lfloor (m-e)/(e-1) \rfloor$th column in the first component,
- ◇ addable node $(1,3,2)$ and removable node $(e-1,1,2)$ if $m \equiv -1 \pmod e$,
- ◇ addable nodes $(e+1,1,1)$ and $(e,1,2)$, and removable node $(1,2,2)$ if $m \equiv 0 \pmod e$,
- ◇ addable node $(2,2,2)$ if $m \equiv 1 \pmod e$.

So $((n-m+1,\{m-e-1\}),(2,1^{e-2}))$ has i-signature $-+$ if $m \not\equiv -1,0,1 \pmod e$, $-++-$ if $m \equiv -1 \pmod e$, $-++-+$ if $m \equiv 0 \pmod e$, and $-++$ if $m \equiv 1 \pmod e$. The highest conormal i-node in each of these sequences corresponds to the addable node at the bottom of the $\lfloor (m-e)/(e-1) \rfloor$th column in the first component, and adding this node we have

$$\begin{aligned}\mu_{n,2m+1} \uparrow_{\kappa_2+1-m} &= ((n-m+1,\{m-e-1\}),(2,1^{e-2})) \uparrow_{\kappa_2+1-m} \\ &= ((n-m+1,\{m-e\}),(2,1^{e-2})) \\ &= \mu_{n+1,2m+1}.\end{aligned}$$

(c) Let $m - \frac{n}{e} < m \leqslant n-1$. We observe that $((\{m+1\}, n-m-1), \varnothing)$ has the following addable/removable i-nodes

- ◇ the removable node at the bottom of the $\lfloor (m+e-1)/(e-1) \rfloor$th column in the first component,
- ◇ addable node $(e, n-m, 1)$,
- ◇ addable node $(e+1,1,1)$ if $m \equiv 0 \pmod e$,
- ◇ addable node $(1,1,2)$ if $m \equiv 1 \pmod e$.

The i-signature of $((\{m+1\}, n-m-1), \varnothing)$ is $-+$ if $m \not\equiv 0,1 \pmod e$, and is $-++$ if $m \equiv 0,1 \pmod e$. The highest conormal i-node in each sequence corresponds to $(e, n-m, 1)$, and adding this node we have

$$\begin{aligned}\mu_{n,2m} \uparrow_{\kappa_2+1-m} &= ((\{m+1\}, n-m-1), \varnothing) \uparrow_{\kappa_2+1-m} \\ &= ((\{m+1\}, n-m), \varnothing)\end{aligned}$$

$$= \mu_{n+1,2m}.$$

We now observe that $((\{m-e+1\}, n-m-1), (2, 1^{e-2}))$ has removable/addable i-nodes

- the removable node at the bottom of the $\lfloor (m-1)/(e-1) \rfloor$th column in the first component,
- addable node $(e, n-m, 1)$,
- addable nodes $(e+1, 1, 1)$ and $(e, 1, 2)$, and removable node $(1, 2, 2)$ if $m \equiv 0 \pmod e$,
- addable node $(1, 3, 2)$ and removable node $(e-1, 1, 2)$ if $m \equiv -1 \pmod e$,
- addable node $(2, 2, 2)$ if $m \equiv 1 \pmod e$.

The i-signature of $((\{m-e+1\}, n-m-1), (2, 1^{e-2}))$ is $-+$ if $m \not\equiv -1, 0, 1 \pmod e$, $-++-+$ if $m \equiv 0 \pmod e$, $-++-$ if $m \equiv -1 \pmod e$, and $-++$ if $m \equiv 1 \pmod e$. The highest conormal i-node in each of these sequences is $(e, n-m, 1)$, and adding this node we have

$$\begin{aligned}\mu_{n,2m+1} \uparrow_{\kappa_2+1-m} &= ((\{m-e+1\}, n-m-1), (2, 1^{e-2}))\\ &= ((\{m-e+1\}, n-m), (2, 1^{e-2}))\\ &= \mu_{n+1,2m+1}.\end{aligned}$$

□

Theorem 9.13. *Suppose that $n \not\equiv 0 \pmod e$ and $1 \leqslant m \leqslant n-1$. Then $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$.*

Moreover, $D_{\mu_{n,2m}} \cong \operatorname{im}(\chi_m)$ and $D_{\mu_{n,2m+1}} \cong S_{((n-m),(1^m))}/\operatorname{im}(\chi_m)$.

Proof. We first show that $D_{\mu_{n,3}}$ is a composition factor of $S_{((n-1),(1))}$. We have

$$f^{(2)}_{\kappa_2-1} S_{((n-1),(1))} \cong S_{((n),(1^2))} \text{ if } n \equiv -1 \pmod e,$$

and

$$f_{\kappa_2-1} S_{((n-1),(1))} \cong S_{((n-1),(1^2))} \text{ if } n \not\equiv -1 \pmod e.$$

For $n \equiv -1 \pmod e$, $S_{((n),(1^2))}$ has composition factor $D_{\mu_{n+2,5}}$, by downwards induction on n. Hence, by Proposition 1.50, $S_{((n-1),(1))}$ has composition factor $D_{\mu_{n+2,5} \downarrow^2_{\kappa_2-1}}$. We have

$$\begin{aligned}((n-1),(1)) \uparrow^2_{\kappa_2-1} = \mu_{n,3} \uparrow^2_{\kappa_2-1} &= \mu_{n+1,5} \uparrow_{\kappa_2-1} && \text{(Equation (9.2.1b))}\\ &= \mu_{n+2,5} && \text{(Equation (9.2.2b))}\\ &= ((n),(1^2)).\end{aligned}$$

Its inverse gives us $\mu_{n,3} = \mu_{n+2,5} \downarrow^2_{\kappa_2-1}$, and hence $D_{\mu_{n,3}}$ is a composition factor of $S_{((n-1),(1))}$.

Similarly, for $n \not\equiv -1 \pmod e$, $S_{((n-1),(1^2))}$ has composition factor $D_{\mu_{n+1,5}}$. So by Proposition 1.50, $S_{((n-1),(1))}$ has composition factor $D_{\mu_{n+1,5}} \downarrow_{\kappa_2-1}$. Observe that

$$((n-1),(1)) \uparrow_{\kappa_2-1} = \mu_{n,3} \uparrow_{\kappa_2-1} = \mu_{n+1,5} = ((n-1),(1^2)),$$

by Equation (9.2.1b). Its inverse gives us $\mu_{n,3} = \mu_{n+1,5} \downarrow_{\kappa_2-1}$, and hence $D_{\mu_{n,3}}$ is a composition factor of $S_{((n-1),(1))}$.

1. Suppose that $n - l \not\equiv 2 \pmod e$. We have $e_{\kappa_2+1-m}S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-1}))}$, and by induction, $S_{((n-m),(1^{m-1}))}$ has composition factors $D_{\mu_{n-1,2m-2}}$ and $D_{\mu_{n-1,2m-1}}$. It follows from Proposition 1.50 that $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n-1,2m-2}\uparrow_{\kappa_2+1-m}}$ and $D_{\mu_{n-1,2m-1}\uparrow_{\kappa_2+1-m}}$. We observe that $\mu_{n-1,2m-2} \uparrow_{\kappa_2+1-m} = \mu_{n,2m}$ by Equation (9.2.1a), and $\mu_{n-1,2m-1} \uparrow_{\kappa_2+1-m} = \mu_{n,2m+1}$ by Equation (9.2.1b). Thus $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$.

2. Suppose that $n-l \equiv 2 \pmod e$. We have $e^{(2)}_{\kappa_2+1-m}S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-1}))}$, and by induction, $S_{((n-m-1),(1^{m-1}))}$ has composition factors $D_{\mu_{n-2,2m-2}}$ and $D_{\mu_{n-2,2m-1}}$. We have

$$\begin{aligned} \mu_{n-2,2m-2} \uparrow^2_{\kappa_2+1-m} &= \mu_{n-1,2m-2} \uparrow_{\kappa_2+1-m} && \text{(Equation (9.2.2a))} \\ &= \mu_{n,2m} && \text{(Equation (9.2.1a))}. \end{aligned}$$

   So, by Proposition 1.50, $D_{\mu_{n,2m}}$ is a composition factor of $S_{((n-m),(1^m))}$.

   We also have that

$$\begin{aligned} \mu_{n-2,2m-1} \uparrow^2_{\kappa_2+1-m} &= \mu_{n-1,2m-1} \uparrow_{\kappa_2+1-m} && \text{(Equation (9.2.2b))} \\ &= \mu_{n,2m+1} && \text{(Equation (9.2.1b))}. \end{aligned}$$

   Thus, by Proposition 1.50, $D_{\mu_{n,2m+1}}$ is another composition factor of $S_{((n-m),(1^m))}$.

Furthermore, from Proposition 7.13, the composition factors $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ of $S_{((n-m),(1^m))}$ must correspond to $\operatorname{im}(\chi_m)$ and $S_{((n-m),(1^m))}/\operatorname{im}(\chi_m)$. By Lemma 6.5,

$$\operatorname{im}(\chi_m) = \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), T(1,1,1) = 1\},$$

and hence

$$S_{((n-m),(1^m))}/\operatorname{im}(\chi_m) = \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), T(1,1,2) = 1\}.$$

Now let $S, T \in \operatorname{Std}((n-m),(1^m))$ such that 1 lies in the arm of T and 1 lies in the leg of S. Clearly, every tableau T has residue sequence $(\kappa_1, i_2, \ldots, i_n)$, and every tableau S has residue sequence $(\kappa_2, i_2, \ldots, i_n)$. The first component of $\mu_{n,2m}$ is its only non-zero component, whereas both of the components of $\mu_{n,2m+1}$ are non-zero. Thus, only the residue sequence of $\mu_{n,2m+1}$ can begin with residue κ_2, and hence $D_{\mu_{n,2m}} \cong \mathrm{im}(\chi_m)$ and $D_{\mu_{n,2m+1}} \cong S_{((n-m),(1^m))}/\,\mathrm{im}(\chi_m)$, as required. □

Theorem 9.14. *If $n \equiv 0 \pmod e$, then the following statements hold.*

1. *$S_{((n-1),(1))}$ has composition factors $S_{((n),\varnothing)}$, $D_{\mu_{n,4}}$ and $D_{\mu_{n,5}}$.*
2. *For $2 \leqslant m \leqslant n-2$, $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n,2m}}$, $D_{\mu_{n,2m+1}}$, $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$.*
3. *$S_{((1),(1^n))}$ has composition factors $S_{(\varnothing,(1^n))}$, $D_{\mu_{n,2n-2}}$ and $D_{\mu_{n,2n-1}}$.*

Moreover, $D_{\mu_{n,2m}} \cong \mathrm{im}(\phi_m)$ and $D_{\mu_{n,2m+1}} \cong \ker(\gamma_m)/\,\mathrm{im}(\phi_m)$.

Proof. (i) Firstly, by removing the foot node of $S_{((n-1),(1))}$, we have

$$e_{\kappa_1} S_{((n-1),(1))} \cong S_{((n-1),\varnothing)} \cong D_{((n-1),\varnothing)}.$$

The κ_2-signature of $((n-1),\varnothing)$ is $++$, corresponding to conormal nodes $(1,n,1)$ and $(1,1,2)$. Adding the higher of these nodes, $((n-1),\varnothing)\uparrow_{\kappa_2}= ((n),\varnothing)$, and by Proposition 1.50, $D_{((n),\varnothing)}$ is a composition factor of $S_{((n-1),(1))}$.

Now suppose that $2 \leqslant m \leqslant n-1$. By removing the foot node of $((n-m),(1^m))$, we have

$$e_{\kappa_2+1-m} S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-1}))}.$$

We know that $S_{((n-m),(1^{m-1}))}$ has composition factors $D_{\mu_{n-1,2m-2}}$ and $D_{\mu_{n-1,2m-1}}$, by Theorem 9.13. Observe that $\mu_{n-1,2m-2}\uparrow_{\kappa_2+1-m}= \mu_{n,2m}$ by Equation (9.2.1a), and that $\mu_{n-1,2m-1}\uparrow_{\kappa_2+1-m}= \mu_{n,2m+1}$ by Equation (9.2.1b). Thus, by Proposition 1.50, $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ are composition factors of $S_{((n-m),(1^m))}$.

(ii) First suppose that $1 \leqslant m \leqslant n-2$. By removing the hand node of $((n-m),(1^m))$, we have

$$e_{\kappa_2-m} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^m))}.$$

By Theorem 9.13, $S_{((n-m-1),(1^m))}$ has composition factors $D_{\mu_{n-1,2m}}$ and $D_{\mu_{n-1,2m+1}}$. Observe that $\mu_{n-1,2m}\uparrow_{\kappa_2-m}$ by Equation (9.2.1a), and that $\mu_{n-1,2m+1}\uparrow_{\kappa_2-m}= \mu_{n,2m+3}$ by Equation (9.2.1b). Thus, $S_{((n-m),(1^m))}$ also has composition factors $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$ by Proposition 1.50.

Secondly, by removing the hand node of $S_{((1),(1^{n-1}))}$, we have

$$e_{\kappa_1} S_{((1),(1^{n-1}))} \cong S_{(\varnothing,(1^{n-1}))} \cong D_{(\{n-e\},(1^{e-1}))},$$

by Lemma 8.2. The κ_1-signature of $(\{n-e\},(1^{e-1}))$ is $+++$, corresponding to conormal nodes $(1,\lfloor (n-2)/(e-1)\rfloor+1,1)$, $(1,2,2)$ and $(e,1,2)$. Adding the highest of these nodes, we have $(\{n-e\},(1^{e-1}))\uparrow_{\kappa_1}= (\{n-e+1\},(1^{e-1}))$. By

Lemma 8.2, $D_{(\{n-e+1\},(1^{e-1}))} \cong S_{(\varnothing,(1^n))}$, and by Proposition 1.50, $S_{(\varnothing,(1^n))}$ is a composition factor of $S_{((1),(1^{n-1}))}$.

Furthermore, from Theorem 7.16, we know that the composition factors $D_{\mu_{n,2m}}$, $D_{\mu_{n,2m+1}}$, $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$ of $S_{((n-m),(1^m))}$ must somehow correspond to $\operatorname{im}(\phi_m)$, $\operatorname{im}(\phi_{m+1})$, $\ker(\gamma_m)/\operatorname{im}(\phi_m)$ and $\ker(\gamma_{m+1})/\operatorname{im}(\phi_{m+1})$, for $2 \leqslant m \leqslant n-2$. Moreover, $\operatorname{im}(\phi_{m+1})$ and $\ker(\gamma_{m+1})/\operatorname{im}(\phi_{m+1})$ are composition factors of both $S_{((n-m),(1^m))}$ and $S_{((n-m-1),(1^{m+1}))}$, and hence must somehow correspond to $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$.

We now let $v_T \in \{\operatorname{im}(\phi_m), \operatorname{im}(\phi_{m+1}), \ker(\gamma_m)/\operatorname{im}(\phi_m), \ker(\gamma_{m+1})/\operatorname{im}(\phi_{m+1})\}$ such that $T \in \operatorname{Std}((n-m),(1^m))$. By Lemma 6.4 and Lemma 6.5, we have that

$$\operatorname{im}(\phi_{m+1}) \cong \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), T(1,1,1) = 1, T(1,n-m,1) = n\}$$

and

$$\ker(\gamma_{m+1}) \cong \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), T(1,n-m,1) = n\}.$$

Hence

$$\ker(\gamma_{m+1})/\operatorname{im}(\phi_{m+1}) \cong \operatorname{span}\{v_T \mid T \in \operatorname{Std}((n-m),(1^m)), T(1,1,2) = 1, T(1,n-m,1) = n\}.$$

It follows, together with Lemma 6.5, that

- ◇ $T(1,1,1) = 1$ if v_T lies in $\operatorname{im}(\phi_m)$ or $\operatorname{im}(\phi_{m+1})$;
- ◇ $T(1,1,2) = 1$ if v_T lies in $\ker(\gamma_m)/\operatorname{im}(\phi_m)$ or $\ker(\gamma_{m+1})/\operatorname{im}(\phi_{m+1})$.

Now observe that only the first component of $\mu_{n,2m}$ is non-empty, whereas both components of $\mu_{n,2m+1}$ are non-empty. It follows that 1 can only lie in the leg of T if v_T lies in $D_{\mu_{n,2m+1}}$ or $D_{\mu_{n,2m+3}}$, and hence $D_{\mu_{n,2m}} \cong \operatorname{im}(\phi_m)$ and $D_{\mu_{n,2m+1}} \cong \ker(\gamma_m)/\operatorname{im}(\phi_m)$, as required.

□

Chapter 10

Ungraded decomposition numbers

In the previous chapter, we found the labels $\lambda \in \mathscr{RP}_n^2$ of the irreducible factors D_λ of Specht modules labelled by hook bipartitions. We now draw on these results to find the *ungraded* multiplicities $[S_{((n-m),(1^m))} : D_\lambda]$ of irreducible $\mathscr{R}_n^\Lambda$-modules D_λ, appearing as composition factors of $S_{((n-m),(1^m))}$, for all $\lambda \in \mathscr{RP}_n^2$.

For all n and κ, recall that the trivial representation for $\mathscr{R}_n^\Lambda$ is $S_{((n),\varnothing)} \cong D_{((n),\varnothing)}$, and by Lemma 8.2, the sign representation is

$$S_{(\varnothing,(1^n))^R} \cong D_{(\varnothing,(1^n))^R} = \begin{cases} D_{(\varnothing,(1^n))} & \text{if } n < l; \\ D_{((\{n-l\}),(1^l))} & \text{if } n \geqslant l, \end{cases}$$

where l is the residue of $\kappa_2 - \kappa_1$ modulo e.

10.1 Case I: when $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv l+1 \pmod e$

We remind the reader that we found that $S_{((n-m),(1^m))}$ is irreducible in Theorem 7.8, for all $0 \leqslant m \leqslant n$, and by Theorem 9.6, $S_{((n-m),(1^m))} \cong D_{\mu_{n,m}}$, leading us to the following result.

Theorem 10.1. *Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv l+1 \pmod e$. Then, by a specific ordering on the columns, part of the (ungraded) decomposition matrix for $\mathscr{R}_n^\Lambda$ comprising of rows corresponding to hook bipartitions is*

$$\begin{array}{c} S_{(n)} \\ S_{(n-1,1)} \\ S_{(n-2,1^2)} \\ \vdots \\ S_{(1^n)} \end{array} \left(\begin{array}{ccccc|c} 1 & & & & & \\ & 1 & & 0 & & \\ & & 1 & & & 0 \\ & 0 & & \ddots & & \\ & & & & 1 & \end{array}\right)$$

10.2 Case II: when $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \equiv l+1 \pmod e$

From Theorem 9.7, the composition factors of $S_{((n-m),(1^m))}$ are $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$, for $1 \leqslant m \leqslant n-1$, and hence $D_{\mu_{n,m}}$ is a composition factor of both $S_{((n-m),(1^m))}$ and $S_{((n-m-1),(1^{m+1}))}$, for $1 \leqslant m \leqslant n-2$. Also, note that $D_{\mu_{n,0}} = D_{((n),\varnothing)}$ and $D_{\mu_{n,n-1}} = D_{(\varnothing,(1^n))^R}$. Furthermore, since the bipartitions $\mu_{n,0}, \mu_{n,1}, \ldots, \mu_{n,n-1}$ are distinct, the irreducible modules $D_{\mu_{n,0}}, D_{\mu_{n,1}}, \ldots, D_{\mu_{n,n-1}}$ are non-isomorphic, which leads us to the following result.

Theorem 10.2. *Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \equiv \kappa_2 - \kappa_1 + 1 \pmod e$. Then, by a specific ordering on the columns, part of the (ungraded) decomposition matrix for $\mathscr{R}_n^\Lambda$ comprising of rows corresponding to hook bipartitions is*

$$\begin{array}{c} S_{((n),\emptyset)} \\ S_{(n-1),(1))} \\ S_{(n-2),(1^2))} \\ S_{((n-3),(1^3))} \\ \vdots \\ S_{((1),(1^{n-1}))} \\ S_{(\emptyset,(1^n))} \end{array} \left(\begin{array}{cccccc|c} 1 & & & & 0 & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & 1 & 1 & & & 0 \\ & & & \ddots & \ddots & & \\ & 0 & & & 1 & 1 & \\ & & & & & 1 & \end{array}\right).$$

10.3 Case III: when $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv 0 \pmod e$

From Theorem 9.13, the composition factors of $S_{((n-m),(1^m))}$ are $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$, for $1 \leqslant m \leqslant n-1$. Furthermore, since the bipartitions $((n),\varnothing), \mu_{n,2}, \mu_{n,3}, \ldots, \mu_{2n-1}, (\varnothing,(1^n))^R$ are distinct, the irreducible modules $D_{((n),\varnothing)}, D_{\mu_{n,2}}, D_{\mu_{n,3}}, \ldots, D_{\mu_{n,2n-1}}, D_{(\varnothing,(1^n))^R}$ are non-isomorphic, leading us to the following result.

Theorem 10.3. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv 0 \pmod e$. Then, by a specific ordering on the columns, part of the (ungraded) decomposition matrix for $\mathscr{R}_n^\Lambda$ comprising* *of rows corresponding to hook bipartitions is*

$$
\begin{array}{c}
S_{((n),\varnothing)} \\
S_{((n-1),(1))} \\
S_{((n-2),(1^2))} \\
S_{((n-3),(1^3))} \\
\vdots \\
S_{((1),(1^{n-1}))} \\
S_{(\varnothing,(1^n))}
\end{array}
\left(\begin{array}{cccccccc|c}
1 & & & & & & & 0 & \\
 & 1 & 1 & & & & & & \\
 & & & 1 & 1 & & & & \\
 & & & & 1 & 1 & & & 0 \\
 & & & & & \ddots & \ddots & & \\
 & 0 & & & & & 1 & 1 & \\
 & & & & & & & 1 &
\end{array}\right).
$$

10.4 Case IV: when $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \equiv 0 \pmod e$

From Theorem 9.14, the composition factors of $S_{((n-m),(1^m))}$ are

- $D_{((n),\varnothing)}$, $D_{\mu_{n,4}}$ and $D_{\mu_{n,5}}$, for $m = 1$;
- $D_{\mu_{n,2m}}$, $D_{\mu_{n,2m+1}}$, $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$, for $2 \leqslant m \leqslant n-2$;
- $D_{\mu_{n,2n-2}}$, $D_{\mu_{n,2n-1}}$ and $D_{(\varnothing,(1^n))^R}$, for $m = n-1$,

and hence $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$ are composition factors of both $S_{((n-m),(1^m))}$ and $S_{((n-m-1),(1^{m+1}))}$, for $1 \leqslant m \leqslant n-2$. Furthermore, since the bipartitions $\mu_{n,2}, \mu_{n,3}, \dots, \mu_{n,2n-1}$ are distinct, the irreducible modules $D_{\mu_{n,2}}, D_{\mu_{n,3}}, \dots, D_{\mu_{n,2n-1}}$ are non-isomorphic, which leads us to the following result.

Theorem 10.4. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \equiv 1 + \kappa_2 - \kappa_1 \pmod e$. Then, by a specific ordering on the columns, part of the (ungraded) decomposition matrix for $\mathscr{R}_n^\Lambda$ comprising of rows corresponding to hook bipartitions is*

$$
\begin{array}{c}
S_{((n),\varnothing)} \\
S_{((n-1),(1))} \\
S_{((n-2),(1^2))} \\
S_{((n-3),(1^3))} \\
S_{((n-4),(1^4))} \\
\vdots \\
S_{((2),(1^{n-2}))} \\
S_{((1),(1^{n-1}))} \\
S_{(\varnothing,(1^n))}
\end{array}
\left(\begin{array}{cccccccccccc|c}
1 & & & & & & & & & 0 & & & \\
1 & 1 & 1 & & & & & & & & & & \\
 & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & 1 & 1 & 1 & 1 & & & & & & \\
 & & & & & 1 & 1 & 1 & 1 & & & & 0 \\
 & & & & & & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & & & & 1 & 1 & 1 & 1 & & \\
 & & 0 & & & & & & & 1 & 1 & 1 & \\
 & & & & & & & & & & & 1 &
\end{array}\right).
$$

Chapter 11

Graded dimensions of $S_{((n-m),(1^m))}$

From now on, we study *graded* Specht modules labelled by hook bipartitions, considering the combinatorial $\mathbb{Z}$-grading defined on these $\mathscr{R}^\Lambda_n$-modules. In this self-contained chapter, we determine the graded dimensions of $S_{((n-m),(1^m))}$, motivated to determine the graded decomposition matrices whose rows correspond to hook bipartitions. Recall from Section 1.3.4 that the graded dimension of S_λ, denoted $\operatorname{grdim}\big(S_{((n-m),(1^m))}\big)$, is the Laurent polynomial

$$\sum_{T\in\operatorname{Std}(\lambda)} v^{\deg(T)} \qquad (\lambda\in\mathscr{P}^l_n).$$

We first understand how the entries in the leg of a standard $((n-m),(1^m))$-tableau affect its degree.

Lemma 11.1. *Suppose $1\leqslant i\leqslant k$. Then $((k-i),(1^i))$ has neither an addable nor a removable (κ_2+1-i)-node in the first row of the first component, except in the following cases.*

(i) If $k\equiv l+1 \pmod e$, then $(1,k-i+1,1)$ is an addable (κ_2+1-i)-node for $((k-i),(1^i))$.

(ii) If $k\equiv l+2 \pmod e$ and $k>i$, then $(1,k-i,1)$ is a removable (κ_2+1-i)-node for $((k-i),(1^i))$.

Proof. Let $T\in\operatorname{Std}((n-m),(1^m))$. Since we want to determine if $((k-i),(1^i))$ has an addable or a removable (κ_2+1-i)-node, which will lie in the first row of the first component if one exists, we have $T(i,1,2)=k$.

1. Suppose that $T(i,1,2)=l+1+\alpha e$ for some $\alpha\geqslant 0$. Then the entries $1,\dots,l+\alpha e$ lie in the set of nodes $\{(1,1,2),\dots,(i-1,1,2)\}\cup\{(1,1,1),\dots,(1,j,1)\}$, where $j=l+\alpha e-i+1$. There are $j=l+\alpha e-i+1$ entries strictly smaller than $l+1+\alpha e$ in the arm of T and there are $i-1=l+\alpha e-j$ entries strictly smaller than $l+1+\alpha e$ in the leg of T. We now observe that

$$\operatorname{res}(1,j+1,1)=\kappa_1+j=\kappa_1+l+\alpha e-i+1=\kappa_2-i+1=\operatorname{res}(i,1,2),$$

and since $T(i,1,2) > T(1,j,1)$, it follows that $(1,j+1,1)$, which equals $(1,k-i+1,1)$, is an addable (κ_2+1-i)-nodes for $((k-i),(1^i))$.

2. Suppose that $T(i,1,2) = l+2+\alpha e$ for some $\alpha \geqslant 0$. Then the entries $1,\dots,l+1+\alpha e$ lie in the set of nodes $\{(1,1,2),\dots,(i-1,1,2)\} \cup \{(1,1,1),\dots,(1,j,1)\}$, where $j = l+2+\alpha e - i$. There are $j = l+2+\alpha e-i$ entries in the arm of T strictly smaller than $l+2+\alpha e$ and there are $i-1 = l+1+\alpha e - j$ entries in the leg of T strictly smaller than $l+2+\alpha e$. We now observe that

$$\mathrm{res}(1,j,1) = \kappa_1 + j - 1 = \kappa_1 + l + 1 + \alpha e - i = \kappa_2 + 1 - i = \mathrm{res}(i,1,2),$$

and since $T(1,j,1) < T(i,1,2)$, it follows that $(1,j,1)$, which equals $(1,k-i)$, is a removable (κ_2+1-i)-node for $((k-i),(1^i))$.

Moreover, if $T(i,1,2) > T(1,1,1)$, then it is clear that $((k-i),(1^i))$ does not have a removable (κ_2+1-i)-node.

3. Suppose that $T(i,1,2) = l+k+\alpha e$ for some $\alpha \geqslant 0$ such that $k \in \{3,\dots,e\}$. Then the entries $1,\dots,l+k+\alpha e-1$ lie in the set of nodes $\{(1,1,2),\dots,(i-1,1,2)\} \cup \{(1,1,1),\dots,(1,j,1)\}$, where $j = l+k+\alpha e-i$. There are $j = l+k+\alpha e-i$ entries strictly smaller than $l+k+\alpha e$ in the arm of T and there are $i = 1 = l+k+\alpha e-i-j$ entries strictly smaller than $l+k+\alpha e$ in the leg of T. We observe that $\mathrm{res}(i,1,2) = \kappa_2+1-i$, whereas

$$\mathrm{res}(1,j,1) = \kappa_1+j-1 = \kappa_1+l+k+\alpha e-i-1 = \kappa_2+k-i-1 = \mathrm{res}(1,j+1,1)-1.$$

Hence $\mathrm{res}(i,1,2) \neq \mathrm{res}(1,j,1), \mathrm{res}(1,j+1,1)$ since $k \neq 1,2$, and it thus follows that $((k-i),(1^i))$ does not have a removable or an addable (κ_2+1-i)-node in the first row of the first component.

□

Now we are able to obtain the degree of an arbitrary standard $((n-m),(1^m))$-tableau.

Lemma 11.2. *Let $T \in \mathrm{Std}((n-m),(1^m))$ and $1 \leqslant i \leqslant m$. Then the degree of T is*

$$\left\lfloor \tfrac{m+e-l-2}{e} \right\rfloor + \left\lfloor \tfrac{l+1}{e} \right\rfloor + \left\lfloor \tfrac{m}{e} \right\rfloor + \#\{i \mid T(i,1,2) \equiv l+1 \pmod e\} \\ - \#\{i \mid T(i,1,2) \equiv l+2 \pmod e\}.$$

Proof. Suppose that $T(i,1,2) = k$ for $i \leqslant k \leqslant n$, so that $T_{\leqslant k}$ has shape $((k-i),(1^i))$. Let $\widehat{\deg}(T_{\leqslant k})$ be the summand in $\deg(T_{\leqslant k})$, defined similarly to $\deg(T_{\leqslant k})$, except that we only attach a non-zero degree to a (κ_2+1-i)-node $(i,1,2)$ in $T_{\leqslant k}$ if $((k-i),(1^i))$ has either an addable or a removable (κ_2+1-i)-node in the first row of the first component.

Thus, it follows from Lemma 11.1 that

$$\widehat{\deg}(T_{\leqslant k}) = \begin{cases} \widehat{\deg}(T_{<k}) + 1 & \text{if } k \equiv l+1 \pmod{e}; \\ \widehat{\deg}(T_{<k}) - 1 & \text{if } k \equiv l+2 \pmod{e} \text{ and } k > i. \end{cases}$$

The two remaining summands in $\deg(T_{\leqslant k})$ are defined similarly to $\deg(T_{\leqslant k})$, except that we only attach a non-zero degree to a $(\kappa_2 + 1 - i)$-node $(i, 1, 2)$ in $T_{\leqslant k}$

- ⋄ if $((k-i), (1^i))$ has addable $(\kappa_1 - 1)$-node $(2, 1, 1)$,
- ⋄ if $((k-i), (1^i))$ has addable $(\kappa_2 + 1)$-node $(1, 2, 2)$,

taken over all k. The former holds if $k > i$ and the latter holds if $i \equiv 0 \pmod{e}$.

The number of nodes in the leg of T with residue $\kappa_1 - 1$ is

$$\left\lfloor \tfrac{m+e-l-2}{e} \right\rfloor + \left\lfloor \tfrac{l+1}{e} \right\rfloor = \begin{cases} \left\lfloor \tfrac{m+e-l-2}{e} \right\rfloor & \text{if } l \neq e-1 \\ \left\lfloor \tfrac{m-1}{e} \right\rfloor + 1 & \text{if } l = e-1, \end{cases}$$

and there are $\left\lfloor \frac{m}{e} \right\rfloor$ nodes in the leg of T with residue $\kappa_2 + 1$. Hence

$$\begin{aligned}
\deg(T) = {} & \#\{i \mid (i,1,2) \text{ has addable } (\kappa_1 - 1)\text{-node } (2,1,1)\} \\
& + \#\{i \mid (i,1,2) \text{ has addable } (\kappa_2 + 1)\text{-node } (1,2,2)\} \\
& + \#\{i \mid (i,1,2) \text{ has addable } (\kappa_2 + 1 - i)\text{-node in the first row of } T\} \\
& - \#\{i \mid (i,1,2) \text{ has removable } (\kappa_2 + 1 - i)\text{-node in the first row of } T\} \\
= {} & \#\{i \mid i \equiv l+2 \pmod{e}, k > i\} \\
& + \#\{i \mid i \equiv 0 \pmod{e}\} \\
& + \#\{i \mid k \equiv l+1 \pmod{e}\} \\
& - \#\{i \mid k \equiv l+2 \pmod{e}, k > i\} \\
= {} & \#\{i \mid i \equiv l+2 \pmod{e}\} - \#\{i \mid i \equiv l+2 \pmod{e}, k = i\} \\
& + \#\{i \mid i \equiv 0 \pmod{e}\} \\
& + \#\{i \mid k \equiv l+1 \pmod{e}\} \\
& - \#\{i \mid k \equiv l+2 \pmod{e}\} + \#\{i \mid k \equiv l+2 \pmod{e}, k = i\} \\
= {} & \#\{i \mid i \equiv l+2 \pmod{e}\} \\
& + \#\{i \mid i \equiv 0 \pmod{e}\} \\
& + \#\{i \mid k \equiv l+1 \pmod{e}\} \\
& - \#\{i \mid k \equiv l+2 \pmod{e}\},
\end{aligned}$$

and thus we obtain our desired result. □

The following result is a trivial consequence of Lemma 11.2.

Lemma 11.3. *We have*

$$\operatorname{grdim}\left(S_{((n),\varnothing)}\right) = 1; \qquad \operatorname{grdim}\left(S_{(\varnothing,(1^n))}\right) = \left\lfloor \tfrac{n}{e} \right\rfloor + \left\lfloor \tfrac{n+e-2-l}{e} \right\rfloor + \left\lfloor \tfrac{l+1}{e} \right\rfloor.$$

To our end, it is sufficient to only obtain the leading and trailing terms, and in some cases the second leading and second tailing terms too, of the graded dimensions of Specht modules labelled by hook bipartitions. For any $T \in \operatorname{Std}((n-m),(1^m))$, we define

$$a_T := \#\{i \mid T(i,1,2) \equiv l+1 \pmod e\} - \#\{i \mid T(i,1,2) \equiv l+2 \pmod e\}.$$

Then, for any non-empty subset $\mathscr{T}$ of $\operatorname{Std}((n-m),(1^m))$, we define the set

$$A_{\mathscr{T}} := \{a_T \mid T \in \mathscr{T}\}.$$

We say that the *maximum degree* of $\mathscr{T}$ is the largest degree of all tableaux in $\mathscr{T}$, written

$$\operatorname{maxdeg}(\mathscr{T}) = \max\{\deg(T) \mid T \in \mathscr{T}\}.$$

Similarly, the *minimum degree* of $\mathscr{T}$ is the smallest degree of all tableaux in $\mathscr{T}$, written

$$\operatorname{mindeg}(\mathscr{T}) = \min\{\deg(T) \mid T \in \mathscr{T}\}.$$

By Lemma 11.2, it follows that

$$\operatorname{maxdeg}(\mathscr{T}) := \left\lfloor \frac{m+e-l-2}{e} \right\rfloor + \left\lfloor \frac{l+1}{e} \right\rfloor + \left\lfloor \frac{m}{e} \right\rfloor + \max(A_{\mathscr{T}})$$

and

$$\operatorname{mindeg}(\mathscr{T}) := \left\lfloor \frac{m+e-l-2}{e} \right\rfloor + \left\lfloor \frac{l+1}{e} \right\rfloor + \left\lfloor \frac{m}{e} \right\rfloor + \min(A_{\mathscr{T}}).$$

We now set

$$\begin{aligned} a &:= \#\{i \mid 1 \leqslant i \leqslant n, i \equiv l+1 \pmod e\},\\ b &:= \#\{i \mid 1 \leqslant i \leqslant n, i \equiv l+2 \pmod e\},\\ c &:= \#\{i \mid 1 \leqslant i \leqslant n, i-l \not\equiv 1,2 \pmod e\}. \end{aligned}$$

Lemma 11.4. *Let* $\mathscr{T} = \operatorname{Std}((n-m),(1^m))$.

1. *If* $1 \leqslant m \leqslant \frac{n}{e}$, *then* $\max(A_{\mathscr{T}}) = m$ *and* $\min(A_{\mathscr{T}}) = -m$.
2. *If* $\frac{n}{e} < m < n - \frac{n}{e}$, *then* $\max(A_{\mathscr{T}}) = a$ *and* $\min(A_{\mathscr{T}}) = -b$.
3. *If* $n - \frac{n}{e} \leqslant m \leqslant n-1$, *then* $\max(A_{\mathscr{T}}) = n-m+a-b$ *and* $\min(A_{\mathscr{T}}) = m-n+a-b$.

Proof. Let $S, T \in \mathscr{T}$ where $\deg(T) = \operatorname{maxdeg}(\mathscr{T})$ and $\deg(S) = \operatorname{mindeg}(\mathscr{T})$. We have $a, b \in \{\lfloor \frac{n}{e} \rfloor, \lfloor \frac{n}{e} \rfloor + 1\}$, depending on κ and n.

1. We can place an entry congruent to $l+1$ modulo e in each node in the leg of T, and similarly, we can place an entry congruent to $l+2$ modulo e in each node in the leg of S.

2. The legs of S and T contain at least $\lfloor \frac{n}{e} \rfloor + 1$ nodes, and so we can place every entry congruent to $l+2$ modulo e in the arm of T, and similarly, we can place every entry congruent to $l+1$ modulo e in the arm of S.

3. The arms of S and T contain at most $\lfloor \frac{n}{e} \rfloor + 1$ nodes. Thus we place an entry congruent to $l+2$ modulo e in every node in the arm of T. Then the leg of T contains a entries congruent to $l+1$ modulo e, $b-n+m$ entries congruent to $l+2$ modulo e, and $n-a-b$ entries congruent to neither $l+1$ nor congruent to $l+2$ modulo e. Similarly, we place an entry congruent to $l+1$ modulo e in every node in the arm of S. Then the leg of S contains $a-n+m$ entries congruent to $l+1$ modulo e, b entries congruent to $l+2$ modulo e, and $n-a-b$ entries congruent to neither $l+1$ nor congruent to $l+2$ modulo e.

□

Proposition 11.5. *Let $\mathscr{T} = \mathrm{Std}((n-m),(1^m))$. Then the graded dimension of $S_{((n-m),(1^m))}$ is*

$$\sum_{i=0}^{\max(A_{\mathscr{T}})-\min(A_{\mathscr{T}})} \left(\sum_{j=0}^{\max(A_{\mathscr{T}})} \left(\binom{a}{m-i+j}\binom{b}{j}\binom{c}{i-2j} \right) v^{\left(m-i+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor + \lfloor \frac{l+1}{e} \rfloor\right)} \right).$$

Proof. Let $T \in \mathscr{T}$. By Lemma 11.2, there are at most $\max(A_{\mathscr{T}})$ entries in the leg of T congruent to $l+1$ modulo e, and at most $-\min(A_{\mathscr{T}})$ entries congruent to $l+2$ modulo e. Thus, there exists a tableau with degree $\max(A_{\mathscr{T}}) - i + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor + \lfloor \frac{l+1}{e} \rfloor$ for all $i \in \{0, \dots, \max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}})\}$, so $\mathrm{grdim}(S_{((n-m),(1^m))})$ has $\max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}}) + 1$ terms.

Suppose that T has degree $\max(A_{\mathscr{T}}) - i + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor + \lfloor \frac{l+1}{e} \rfloor$ for some i, and suppose that there are j entries congruent to $l+2$ modulo e in the leg of T. These j entries contribute $-j$ to the degree of T. Hence, there must be $m-i+j$ entries congruent to $l+1$ modulo e in the leg of T, and the remaining $i-2j$ nodes in the leg of T must contain entries congruent to neither $l+1$ modulo e nor congruent to $l+2$ modulo e. Thus, there are $\binom{a}{m-i+j}\binom{b}{j}\binom{c}{i-2j}$ standard $((n-m),(1^m))$-tableaux with this combination of entries in its leg for some $j \in \left\{0, \dots, \lfloor \frac{i}{2} \rfloor\right\}$, and summing over j gives the number of standard $((n-m),(1^m))$-tableaux with degree $\max(A_{\mathscr{T}}) - i + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor + \lfloor \frac{l+1}{e} \rfloor$. □

Corollary 11.6. *The first and last two terms in the graded dimension of $S_{((n-m),(1^m))}$, respectively, are given in the following cases.*

1. *For* $1 \leqslant m \leqslant \frac{n}{e}$,

$$\binom{a}{m} v^{\left(m+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} + c\binom{a}{m-1} v^{\left(m-1+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} + \cdots$$
$$\cdots + c\binom{b}{m-1} v^{\left(1-m+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} + \binom{b}{m} v^{\left(-m+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)}.$$

2. *For* $\frac{n}{e} < m < n - \frac{n}{e}$,

$$\begin{aligned}
&\binom{c}{m-a} v^{\left(a+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} \\
&+ \left(a\binom{c}{m-a+1} + b\binom{c}{m-a-1}\right) v^{\left(a-1+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} + \cdots \\
&\cdots + \left(b\binom{c}{m-b+1} + a\binom{c}{m-b-1}\right) v^{\left(1-b+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} \\
&+ \binom{c}{m-b} v^{\left(-b+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)}.
\end{aligned}$$

3. *For* $n - \frac{n}{e} \leqslant m \leqslant n-1$,

$$\begin{aligned}
&\binom{b}{n-m} v^{\left(n-m+a-b+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} \\
&+ c\binom{b}{n-m-1} v^{\left(n-m+a-b-1+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} + \cdots \\
&\cdots + c\binom{a}{n-m-1} v^{\left(1+m-n+a-b+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)} \\
&+ \binom{a}{n-m} v^{\left(m-n+a-b+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m+e-l-2}{e}\rfloor+\lfloor\frac{l+1}{e}\rfloor\right)}.
\end{aligned}$$

Proof. 1. We obtain the leading term in the graded dimension of $S_{((n-m),(1^m))}$ by setting $i = j = 0$ in Proposition 11.5, the second term term by setting $i = 1$ and $j = 0$, the trailing term by setting $i = 2m$ and $j = m$, and the second trailing term by setting $i = 2m - 1$ and $j = m - 1$.

2. We obtain the leading term in the graded dimension of $S_{((n-m),(1^m))}$ by setting $i = m - a$ and $j = 0$ in Proposition 11.5, the second leading term by setting $i = m - a + 1$ and $j \in \{0, 1\}$, the trailing term by setting $i = m + b$ and $j = b$, and the second leading term by setting $i = m + b - 1$ and $j \in \{b - 1, b\}$.

3. We obtain the leading term in the graded dimension of $S_{((n-m),(1^m))}$ by setting $i = 2m - 2a - c$ and $j = m - a - c$ in Proposition 11.5, the second leading term by setting $i = 2m - 2a - c + 1$ and $j = m - a - c + 1$, the trailing term by setting $i = 2b + c$ and $j = b$, the second trailing term by setting $i = 2b + c - 1$ and $j = b$.

□

We now apply this result to Specht modules labelled by hook bipartitions dependent on whether $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ or not and whether $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$ or not, explicitly giving the corresponding result in each case, which will be useful to refer to in Chapter 13. We let $T \in \mathrm{Std}((n-m),(1^m))$.

11.1 Case I: $\mathrm{grdim}\left(S_{((n-m),(1^m))}\right)$ when $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv l + 1 \pmod{e}$

For $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv l+1 \pmod{e}$, there are $\lfloor \frac{n}{e} \rfloor$ entries in T that are congruent to $l+1$ modulo e, and $\lfloor \frac{n}{e} \rfloor$ entries in T that are congruent to $l+2$ modulo e, leading us to the following result by Corollary 11.6.

Proposition 11.7. *Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv l+1 \pmod{e}$. Then the leading and trailing terms of $\mathrm{grdim}(S_{((n-m),(1^m))})$, respectively, are as follows.*

1. $\binom{\lfloor \frac{n}{e} \rfloor + 1}{m} v^{\left(m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor\right)}$ *and* $\binom{\lfloor \frac{n}{e} \rfloor + 1}{m} v^{\left(-m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor\right)}$ *if* $1 \leqslant m \leqslant \frac{n}{e} + 1$,
2. $\binom{n - 2(\lfloor \frac{n}{e} \rfloor - 1)}{m - \lfloor \frac{n}{e} \rfloor - 1} v^{\left(\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor\right)}$ *and* $\binom{n - 2(\lfloor \frac{n}{e} \rfloor - 1)}{m - \lfloor \frac{n}{e} \rfloor - 1} v^{\left(-\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor\right)}$ *if* $\frac{n}{e} + 1 < m < n - \frac{n}{e} - 1$,
3. $\binom{\lfloor \frac{n}{e} \rfloor + 1}{n-m} v^{\left(n - m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor\right)}$ *and* $\binom{\lfloor \frac{n}{e} \rfloor + 1}{n-m} v^{\left(m - n + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor\right)}$ *if* $n - \frac{n}{e} - 1 \leqslant m \leqslant n-1$.

Example 11.8. *Let $e = 3$, $\kappa = (0,1)$. There are six tableaux that index the basis vectors of $S_{((2),(1^2))}$, namely*

$$T_1 = \left(\begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right),\ T_2 = \left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}\right),\ T_3 = \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline\end{array}\right),\ T_4 = \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}\right),\ T_5 = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array}\right),\ T_6 = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}\right).$$

It is easy to check that $\deg(T_1) = \deg(T_5) = 1$, $\deg(T_2) = \deg(T_6) = -1$ and $\deg(T_3) = \deg(T_4) = 0$, so that $\mathrm{grdim}\left(S_{((2),(1^2))}\right) = 2v + 2 + 2v^{-1}$.

11.2 Case II: $\mathrm{grdim}\left(S_{((n-m),(1^m))}\right)$ when $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv l + 1 \pmod{e}$

For $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv l+1 \pmod{e}$, there are $\lfloor \frac{n}{e} \rfloor + 1$ entries in T congruent to $l+1$ modulo e, and $\lfloor \frac{n}{e} \rfloor$ entries in T congruent to $l+2$ modulo e, which leads us to the following result by Corollary 11.6.

Proposition 11.9. *Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv l+1 \pmod{e}$. Then the leading and trailing terms of* $\mathrm{grdim}(S_{((n-m),(1^m))})$*, respectively, are as follows.*

1. $\binom{\lfloor \frac{n}{e} \rfloor + 1}{m} v^{\left(m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)}$ *and* $\binom{\lfloor \frac{n}{e} \rfloor}{m} v^{\left(-m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)}$ *if* $1 \leqslant m \leqslant \frac{n}{e}$,
2. $\binom{n - 2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor - 1} v^{\left(\lfloor \frac{n}{e} \rfloor + 1 + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)}$ *and* $\binom{n - 2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{\left(-\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)}$ *if* $\frac{n}{e} < m < n - \frac{n}{e}$,
3. $\binom{\lfloor \frac{n}{e} \rfloor}{n - m} v^{\left(n - m + 1 + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)}$ *and* $\binom{\lfloor \frac{n}{e} \rfloor + 1}{n - m} v^{\left(m - n + 1 + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)}$ *if* $n - \frac{n}{e} \leqslant m \leqslant n - 1$.

11.3 Case III: $\mathrm{grdim}\left(S_{((n-m),(1^m))}\right)$ when $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$

For $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$, there are $\lfloor \frac{n}{e} \rfloor$ entries in T congruent to $l+1$ modulo e, and $\lfloor \frac{n}{e} \rfloor + 1$ entries in T congruent to $l+2$ modulo e, which leads us to the following result by Corollary 11.6.

Proposition 11.10. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$. Then the leading and trailing terms of* $\mathrm{grdim}(S_{((n-m),(1^m))})$*, respectively, are as follows.*

1. $\binom{\lfloor \frac{n}{e} \rfloor}{m} v^{\left(m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m-1}{e} \rfloor + 1\right)}$ *and* $\binom{\lfloor \frac{n}{e} \rfloor + 1}{m} v^{\left(-m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m-1}{e} \rfloor + 1\right)}$ *if* $1 \leqslant m \leqslant \frac{n}{e}$,
2. $\binom{n - 2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{\left(\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m-1}{e} \rfloor + 1\right)}$ *and* $\binom{n - 2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor - 1} v^{\left(-\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m-1}{e} \rfloor + 1\right)}$ *if* $\frac{n}{e} < m < n - \frac{n}{e}$,
3. $\binom{\lfloor \frac{n}{e} \rfloor + 1}{n - m} v^{\left(n - m - 1 + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m-1}{e} \rfloor + 1\right)}$ *and* $\binom{\lfloor \frac{n}{e} \rfloor}{n - m} v^{\left(m - n - 1 + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m-1}{e} \rfloor + 1\right)}$ *if* $n - \frac{n}{e} \leqslant m \leqslant n - 1$.

11.4 Case IV: $\mathrm{grdim}\left(S_{((n-m),(1^m))}\right)$ when $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$

For $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$, there are $\frac{n}{e}$ entries in T congruent to $l+1$ modulo e, and $\frac{n}{e}$ entries in T congruent to $l+2$ modulo e, leading us to the following result by Corollary 11.6.

Proposition 11.11. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$. Then the first and last two terms of* $\mathrm{grdim}(S_{((n-m),(1^m))})$ *are given in the following cases.*

1. *For* $1 \leqslant m \leqslant \frac{n}{e}$,

$$\binom{\frac{n}{e}}{m} v^{\left(m+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)} + \frac{(e-2)n}{e}\binom{\frac{n}{e}}{m-1} v^{\left(m-1+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)} + \cdots$$
$$\cdots + \frac{(e-2)n}{e}\binom{\frac{n}{e}}{m-1} v^{\left(1-m+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)} + \binom{\frac{n}{e}}{m} v^{\left(-m+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)}.$$

2. *For* $\frac{n}{e} < m < \frac{n(e-1)+e}{e}$,

$$\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\left(\frac{n}{e}+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)}$$
$$+\frac{n}{e}\left(\binom{\frac{(e-2)n}{e}}{m-\frac{n}{e}+1} + \binom{\frac{(e-2)n}{e}}{m-\frac{n}{e}-1}\right) v^{\left(\frac{n}{e}-1+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)} + \cdots$$
$$\cdots + \frac{n}{e}\left(\binom{\frac{(e-2)n}{e}}{m-\frac{n}{e}+1} + \binom{\frac{(e-2)n}{e}}{m-\frac{n}{e}-1}\right) v^{\left(1-\frac{n}{e}+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)}$$
$$+\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\left(-\frac{n}{e}+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)}.$$

3. *For* $\frac{n(e-1)+e}{e} \leqslant m \leqslant n-1$,

$$\binom{\frac{n}{e}}{n-m} v^{\left(n-m+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)}$$
$$+\frac{(e-2)n}{e}\binom{\frac{n}{e}}{n-m-1} v^{\left(n-m-1+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)} + \cdots$$
$$\cdots + \frac{(e-2)n}{e}\binom{\frac{n}{e}}{n-m-1} v^{\left(1+m-n+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)}$$
$$+\binom{\frac{n}{e}}{n-m} v^{\left(m-n+\lfloor\frac{m}{e}\rfloor+\lfloor\frac{m-1}{e}\rfloor+1\right)}.$$

Chapter 12

Graded dimensions of composition factors of $S_{((n-m),(1^m))}$

In this chapter, we study the *graded* composition factors of Specht modules labelled by hook bipartitions, which arise as ungraded composition factors together with a grading shift, and determine results concerning their graded dimensions. Our results rely on the basis elements of these irreducible $\mathscr{R}_n^\Lambda$-modules, which we deduce from the bases of the images and kernels given in Lemma 6.5.

For $\lambda \in \mathscr{RP}_n^l$, recall from Proposition 1.36 that the graded dimension of the irreducible $\mathscr{R}_n^\Lambda$-module D_λ is symmetric in v and v^{-1}, and by Corollary 1.37, the graded dimension of D_λ that is spanned by $\{v_T \mid T \in \mathscr{T}\}$ is

$$\operatorname{grdim}(D_\lambda) = v^i \sum_{T \in \mathscr{T}} v^{\deg(T)},$$

where $2i = -\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})$.

We now determine the leading terms in the graded dimensions of composition factors of $S_{((n-m),(1^m))}$, dependent on whether or not $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and whether or not $n \equiv l+1 \pmod e$ or not. By symmetry of the graded dimensions of irreducible $\mathscr{R}_n^\Lambda$-modules, we automatically recover their trailing terms.

12.1 Case I: $\kappa_2 \not\equiv \kappa_1 - 1$ and $n \not\equiv l+1$ modulo e

Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv l+1 \pmod e$. Recall from Theorem 7.8 that $S_{((n-m),(1^m))}$ is an irreducible $\mathscr{R}_n^\Lambda$-module, and by Theorem 9.6, $S_{((n-m),(1^m))} \cong D_{\mu_{n,m}}\langle i\rangle$ as graded $\mathscr{R}_n^\Lambda$-modules for some $i \in \mathbb{Z}$.

Proposition 12.1. *Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv l+1 \pmod e$. Then the leading term of $\operatorname{grdim}(D_{\mu_{n,m}})$ is*

1. $\binom{\lfloor \frac{n}{e} \rfloor + 1}{m} v^m$ *if $1 \leqslant m \leqslant \frac{n}{e} + 1$,*

2. $\binom{n-2\left(\lfloor \frac{n}{e} \rfloor - 1\right)}{m - \lfloor \frac{n}{e} \rfloor - 1}$ *if* $\frac{n}{e} + 1 < m < n - \frac{n}{e} - 1$,

3. $\binom{\lfloor \frac{n}{e} \rfloor + 1}{n-m} v^{n-m}$ *if* $n - \frac{n}{e} - 1 \leqslant m \leqslant n-1$.

Moreover, $D_{\mu_{n,m}} \cong S_{((n-m),(1^m))} \left\langle -\lfloor \frac{m}{e} \rfloor - \lfloor \frac{m+e-l-2}{e} \rfloor \right\rangle$.

Proof. Since $S_{((n-m),(1^m))}$ is irreducible, the coefficient of the leading degree in the graded dimension of $D_{\mu_{n,m}}$ equals the coefficient of the leading degree in the graded dimension of $S_{((n-m),(1^m))}$, established in Proposition 11.7.

Let $\mathscr{T}$ be the set of all standard $((n-m),(1^m))$-tableaux. If $1 \leqslant m \leqslant \frac{n}{e} + 1$, then $\operatorname{maxdeg}(\mathscr{T}) = m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor$ and $\operatorname{mindeg}(\mathscr{T}) = -m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor$, by Proposition 11.7. So, the highest degree in the graded dimension of $D_{((n-m),(1^m))}$ is $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})) = m$. One can similarly deduce the leading degree in the other two cases.

Moreover, we determine $i \in \mathbb{Z}$ where $D_{((n-m),(1^m))} \cong S_{((n-m),(1^m))}\langle i \rangle$. If $1 \leqslant m \leqslant \frac{n}{e} + 1$, then $\operatorname{maxdeg}(\mathscr{T}) = m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor$ and $\operatorname{mindeg}(\mathscr{T}) = -m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor$, by Proposition 11.7. By the definition of the graded dimension of $D_{((n-m),(1^m))}$, we know that $i = -\frac{1}{2}\operatorname{maxdeg}(\mathscr{T}) - \frac{1}{2}\operatorname{mindeg}(\mathscr{T}) = -\lfloor \frac{m}{e} \rfloor - \lfloor \frac{m+e-l-2}{e} \rfloor$, as required. We similarly determine the same grading shift for $\frac{n}{e} + 1 < m \leqslant n-1$. □

12.2 Case II: $\kappa_2 \not\equiv \kappa_1 - 1$ and $n \equiv l+1$ modulo e

Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \equiv l + 1 \pmod e$. For $1 \leqslant m \leqslant n-1$, recall from Theorem 9.7 that $S_{((n-m),(1^m))}$ has graded composition factors $D_{\mu_{n,m-1}} \cong \operatorname{im}(\gamma_{m-1})\langle i \rangle$ and $D_{\mu_{n,m}} \cong \operatorname{im}(\gamma_m)\langle j \rangle$, for some $i, j \in \mathbb{Z}$.

Proposition 12.2. *Let* $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ *and* $n \equiv l+1 \pmod e$. *Then the leading term of* $\operatorname{grdim}(D_{\mu_{n,m}})$ *is*

1. $\binom{\lfloor \frac{n}{e} \rfloor}{m} v^m$ *if* $0 \leqslant m \leqslant \frac{n}{e}$,

2. $\binom{n - 2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{\lfloor \frac{n}{e} \rfloor}$ *if* $\frac{n}{e} < m < n - \frac{n}{e}$,

3. $\binom{\lfloor \frac{n}{e} \rfloor}{n-m-1} v^{n-m-1}$ *if* $\frac{n}{e} \leqslant m \leqslant n-1$.

Moreover, $D_{\mu_{n,m}} \cong \operatorname{im}(\gamma_m) \left\langle -\lfloor \frac{m+e-l-2}{e} \rfloor - \lfloor \frac{m}{e} \rfloor \right\rangle$.

Proof. Let $\mathscr{T} = \{T \in \operatorname{Std}((n-m),(1^m)) \mid T(1, n-m, 1) = n\}$. By Lemma 6.5, we know $\{v_T \mid T \in \mathscr{T}\}$ is a basis for $\operatorname{im}(\gamma_m)$, where $T \in \mathscr{T}$. By Corollary 1.37, we have

$$\operatorname{grdim}\left(D_{\mu_{n,m}}\right) = v^i \operatorname{grdim}\left(\operatorname{im}(\gamma_m)\right) = v^i \sum_{T \in \mathscr{T}} v^{\deg(T)},$$

where $2i = -\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})$. Now suppose that $T \in \mathscr{T}$. We note that there are $\lfloor \frac{n}{e} \rfloor + 1$ entries in T congruent to $l+1$ modulo e, n being one such entry, and $\lfloor \frac{n}{e} \rfloor$ entries in T congruent to $l+1$ modulo e, and hence there are $n - 2\lfloor \frac{n}{e} \rfloor - 1$ entries in T congruent to neither $l+1$ nor $l+2$ modulo e. We now consider the three cases in the proposition.

1. The tableaux in $\mathscr{T}$ with the maximum degree are those tableaux formed by placing m of the remaining $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+1$ modulo e in the m nodes in their legs. Hence, $\max(A_{\mathscr{T}}) = m$. Whereas, the tableaux in $\mathscr{T}$ with the minimum degree are those tableaux formed by placing m of the $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+2$ modulo e in the m nodes in their legs. Hence $\min(A_{\mathscr{T}}) = -m$. Thus, $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})) = \frac{1}{2}(\max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}})) = m$.

2. The tableaux in $\mathscr{T}$ with the maximum degree are those tableaux formed by placing the remaining $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+1$ modulo e in their legs, together with $m - \lfloor \frac{n}{e} \rfloor$ entries congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\max(A_{\mathscr{T}}) = \lfloor \frac{n}{e} \rfloor$. Whereas, the tableaux in $\mathscr{T}$ with the minimum degree are those tableaux formed by placing the $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+2$ modulo e in their legs, together with $m - \lfloor \frac{n}{e} \rfloor$ entries congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\min(A_{\mathscr{T}}) = -\lfloor \frac{n}{e} \rfloor$. Thus, $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})) = \frac{1}{2}(\max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}})) = \lfloor \frac{n}{e} \rfloor$.

3. The tableaux in $\mathscr{T}$ with the maximum degree are those tableaux with the least amount of entries in their legs congruent to $l+2$ modulo e, that is, we place $n-m-1$ of the $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+2$ modulo e in the remaining $n-m-1$ nodes in their arms. Thus in the legs of these tableaux there are $\lfloor \frac{n}{e} \rfloor$ nodes congruent to $l+1$ modulo e, $\lfloor \frac{n}{e} \rfloor - n + m + 1$ nodes congruent to $l+2$ modulo e, and $n - 2\lfloor \frac{n}{e} \rfloor - 1$ nodes congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\max(A_{\mathscr{T}}) = n-m-1$. Whereas, the minimum degree are those tableaux with the least amount of entries in their legs congruent to $l+1$ modulo e, that is, we place $n-m-1$ of the remaining $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+2$ modulo e in the remaining $n-m-1$ nodes in their arms. So in the legs of these tableaux there are $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+2$ modulo e, $\lfloor \frac{n}{e} \rfloor - n + m + 1$ nodes congruent to $l+1$ modulo e, and $n - 2\lfloor \frac{n}{e} \rfloor - 1$ nodes congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\min(A_{\mathscr{T}}) = m - n + 1$. Thus, $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})) = \frac{1}{2}(\max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}})) = n - m - 1$.

Moreover, notice that $\min(A_{\mathscr{T}}) = -\max(A_{\mathscr{T}})$, for all m. Thus, $2i = -\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T}) = -2\lfloor \frac{m+e-l-2}{e} \rfloor - 2\lfloor \frac{m}{e} \rfloor$, as required. □

Example 12.3. *Let* $e = 3$, $\kappa = (0,0)$, $n = 7$ *and* $\mathscr{T} = \{T \in \operatorname{Std}((5),(1^2)) \mid T(2,1,2) = 7\}$. *By Lemma 6.5, it follows that* $\operatorname{im}(\gamma_1)$ *is spanned by* $\{v_T \mid T \in \mathscr{T}\}$. *There are six* *possible such tableaux, namely*

$$S = \left(\begin{array}{|c|c|c|c|c|}\hline 2&3&4&5&6\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 7\\\hline\end{array}\right),\quad s_1S = \left(\begin{array}{|c|c|c|c|c|}\hline 1&3&4&5&6\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 7\\\hline\end{array}\right),\quad s_2s_1S = \left(\begin{array}{|c|c|c|c|c|}\hline 1&2&4&5&6\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 7\\\hline\end{array}\right),$$

$$s_3s_2s_1S = \left(\begin{array}{|c|c|c|c|c|}\hline 1&2&3&5&6\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline 7\\\hline\end{array}\right),\quad s_4s_3s_2s_1S = \left(\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&6\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 7\\\hline\end{array}\right),\quad s_5s_4s_3s_2s_1S = \left(\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline\end{array}, \begin{array}{|c|}\hline 6\\\hline 7\\\hline\end{array}\right).$$

By Lemma 11.1, if $T \in \mathscr{T}$ *where* $\deg(T) = \operatorname{maxdeg}(\mathscr{T})$*, then* $T(1,1,2) \in \{1,4\}$*, whereas, if* $T \in \mathscr{T}$ *where* $\deg(T) = \operatorname{mindeg}(\mathscr{T})$*, then* $T(1,1,2) \in \{2,5\}$*. Hence,* $\deg(S) = \deg(s_3s_2s_1S) > \deg(s_2s_1S) = \deg(s_5s_4s_3s_2s_1S) > \deg(s_1S) = \deg(s_4s_3s_2s_1S)$*. One can check that* $\deg(S) = 3$*,* $\deg(s_1S)\, 1$ *and* $\deg(s_2s_1S) = 2$*, so that*

$$\operatorname{grdim}(\operatorname{im}(\gamma_1)) = 2v^3 + 2v^2 + 2v.$$

By Lemma 9.12, $\operatorname{im}(\gamma_1) \cong D_{\mu_{7,1}} = D_{((6),(1))}$ *as ungraded* $\mathscr{R}_7^\Lambda$*-modules, and so by shifting the grading for* $\operatorname{im}(\gamma_1)$ *we see that*

$$\operatorname{grdim}\left(D_{((6),(1))}\right) = \operatorname{grdim}(\operatorname{im}(\gamma_1)\langle -2\rangle) = 2v + 2 + 2v^{-1}.$$

12.3 Case III: $\kappa_2 \equiv \kappa_1 - 1$ and $n \not\equiv 0$ modulo e

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \not\equiv 0 \pmod e$. For $1 \leqslant m \leqslant n-1$, we recall from Theorem 9.13 that $S_{((n-m),(1^m))}$ has graded composition factors $D_{\mu_{n,2m}} \cong \operatorname{im}(\chi_m)\langle i\rangle$ and $D_{\mu_{n,2m+1}} \cong (S_{((n-m),(1^m))}/\operatorname{im}(\phi_m))\langle j\rangle$, for some $i, j \in \mathbb{Z}$.

Proposition 12.4. *Let* $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ *and* $n \not\equiv 0 \pmod e$.

1. *The leading term of* $\operatorname{grdim}(D_{\mu_{n,2m}})$ *is*
   (a) $\binom{\lfloor \frac{n}{e}\rfloor}{m} v^m$ *if* $1 \leqslant m \leqslant \frac{n}{e}$,
   (b) $\binom{n-2\lfloor \frac{n}{e}\rfloor - 1}{m - \lfloor \frac{n}{e}\rfloor} v^{\lfloor \frac{n}{e}\rfloor}$ *if* $\frac{n}{e} < m < n - \frac{n}{e}$,
   (c) $\binom{\lfloor \frac{n}{e}\rfloor}{n-m-1} v^{(n-m-1)}$ *if* $n - \frac{n}{e} \leqslant m \leqslant n-1$.

   Moreover, $D_{\mu_{n,2m}} \cong \operatorname{im}(\chi_m)\left\langle -\lfloor \frac{m}{e}\rfloor - \lfloor \frac{m-1}{e}\rfloor - 1\right\rangle$.
2. *The leading term of* $\operatorname{grdim}(D_{\mu_{n,2m+1}})$ *is*
   (a) $\binom{\lfloor \frac{n}{e}\rfloor}{m-1} v^{m-1}$ *if* $1 \leqslant m \leqslant \frac{n}{e}$,

(b) $\binom{n-2\lfloor\frac{n}{e}\rfloor-1}{m-1-\lfloor\frac{n}{e}\rfloor} v^{\lfloor\frac{n}{e}\rfloor}$ *if* $\frac{n}{e} < m < n - \frac{n}{e}$,

(c) $\binom{\lfloor\frac{n}{e}\rfloor}{n-m} v^{n-m}$ *if* $n - \frac{n}{e} \leqslant m \leqslant n-1$

Moreover, $D_{\mu_{n,2m+1}} \cong S_{((n-m),(1^m))}/\operatorname{im}(\chi_m)\left\langle -\lfloor\frac{m-1}{e}\rfloor - \lfloor\frac{m}{e}\rfloor\right\rangle$.

Proof. We note that in the set $\{1,\ldots,n\}$ there are $\lfloor\frac{n}{e}\rfloor$ entries congruent to $l+1$ modulo e, $\lfloor\frac{n}{e}\rfloor+1$ entries congruent to $l+2$ modulo e (including 1), and hence $n-2\lfloor\frac{n}{e}\rfloor-1$ congruent to neither $l+1$ nor $l+2$ modulo e.

1. Let $\mathscr{T} = \{T \in \operatorname{Std}((n-m),(1^m)) \mid T(1,1,1) = 1\}$. By Lemma 6.5, we know that $\operatorname{im}(\chi_m)$ is spanned by $\{v_T \mid T \in \mathscr{T}\}$. By Corollary 1.37, we have

$$\operatorname{grdim}\left(D_{\mu_{n,2m}}\right) = v^i \operatorname{grdim}\left(\operatorname{im}(\chi_m)\right) = v^i \sum_{T\in\mathscr{T}} v^{\deg(T)},$$

where $2i = -\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})$.

(a) The tableaux in $\mathscr{T}$ with the maximum degree are those tableaux constructed by placing m of the $\lfloor\frac{n}{e}\rfloor$ entries congruent to $l+1$ modulo e in the m nodes in their legs. Hence, $\max(A_{\mathscr{T}}) = m$. Whereas, the tableaux in $\mathscr{T}$ with the minimum degree are those tableaux constructed by placing m of the remaining $\lfloor\frac{n}{e}\rfloor$ entries congruent to $l+2$ modulo e in the m nodes in their legs. Hence, $\min(A_{\mathscr{T}}) = -m$.

(b) The tableaux in $\mathscr{T}$ with the maximum degree are those tableaux constructed by placing the $\lfloor\frac{n}{e}\rfloor$ entries congruent to $l+1$ in their legs, together with $m-\lfloor\frac{n}{e}\rfloor$ of the $n-2\lfloor\frac{n}{e}\rfloor-1$ entries congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\max(A_{\mathscr{T}}) = \lfloor\frac{n}{e}\rfloor$. Whereas, the tableaux in $\mathscr{T}$ with the minimum degree are those tableaux constructed by placing the remaining $\lfloor\frac{n}{e}\rfloor$ entries congruent to $l+2$ in their legs, together with $m-\lfloor\frac{n}{e}\rfloor$ of the $n-2\lfloor\frac{n}{e}\rfloor-1$ entries congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\min(A_{\mathscr{T}}) = -\lfloor\frac{n}{e}\rfloor$.

(c) The tableaux in $\mathscr{T}$ with the maximum degree are those tableaux with the least amount of entries congruent to $l+2$ modulo e in their legs, that is, we place $n-m-1$ of the remaining $\lfloor\frac{n}{e}\rfloor$ nodes congruent to $l+2$ modulo e in the remaining $n-m-1$ nodes in their arms. Thus, in the legs of these tableaux there are $\lfloor\frac{n}{e}\rfloor$ entries congruent to $l+1$ modulo e, $\lfloor\frac{n}{e}\rfloor-n+m+1$ entries congruent to $l+2$ modulo e, and $n-2\lfloor\frac{n}{e}\rfloor+1$ entries congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\max(A_{\mathscr{T}}) = n-m-1$. Whereas, the tableaux in $\mathscr{T}$ with the minimum degree are those tableaux with the least amount of entries congruent to $l+1$ modulo e in their legs, that is, we place $n-m-1$ of the $\lfloor\frac{n}{e}\rfloor$ entries congruent to $l+1$ modulo e in the remaining $n-m-1$ nodes in their arms. Thus, in the legs of these tableaux there are $\lfloor\frac{n}{e}\rfloor$ entries congruent to $l+2$ modulo e, $\lfloor\frac{n}{e}\rfloor-n+m+1$ entries congruent to

$l+1$ modulo e, and $n - 2\lfloor \frac{n}{e} \rfloor + 1$ entries congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\max(A_{\mathscr{T}}) = -n+m+1$.

Moreover, notice that $\min(A_{\mathscr{T}}) = -\max(A_{\mathscr{T}})$ for all m. Thus, $2i = -\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T}) = -2\lfloor \frac{m}{e} \rfloor - 2\lfloor \frac{m-1}{e} \rfloor - 2$, as required.

2. Let $\mathscr{S} = \{T \in \operatorname{Std}((n-m),(1^m)) \mid T(1,1,2) = 1\}$. By Lemma 6.5, we know that $S_{((n-m),(1^m))}/\operatorname{im}(\chi_m)$ is spanned by $\{v_T \mid T \in \mathscr{S}\}$. By Corollary 1.37, we have

$$\operatorname{grdim}\left(D_{\mu_{n,2m+1}}\right) = v^i \operatorname{grdim}\left(S_{((n-m),(1^m))}/\operatorname{im}(\chi_m)\right) = v^i \sum_{T \in \mathscr{S}} v^{\deg(T)},$$

where $2i = -\operatorname{maxdeg}(\mathscr{S}) - \operatorname{mindeg}(\mathscr{S})$.

(a) The tableaux in $\mathscr{S}$ with the maximum degree are those tableaux constructed by placing $m-1$ of the $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+1$ modulo e in the remaining $m-1$ nodes in their legs. Hence, $\max(A_{\mathscr{S}}) = m-2$. Whereas, the tableaux in $\mathscr{S}$ with the minimum degree are those tableaux constructed by placing $m-1$ of the remaining $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+2$ modulo e in the remaining $m-1$ nodes in their legs. Hence, $\min(A_{\mathscr{S}}) = -m$.

(b) The tableaux in $\mathscr{S}$ with the maximum degree are those tableaux constructed by placing the $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+1$ modulo e, together with $m - \lfloor \frac{n}{e} \rfloor - 1$ of the $n - 2\lfloor \frac{n}{e} \rfloor - 1$ entries congruent to neither $l+1$ nor $l+2$ modulo e, in the remaining $m-1$ nodes in their legs. Hence, $\max(A_{\mathscr{S}}) = \lfloor \frac{n}{e} \rfloor - 1$. Whereas, the tableaux in $\mathscr{S}$ with the minimum degree are those tableaux constructed by placing the remaining $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+2$ modulo e, together with $m - \lfloor \frac{n}{e} \rfloor - 1$ of the $n - 2\lfloor \frac{n}{e} \rfloor - 1$ entries congruent to neither $l+1$ nor $l+2$ modulo e, in the remaining $m-1$ nodes in their legs. Hence, $\min(A_{\mathscr{S}}) = -\lfloor \frac{n}{e} \rfloor - 1$.

(c) The tableaux in $\mathscr{T}$ with the maximum degree are those tableaux with the least amount of entries congruent to $l+2$ modulo e in their legs, that is, we place $n-m-1$ entries of the remaining $\lfloor \frac{n}{e} \rfloor$ congruent to $l+2$ modulo e in their arms. So, there are $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+1$ modulo e, $\lfloor \frac{n}{e} \rfloor - n + m + 1$ entries congruent to $l+2$ modulo e, and $n - 2\lfloor \frac{n}{e} \rfloor - 1$ entries congruent to neither $l+1$ nor $l+2$ modulo e. Hence, $\max(A_{\mathscr{S}}) = n-m-1$. Whereas, the tableaux in $\mathscr{T}$ with the maximum degree are those tableaux with the least amount of entries congruent to $l+1$ modulo e in their legs, that is, we place $n-m-1$ entries of the $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l+1$ modulo e in their arms. Hence, $\min(A_{\mathscr{S}}) = m-n-1$.

Moreover, notice that $\min(A_{\mathscr{S}}) = -\max(A_{\mathscr{S}}) - 2$. Thus, $2i = -\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T}) = -2\lfloor \frac{m-1}{e} \rfloor - 2\lfloor \frac{m}{e} \rfloor$, as required.

□

Example 12.5. *Let $e = 3$, $\kappa = (0,2)$, $n = 5$ and $\mathscr{T} = \{T \in \mathrm{Std}((3),(1^2)) \mid T(1,1,1) = 1\}$. By Lemma 6.5, $\mathrm{im}(\chi_2)$ is spanned by $\{v_T \mid T \in \mathscr{T}\}$. There are six tableaux in $\mathscr{T}$, namely*

$$T = \begin{array}{c}\begin{array}{|c|c|c|}\hline 1&4&5\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\end{array},\quad s_3T = \begin{array}{c}\begin{array}{|c|c|c|}\hline 1&3&5\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\end{array},\quad s_4s_3T = \begin{array}{c}\begin{array}{|c|c|c|}\hline 1&3&4\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 2\\\hline 5\\\hline\end{array}\end{array},$$

$$s_2s_3T = \begin{array}{c}\begin{array}{|c|c|c|}\hline 1&2&5\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\end{array},\quad s_2s_4s_3T = \begin{array}{c}\begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array}\end{array},\quad s_3s_2s_4s_3T = \begin{array}{c}\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}\end{array}.$$

By Lemma 11.1, one can check that $\deg(T) = \deg(s_2s_4s_3T) = 2$, $\deg(s_3T) = \deg(s_3s_2s_4s_3T) = 0$ and $\deg(s_4s_3T) = \deg(s_2s_3T) = 1$, and hence

$$\mathrm{grdim}\,(\mathrm{im}(\chi_2)) = 2v^2 + 2v + 2.$$

By Theorem 9.13, $\mathrm{im}(\chi_2) \cong D_{\mu_{5,4}} = D_{((3,1^2),\varnothing)}$ as ungraded $\mathscr{R}_5^\Lambda$-modules, and one sees that we obtain $D_{((3,1^2),\varnothing)}$ by shifting the degree of $\mathrm{im}(\chi_2)$ by -1. In other words,

$$\mathrm{grdim}\left(S_{((3,1^2),\varnothing)}\right) = \mathrm{grdim}\,(\mathrm{im}(\chi_2)\langle -1\rangle) = 2v + 2 + 2v^{-1}.$$

Let $\mathscr{S} = \{S \in \mathrm{Std}((3),(1^2)) \mid S(1,1,2) = 1\}$. It follows from Lemma 6.5 that, $S_{((3),(1^2))}/\,\mathrm{im}(\chi_2)$ is spanned by $\{v_S \mid S \in \mathscr{S}\}$. There are four tableaux in $\mathscr{S}$, namely

$$S_1 = \begin{array}{c}\begin{array}{|c|c|c|}\hline 3&4&5\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{array},\quad s_2S = \begin{array}{c}\begin{array}{|c|c|c|}\hline 2&4&5\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\end{array},\quad s_3s_2S = \begin{array}{c}\begin{array}{|c|c|c|}\hline 2&3&5\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\end{array},\quad s_4s_3s_2S = \begin{array}{c}\begin{array}{|c|c|c|}\hline 2&3&4\\\hline\end{array}\\[1em] \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array}\end{array}.$$

It can be easily checked that $\deg(S) = \deg(s_4s_3s_2S) = 0$, $\deg(s_2S) = 1$ and $\deg(s_3s_2S) = -1$, and hence

$$\mathrm{grdim}\left(S_{((3),(1^2))}/\,\mathrm{im}(\chi_2)\right) = v + 2 + v^{-1}.$$

We know $S_{((3),(1^2))}/\,\mathrm{im}(\chi_2) \cong D_{\mu_{5,5}} = D_{((3),(1^2))}$ as ungraded $\mathscr{R}_5^\Lambda$-modules by Theorem 9.13. Since $\mathrm{grdim}\left(S_{((3),(1^2))}/\,\mathrm{im}(\chi_2)\right)$ is symmetric in v and v^{-1}, $S_{((3),(1^2))}/\,\mathrm{im}(\chi_2) \cong D_{((3),(1^2))}$ as graded $\mathscr{R}_5^\Lambda$-modules.

12.4 Case IV: $\kappa_2 \equiv \kappa_1 - 1$ and $n \equiv 0$ modulo e

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \equiv 0 \pmod e$. We know that $S_{((n-m),(1^m))}$ has ungraded composition factors $D_{\mu_{n,2m}} \cong \mathrm{im}(\phi_m)$ and $D_{\mu_{n,2m+1}} \cong \ker(\gamma_m)/\,\mathrm{im}(\phi_m)$ by Theorem 9.14. Under grading shifts, $D_{\mu_{n,2m}}\langle i\rangle$ and $D_{\mu_{n,2m+1}}\langle j\rangle$ are *graded* composition factors of $S_{((n-m),(1^m))}$, for some $i, j \in \mathbb{Z}$, which we determine. In this section, we not only find the leading terms in the graded dimensions of the graded composition factors $D_{\mu_{n,2m}}\langle i\rangle$ and $D_{\mu_{n,2m+1}}\langle j\rangle$ of $S_{((n-m),(1^m))}$, and hence the trailing terms, but the second leading terms, and hence the second trailing terms too. We will see in Chapter 13 that these extra terms are necessary to determine the graded decomposition numbers in this case, since from Theorem 9.14 we know that $S_{((n-m),(1^m))}$ has up to four composition factors.

Proposition 12.6. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod e$ and $n \equiv 0 \pmod e$.*

1. *The first two leading terms of $\operatorname{grdim}(D_{\mu_{n,2m}})$ are*

   (a) $\binom{\frac{n-e}{e}}{m-1}v^{m-1}$, $\frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{m-2}v^{m-2}$ *if* $1 \leqslant m \leqslant \frac{n}{e}$,

   (b) $\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}}v^{\frac{n-e}{e}}$, $\frac{n-e}{e}\left(\binom{\frac{(e-2)n}{e}}{m-\frac{n}{e}+1}+\binom{\frac{(e-2)n}{e}}{m-\frac{n}{e}-1}\right)v^{\frac{n-2e}{e}}$ *if* $\frac{n}{e} < m \leqslant \frac{n(e-1)}{e}$,

   (c) $\binom{\frac{n-e}{e}}{n-m-1}v^{n-m-1}$, $\frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{n-m-2}v^{n-m-2}$ *if* $\frac{n(e-1)+e}{e} \leqslant m \leqslant n-1$.

   Moreover, $D_{\mu_{n,2m}} \cong \operatorname{im}(\phi_m)\left\langle -\lfloor\frac{m-1}{e}\rfloor - \lfloor\frac{m}{e}\rfloor - 2\right\rangle$.

2. *The first two leading terms of $\operatorname{grdim}(D_{\mu_{n,2m+1}})$ are*

   (a) $\binom{\frac{n-e}{e}}{m-2}v^{m-2}$, $\frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{m-3}v^{m-3}$ *if* $2 \leqslant m \leqslant \frac{n}{e}$,

   (b) $\binom{\frac{(e-2)n}{e}}{\frac{e(m-1)-n}{e}}v^{\frac{n-e}{e}}$, $\frac{n-e}{e}\left(\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}}+\binom{\frac{(e-2)n}{e}}{\frac{e(m-2)-n}{e}}\right)v^{\frac{n-2e}{e}}$ *if* $\frac{n}{e} < m \leqslant \frac{n(e-1)}{e}$,

   (c) $\binom{\frac{n-e}{e}}{n-m}v^{n-m}$, $\frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{n-m-1}v^{n-m-1}$ *if* $\frac{n(e-1)+e}{e} \leqslant m \leqslant n-1$.

   Moreover, $D_{\mu_{n,2m+1}} \cong \ker(\gamma_m)/\operatorname{im}(\phi_m)\left\langle -\lfloor\frac{m-1}{e}\rfloor - \lfloor\frac{m}{e}\rfloor - 1\right\rangle$.

Proof. In the set $\{1,\dots,n\}$, there are $\frac{n}{e}$ entries congruent to $l+1$ modulo e (including n), and there are $\frac{n}{e}$ entries congruent to $l+2$ modulo e (including 1).

1. Let $\mathscr{T} = \{T \in \operatorname{Std}((n-m),(1^m)) \mid T(1,1,1) = 1, T(m,1,2) = n\}$. By Lemma 6.5, we know that $\{v_T \mid T \in \mathscr{T}\}$ is a basis for $\operatorname{im}(\phi_m)$. By Corollary 1.37, we have

$$\operatorname{grdim}(D_{\mu_{n,2m}}) = v^i \operatorname{grdim}(\operatorname{im}(\phi_m)) = v^i \sum_{T\in\mathscr{T}} v^{\deg(T)},$$

   where $2i = -\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})$. We let $S, T \in \mathscr{T}$ such that $\max(A_{\mathscr{T}}) = a_T$ and $\min(A_{\mathscr{T}}) = a_S$.

   (a) The degree of T is obtained by placing the remaining $\frac{n-e}{e}$ entries congruent to $l+1$ modulo e in the remaining $m-1$ nodes in the leg of T. Thus every entry in the leg of T is congruent to $l+1$ modulo e. Hence $\max(A_{\mathscr{T}}) = m$.

Whereas, we obtain the degree of S by placing the remaining $\frac{n-e}{e}$ entries congruent to $l+2$ modulo e in the remaining $m-1$ nodes in the leg of T. Thus there is one entry in the leg of S congruent to $l+1$ modulo e in the leg of S, and there are $\frac{n}{e}$ entries congruent to $l+2$ modulo e in the leg of S. Hence $\min(A_{\mathscr{T}}) = 2-m$.

Thus, $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})) = \frac{1}{2}(\max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}})) = m-1$, as required.

(b) We obtain the degree of T by placing the remaining $\frac{n-e}{e}$ entries congruent to $l+1$ modulo e, together with $m-\frac{n}{e}$ entries congruent to neither $l+1$ modulo e nor congruent to $l+2$ modulo e, in the remaining $m-1$ nodes in the leg of T. Thus there are $\frac{n}{e}$ entries in the leg of T congruent to $l+1$ modulo e. Hence, $\max(A_{\mathscr{T}}) = \frac{n}{e}$.

Wheareas, we obtain the degree of S by placing the remaining $\frac{n-e}{e}$ entries congruent to $l+2$ modulo e, together with $m-\frac{n}{e}$ entries congruent to neither $l+1$ modulo e nor $l+2$ modulo e, in the remaining $m-1$ nodes in the leg of S. Thus there is one entry in the leg of S congruent to $l+1$ modulo e, and there are $\frac{n-e}{e}$ entries in the leg of S congruent to $l+2$ modulo e. Hence, $\min(A_{\mathscr{T}}) = 2-\frac{n}{e}$.

Thus, $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{T}) - \operatorname{mindeg}(\mathscr{T})) = \frac{1}{2}(\max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}})) = \frac{n-e}{e}$, as required.

(c) We obtain the degree of T by placing the remaining $\frac{n}{e}-1$ entries congruent to $l+2$ modulo e in the remaining $n-m-1$ nodes in the arm of T. Thus there are $\frac{n}{e}$ entries congruent to $l+1$ modulo e in the leg of T, and there are $\frac{n}{e}-n+m$ entries congruent to $l+2$ modulo e in the leg of T. Hence, $\max(A_{\mathscr{T}}) = n-m$.

Whereas, we obtain the degree of S by placing the remaining $\frac{n}{e}-1$ entries congruent to $l+1$ modulo e in the remaining $n-m-1$ nodes in the arm of S. So, there are $\frac{n}{e}-1$ entries congruent to $l+2$ modulo e in the leg of S, and there are $\frac{n}{e}-n+m+1$ nodes congruent to $l+1$ modulo e in the leg of S. Hence, $\min(A_{\mathscr{T}}) = m-n+2$.

Thus, $\frac{1}{2}(\deg(\mathscr{T}) - \deg(\mathscr{T})) = \frac{1}{2}(\max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}})) = n-m-1$, as required.

Moreover, notice that $\min(A_{\mathscr{T}}) = -\max(A_{\mathscr{T}}) + 2$, and thus $2i = -\max(A_{\mathscr{T}}) - \min(A_{\mathscr{T}}) = -2\lfloor\frac{m}{e}\rfloor - 2\lfloor\frac{m-1}{e}\rfloor - 4$, as required.

2. Let $\mathscr{S} = \{T \in \operatorname{Std}((n-m),(1^m)) \mid T(1,1,2) = 1, T(m,1,2) = n\}$. By Lemma 6.5, we know that $\ker(\gamma_m)/\operatorname{im}(\phi_m)$ is spanned by $\{v_T \mid T \in \mathscr{S}\}$. By Corollary 1.37, we have

$$\operatorname{grdim}\left(D_{\mu_{n,2m+1}}\right) = v^i \operatorname{grdim}(\ker(\gamma_m)/\operatorname{im}(\phi_m)) = v^i \sum_{T\in\mathscr{S}} v^{\deg(T)},$$

where $2i = -\operatorname{maxdeg}(\mathscr{S}) - \operatorname{mindeg}(\mathscr{S})$. We let $S, T \in \mathscr{S}$ such that $\max(A_{\mathscr{S}}) = a_T$ and $\min(A_{\mathscr{S}}) = a_S$.

(a) We obtain the degree of T by placing the remaining $\frac{n-e}{e}$ nodes congruent to $l+1$ modulo e in the remaining $m-2$ nodes in the leg of T. Thus there are $m-1$ entries in the leg of T congruent to $l+1$ modulo e, and there is one entry in the leg of T congruent to $l+2$ modulo e. Hence, $\max(A) = m-2$.

Whereas, we obtain the degree of S by placing the remaining $\frac{n-e}{e}$ nodes congruent to $l+2$ modulo e in the remaining $m-2$ nodes in the leg of S. Thus there is one entry in the leg of S congruent to $l+1$ modulo e, and there are $m-1$ entries in the leg of S congruent to $l+2$ modulo e. Hence, $\min(A) = 2-m$.

Thus, $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{S}) - \operatorname{mindeg}(\mathscr{S})) = \frac{1}{2}(\max(A_{\mathscr{S}}) - \min(A_{\mathscr{S}})) = m-2$, as required.

(b) We obtain the degree of T by placing the remaining $\frac{n-e}{e}$ entries congruent to $l+1$ modulo e in the leg of T, and then place $\frac{n(e-2)}{e}$ entries congruent to neither $l+1$ modulo e nor congruent to $l+2$ modulo e in the remaining $m - \frac{n}{e} - 1$ nodes in the leg of T. Hence, $\max(A_{\mathscr{S}}) = \frac{n-e}{e}$.

Whereas, we obtain the degree of S by placing all of the remaining $\frac{n-e}{e}$ entries congruent to $l+2$ modulo e in the leg of S, together with $\frac{n(e-2)}{e}$ entries congruent to neither $l+1$ modulo e nor congruent to $l+2$ modulo e in the remaining $m - \frac{n}{e} - 1$ nodes in the leg of S. Hence, $\min(A_{\mathscr{S}}) = \frac{e-n}{e}$.

Thus, $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{S}) - \operatorname{mindeg}(\mathscr{S})) = \frac{1}{2}(\max(A_{\mathscr{S}}) - \min(A_{\mathscr{S}})) = \frac{n-e}{e}$, as required.

(c) We obtain the degree of T by placing the remaining $\frac{n-e}{e}$ entries congruent to $l+2$ modulo e in the arm of T. Thus T has $\frac{n}{e}$ entries congruent to $l+1$ modulo e in the leg of T, together with $\frac{n}{e} - n + m$ entries congruent to $l+2$ modulo e in the leg of T. Hence, $\max(A_{\mathscr{S}}) = n - m$.

Whereas, we obtain the degree of S by placing the remaining $\frac{n-e}{e}$ entries congruent to $l+1$ modulo e in the arm of S. Thus S has $\frac{n}{e}$ entries congruent to $l+2$ modulo e in the leg of S, together with $\frac{n}{e} - n + m$ entries congruent to $l+1$ modulo e in the leg of S. Hence, $\min(A_{\mathscr{S}}) = m - n$.

Thus, $\frac{1}{2}(\operatorname{maxdeg}(\mathscr{S}) - \operatorname{mindeg}(\mathscr{S})) = \frac{1}{2}(\max(A_{\mathscr{S}}) - \min(A_{\mathscr{S}})) = n - m$, as required.

Moreover, notice that $\min(A_{\mathscr{S}}) = -\max(A_{\mathscr{S}})$, and thus $2i = -\max(A_{\mathscr{S}}) - \min(A_{\mathscr{T}}) = -2\lfloor \frac{m}{e} \rfloor - 2\lfloor \frac{m-1}{e} \rfloor - 2$, as required.

□

Example 12.7. *Let $e = 3$, $\kappa = (0,2)$, $n = 6$ and $\mathscr{T} = \{T \in \operatorname{Std}((3),(1^3)) \mid T(1,1,1) = 1, T(3,1,2) = 6\}$. From Example 7.17, $\operatorname{im}(\phi_3)$ is spanned by $\{v_T \mid T \in \mathscr{T}\}$. There are* *six tableaux in* $\mathscr{T}$, *namely*

$$T = \begin{array}{|c|c|c|}\hline 1&4&5\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline 6\\\hline\end{array},\quad s_3T = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline 4\\\hline 6\\\hline\end{array},\quad s_4s_3T = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline 5\\\hline 6\\\hline\end{array},$$

$$s_2s_3T = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline 4\\\hline 6\\\hline\end{array},\quad s_2s_4s_3T = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline 5\\\hline 6\\\hline\end{array},\quad s_3s_2s_4s_3T = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|}\hline 4\\\hline 5\\\hline 6\\\hline\end{array}.$$

One can check that $\deg(T) = \deg(s_2s_4s_3) = 4$, $\deg(s_3T) = \deg(s_3s_2s_4s_3T) = 2$ *and* $\deg(s_4s_3T) = \deg(s_2s_3T) = 3$, *and hence*

$$\operatorname{grdim}(\operatorname{im}(\phi_3)) = 2v^4 + 2v^3 + 2v^2.$$

We know $\operatorname{im}(\phi_3) \cong D_{\mu_{6,6}} = D_{((4,1^2),\varnothing)}$ *as ungraded* $\mathscr{R}_6^\Lambda$*-modules. Now, by shifting the grading on* $\operatorname{im}(\phi_3)$ *so that its graded dimension is symmetric in* v *and* v^{-1}, *we have* $D_{((4,1^2),\varnothing)} \cong \operatorname{im}(\phi_3)\langle -3\rangle$ *as graded* $\mathscr{R}_6^\Lambda$*-modules.*

Let $\mathscr{S} = \{S \in \operatorname{Std}((3),(1^3)) \mid S(1,1,2) = 1, S(3,1,2) = 6\}$. *Also from Example 7.17,* $\ker(\gamma_3)/\operatorname{im}(\phi_3)$ *is spanned by* $\{v_S \mid S \in \mathscr{S}\}$. *There are four tableaux in* $\mathscr{S}$, *namely*

$$S = \begin{array}{|c|c|c|}\hline 3&4&5\\\hline\end{array}\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 6\\\hline\end{array},\quad s_2S = \begin{array}{|c|c|c|}\hline 2&4&5\\\hline\end{array}\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline 6\\\hline\end{array},\quad s_3s_2S = \begin{array}{|c|c|c|}\hline 2&3&5\\\hline\end{array}\ \begin{array}{|c|}\hline 1\\\hline 4\\\hline 6\\\hline\end{array},\quad s_4s_3s_2S = \begin{array}{|c|c|c|}\hline 2&3&4\\\hline\end{array}\ \begin{array}{|c|}\hline 1\\\hline 5\\\hline 6\\\hline\end{array}.$$

One can check that $\deg(S) = \deg(s_4s_3s_2) = 2$, $\deg(s_2S) = 3$ *and* $\deg(s_3s_2S) = 1$, *and hence*

$$\operatorname{grdim}(\ker(\gamma_3)/\operatorname{im}(\phi_3)) = v^3 + 2v^2 + v.$$

We have $\ker(\gamma_3)/\operatorname{im}(\phi_3) \cong D_{\mu_{6,7}} = D_{((4),(1^2))}$ *as ungraded* $\mathscr{R}_6^\Lambda$*-modules. By shifting the grading on* $\ker(\gamma_3)/\operatorname{im}(\phi_3)$ *so that its graded dimension is symmetric in* v *and* v^{-1}, *we have* $D_{((4),(1^2))} = \ker(\gamma_3)/\operatorname{im}(\phi_3)\langle -2\rangle$ *as graded* $\mathscr{R}_6^\Lambda$*-modules.*

Chapter 13

Graded decomposition numbers

In this chapter we deduce the *graded* decomposition matrices for $\mathscr{R}_n^\Lambda$ comprising rows corresponding to Specht modules labelled by hook bipartitions. We draw on Chapter 10, where we determined the analogous *ungraded* decomposition matrices for $\mathscr{R}_n^\Lambda$, together with Chapter 11 and Chapter 12, where we found the graded dimensions of $S_{((n-m),(1^m))}$ and of their composition factors, respectively. In fact, we observe that knowing these findings in Chapters 10 to 12 is equivalent to solving part of the Graded Decomposition Problem, which we now provide an answer to.

Recall from Section 1.5 that the *graded decomposition numbers* are defined to be the Laurent polynomials $[S_\lambda : D_\mu]_v = \sum_{i\in\mathbb{Z}}[S_\lambda : D_\mu\langle i\rangle]v^i$, for $\lambda \in \mathscr{P}_n^l$ and $\mu \in \mathscr{RP}_n^l$.

We first determine the grading shifts on the ungraded trivial and sign representations to obtain the analogous graded representations. The trivial representation $S_{((n),\varnothing)}$ is generated by $v_{T_{((n),\varnothing))}}$. It is clear that $\deg(t_{((n),\varnothing))}) = 0$, so $S_{((n),\varnothing)} \cong D_{((n),\varnothing)}$ as graded $\mathscr{R}_n^\Lambda$-modules, and hence $[S_{((n),\varnothing))} : D_{((n),\varnothing))}]_v = 1$. By Lemma 8.2, $S_{(\varnothing,(1^n))} \cong D_\lambda$ where

$$\lambda = \begin{cases} ((\{n-l\}),(1^l)) & \text{if } n \geqslant l,\\ (\varnothing,(1^n)) & \text{if } n < l.\end{cases}$$

We now find $i \in \mathbb{Z}$ where $S_{(\varnothing,(1^n))} \cong D_\lambda\langle i\rangle$.

Lemma 13.1. *1. If $\kappa_2 \equiv \kappa_1 - 1 \pmod e$, then $[S_{(\varnothing,(1^n))} : D_\lambda]_v = v^{2\lfloor \frac{m}{e}\rfloor}$,*

2. If $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$, then $[S_{(\varnothing,(1^n))} : D_\lambda]_v = v^{\left(\lfloor \frac{m}{e}\rfloor + \lfloor \frac{m-l-1}{e}\rfloor + 1\right)}$.

Proof. We have $\operatorname{grdim}(D_\lambda) = 1$, whereas $\operatorname{grdim}\left(S_{(\varnothing,(1^n))}\right) = \deg(t_{(\varnothing,(1^n))})$, so that

$$[S_{(\varnothing,(1^n))} : D_{(\varnothing,(1^n))}]_v = v^{\deg(t_{(\varnothing,(1^n))})}.$$

If $e \mid i$, then $(\varnothing,(1^i))$ has addable (κ_2+1)-node $(1,2,2)$ strictly above $(i,1,2)$, for $1 \leqslant i \leqslant n$. Hence, $d^{(i,1,2)}\left(\varnothing,(1^i)\right) = 1$ when $e \mid i$. Further, there are $\lfloor \frac{m}{e}\rfloor$ nodes $(i,1,2)$ where $e \mid i$.

1. If $e \mid i$, then $(\varnothing, (1^i))$ also has addable κ_1-node $(1,1,1)$ strictly above $(i,1,2)$, for $1 \leqslant i \leqslant n$. So $d^{(i,1,2)}(\varnothing,(1^i)) = 2$ when $e \mid i$, and hence, $\deg(t_{(\varnothing,(1^n))}) = 2\lfloor \frac{m}{e} \rfloor$.

2. For $\alpha \geqslant 0$, $(\varnothing, (1^{(1+l+\alpha e)}))$ has addable κ_2-node $(1,1,1)$ strictly above $(1+l+\alpha e, 1, 2)$, so $d^{(1+l+\alpha e,1,2)}\left(\varnothing,(1^{(1+l+\alpha e)})\right) = 1$. There are $\lfloor \frac{m-l-1}{e} \rfloor + 1$ nodes $(i,1,2)$ with $i \equiv l+1 \pmod{e}$. Thus, $\deg(t_{(\varnothing,(1^n))}) = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m-l-1}{e} \rfloor + 1$.

□

In other words, if $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$, then $S_{(\varnothing,(1^n))} \cong D_\lambda \left\langle 2\lfloor \frac{m}{e} \rfloor \right\rangle$, and if $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$, then $S_{(\varnothing,(1^n))} \cong D_\lambda \left\langle \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m-l-1}{e} \rfloor + 1 \right\rangle$, as graded $\mathscr{R}^\Lambda_n$-modules.

For $\lambda \in \mathscr{K}\mathscr{P}^2_n$, we now establish the *graded* composition multiplicities $[S_{((n-m),(1^m))} : D_\lambda]_v$ of irreducible $\mathscr{R}^\Lambda_n$-modules D_λ arising as composition factors of $S_{((n-m),(1^m))}$, for $1 \leqslant m \leqslant n-1$, depending on whether or not $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ or not and whether or not $n \equiv l+1 \pmod{e}$ or not.

13.1 Case I: $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv l+1 \pmod{e}$

Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv l+1 \pmod{e}$. We recall from Theorem 9.6 that $S_{((n-m),(1^m))}$ is irreducible and isomorphic to $D_{\mu_{n,m}}$ as an ungraded $\mathscr{R}^\Lambda_n$-module, for all $m \in \{1,\dots,n\}$. To find the graded multiplicity of $D_{((n-m),(1^m))}$ arising as a composition factor of $S_{((n-m),(1^m))}$, it suffices to find the grading shift on $D_{((n-m),(1^m))}$ so that it is isomorphic to $S_{((n-m),(1^m))}$ as a graded $\mathscr{R}^\Lambda_n$-module.

Theorem 13.2. *Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv l+1 \pmod{e}$. Then*

$$[S_{((n-m),(1^m)} : D_{((n-m),(1^m)}]_v = v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-l-2}{e} \rfloor\right)}.$$

Proof. We determine $i \in \mathbb{Z}$ where $\left[S_{((n-m),(1^m))} : D_{((n-m),(1^m))}\right]_v = v^i$, which is equivalent to finding $i \in \mathbb{Z}$ where $S_{((n-m),(1^m))} \cong D_{((n-m),(1^m))}\langle i \rangle$ as graded $\mathscr{R}^\Lambda_n$-modules. Thus, the result follows from Proposition 12.1. □

Example 13.3. *Let $e = 3$, $\kappa = (0,0)$. Then the submatrix with rows corresponding to*

Specht modules labelled by hook bipartitions is

$$\begin{array}{c} S_{((8),\varnothing)} \\ S_{((7),(1))} \\ S_{((6),(1^2))} \\ S_{((5),(1^3))} \\ S_{((4),(1^4))} \\ S_{((3),(1^5))} \\ S_{((2),(1^6))} \\ S_{((1),(1^7))} \\ S_{(\varnothing,(1^8))} \end{array} \left(\begin{array}{ccccccccc|c} 1 & & & & & & & & & \\ & 1 & & & & & 0 & & & \\ & & v & & & & & & & \\ & & & v^2 & & & & & & \\ & & & & v^2 & & & & & 0 \\ & & & & & v^3 & & & & \\ & & & & & & v^4 & & & \\ & 0 & & & & & & v^4 & & \\ & & & & & & & & v^5 & \end{array}\right).$$

13.2 Case II: $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \equiv l+1 \pmod e$

Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \equiv l+1 \pmod e$. Recall from Theorem 9.7 that $S_{((n-m),(1^m))}$ has ungraded composition factors $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$, for $1 \leqslant m \leqslant n-1$. We now determine the grading shifts $i, j \in \mathbb{Z}$ so that $D_{\mu_{n,m-1}}\langle i\rangle$ and $D_{\mu_{n,m}}\langle j\rangle$ are graded composition factors of $S_{((n-m),(1^m))}$.

Theorem 13.4. *Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$ and $n \equiv l+1 \pmod e$. Then, for all $m \in \{1,\dots,n-1\}$,*

1. $[S_{((n-m),(1^m))} : D_{\mu_{n,m-1}}]_v = v^{\left(\lfloor \frac{m}{e}\rfloor + \lfloor \frac{m+e-2-l}{e}\rfloor + 1\right)}$,
2. $[S_{((n-m),(1^m))} : D_{\mu_{n,m}}]_v = v^{\left(\lfloor \frac{m}{e}\rfloor + \lfloor \frac{m+e-2-l}{e}\rfloor\right)}$.

Proof. We determine $x, y \in \mathbb{Z}$ where $\operatorname{grdim}(S_{((n-m),(1^m))}) = v^x \operatorname{grdim}(D_{\mu_{n,m-1}}) + v^y \operatorname{grdim}(D_{\mu_{n,m}})$.

1. Let $0 \leqslant m \leqslant \lfloor \frac{n}{e}\rfloor$. By Proposition 11.9, the leading and trailing terms, respectively, in the graded dimension of $S_{((n-m),(1^m))}$ are

$$\binom{\lfloor \frac{n}{e}\rfloor + 1}{m} v^{\left(m + \lfloor \frac{m}{e}\rfloor + \lfloor \frac{m+e-2-l}{e}\rfloor\right)} \text{ and } \binom{\lfloor \frac{n}{e}\rfloor}{m} v^{\left(-m + \lfloor \frac{m}{e}\rfloor + \lfloor \frac{m+e-2-l}{e}\rfloor\right)},$$

and by Proposition 12.2, the leading terms in the graded dimensions of $\operatorname{im}(\gamma_{m-1})$ and $\operatorname{im}(\gamma_m)$, respectively are

$$\binom{\lfloor \frac{n}{e}\rfloor}{m-1} v^{m-1} \text{ and } \binom{\lfloor \frac{n}{e}\rfloor}{m} v^{m}.$$

Firstly, the graded dimensions of $D_{\mu_{n,m}}$ and $S_{((n-m),(1^m))}$ both have $2m+1$ terms, and hence $y = \lfloor \frac{m}{e}\rfloor + \lfloor \frac{m+e-2-l}{e}\rfloor$. Thus, $x - \lfloor \frac{m}{e}\rfloor - \lfloor \frac{m+e-2-l}{e}\rfloor = 0, 1$ since the trailing coefficients in the graded dimensions of $D_{\mu_{n,m}}$ and $S_{((n-m),(1^m))}$ are equal. Now observe that the sum of the leading coefficients in the graded dimensions of $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$ give the leading coefficient in the graded dimension of $S_{((n-m),(1^m))}$. Hence, $x = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor + 1$.

2. Let $\lfloor \frac{n}{e} \rfloor < m < n - \lfloor \frac{n}{e} \rfloor$. By Proposition 11.9, the leading and trailing terms in the graded dimension of $S_{((n-m),(1^m))}$, respectively are

$$\binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor - 1} v^{\left(\lfloor \frac{n}{e} \rfloor + 1 + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)} \text{ and } \binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{\left(-\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)}.$$

By Proposition 12.2, the leading terms in the graded dimensions of $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$, respectively, are

$$\binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor - 1} v^{\lfloor \frac{n}{e} \rfloor} \text{ and } \binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{\lfloor \frac{n}{e} \rfloor}.$$

Observing that the leading coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,m-1}}$ are equal, we deduce that $x = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor + 1$. Similarly, observing that the trailing coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,m}}$ are equal, we deduce that $y = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor$.

3. Let $\lfloor \frac{n}{e} \rfloor \leqslant m \leqslant n-1$. By Proposition 11.9, the leading and trailing terms in the graded dimension of $S_{((n-m),(1^m))}$ are

$$\binom{\lfloor \frac{n}{e} \rfloor}{n-m} v^{\left(n-m+1+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)} \text{ and } \binom{\lfloor \frac{n}{e} \rfloor + 1}{n-m} v^{\left(m-n+1+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor\right)},$$

respectively, and by Proposition 12.2, the leading terms in the graded dimension of $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$, respectively, are

$$\binom{\lfloor \frac{n}{e} \rfloor}{n-m} v^{(n-m)} \text{ and } \binom{\lfloor \frac{n}{e} \rfloor}{n-m-1} v^{(n-m-1)}.$$

Firstly, the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,m-1}}$ both have $2n-2m+1$ terms, and hence $x = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor + 1$. Thus, $y - \lfloor \frac{m}{e} \rfloor - \lfloor \frac{m+e-2-l}{e} \rfloor = 0, 1$, since the leading coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,m-1}}$ are equal. Now observe that the sum of the trailing coefficients in the graded dimensions of $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$ gives the trailing coefficient in the graded dimension of $S_{((n-m),(1^m))}$. Hence, $y = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-2-l}{e} \rfloor$.

□

Example 13.5. *Let $e = 3$, $\kappa = (0,0)$. Then the submatrix with rows corresponding to*

Specht modules labelled by hook bipartitions is

$$\begin{matrix} S_{((7),\varnothing)} \\ S_{((6),(1))} \\ S_{((5),(1^2))} \\ S_{((4),(1^3))} \\ S_{((3),(1^4))} \\ S_{((2),(1^5))} \\ S_{((1),(1^6))} \\ S_{(\varnothing,(1^7))} \end{matrix} \left(\begin{array}{ccccccc|c} 1 & & & & & & & \\ v & 1 & & & 0 & & & \\ & v^2 & v & & & & & \\ & & v^3 & v^2 & & & & 0 \\ & & & v^3 & v^2 & & & \\ & & & & v^4 & v^3 & & \\ & 0 & & & & v^5 & v^4 & \\ & & & & & & v^5 & \end{array}\right).$$

13.3 Case III: $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$. Recall from Theorem 9.13 that the ungraded composition factors of $S_{((n-m),(1^m))}$ are $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$, for all $m \in \{1, \dots, n-1\}$. Hence as graded $\mathscr{R}_n^\Lambda$-modules, the composition factors of $S_{((n-m),(1^m))}$ are $D_{\mu_{n,2m}}\langle i\rangle$ and $D_{\mu_{n,2m+1}}\langle j\rangle$ for some integers i and j, which we now determine.

Theorem 13.6. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$. Then, for all $m \in \{1, \dots, n-1\}$,*

1. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m}}]_v = v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)}$,
2. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m+1}}]_v = v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor - 1\right)}$.

Proof. We determine $x, y \in \mathbb{Z}$ where

$$\operatorname{grdim}(S_{((n-m),(1^m))}) = v^x \operatorname{grdim}(D_{\mu_{n,2m}}) + v^y \operatorname{grdim}(D_{\mu_{n,2m+1}}).$$

1. Let $1 \leqslant m \leqslant \lfloor \frac{n}{e} \rfloor$. By Proposition 11.10, the leading and trailing terms in the graded dimension of $S_{((n-m),(1^m))}$, respectively, are

   $$\binom{\lfloor \frac{n}{e} \rfloor}{m} v^{\left(m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)} \text{ and } \binom{\lfloor \frac{n}{e} \rfloor + 1}{m} v^{\left(-m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)}.$$

   By Proposition 12.4, the leading terms in the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$, respectively, are

   $$\binom{\lfloor \frac{n}{e} \rfloor}{m} v^m \text{ and } \binom{\lfloor \frac{n}{e} \rfloor}{m-1} v^{(m-1)}.$$

   Firstly, the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m}}$ both have $2m+1$ terms, and hence $x = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor$. Thus, we have $y - \lfloor \frac{m}{e} \rfloor - \lfloor \frac{m+e-1}{e} \rfloor = -1, 0$ since the leading coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m}}$ are equal. Observing that the sum of the trailing coefficients in the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ equals the trailing coefficient in the graded dimension of $S_{((n-m),(1^m))}$, and hence $y = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor - 1$.

2. Let $\lfloor \frac{n}{e} \rfloor < m < n - \lfloor \frac{n}{e} \rfloor$. By Proposition 11.10, the leading and trailing terms in the graded dimension of $S_{((n-m),(1^m))}$ are

$$\binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{\left(\lfloor \frac{n}{e} \rfloor + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)} \text{ and } \binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor - 1} v^{\left(-\lfloor \frac{n}{e} \rfloor - 1 + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)},$$

respectively. By Proposition 12.4, the leading terms in the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ are

$$\binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor} v^{\lfloor \frac{n}{e} \rfloor} \text{ and } \binom{n-2\lfloor \frac{n}{e} \rfloor - 1}{m - \lfloor \frac{n}{e} \rfloor - 1} v^{\lfloor \frac{n}{e} \rfloor},$$

respectively. Observe that the graded dimension of $S_{((n-m),(1^m))}$ has $2\lfloor \frac{n}{e} \rfloor + 2$ terms, whereas the graded dimension of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ both have $2\lfloor \frac{n}{e} \rfloor + 1$ terms. Hence, $x = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor = y + 1$ or $y = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor = x + 1$. Observing that the leading coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m}}$ are equal, and the trailing coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m+1}}$ are equal, the former case holds.

3. Let $n - \lfloor \frac{n}{e} \rfloor \leqslant m \leqslant n - 1$. By Proposition 11.10, the leading and trailing terms in the graded dimension of $S_{((n-m),(1^m))}$ are

$$\binom{\lfloor \frac{n}{e} \rfloor + 1}{n-m} v^{\left(n-m-1+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)} \text{ and } \binom{\lfloor \frac{n}{e} \rfloor}{n-m} v^{\left(m-n-1+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)},$$

respectively. By Proposition 12.4, the leading terms in the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ are

$$\binom{\lfloor \frac{n}{e} \rfloor}{n-m-1} v^{(n-m-1)} \text{ and } \binom{\lfloor \frac{n}{e} \rfloor}{n-m} v^{(n-m)},$$

respectively. The graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m+1}}$ both have $2n - 2m + 1$ terms, and hence $y = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor - 1$. Observing that the sum of the leading coefficients in the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ equals the leading coefficient in the graded dimension of $S_{((n-m),(1^m))}$, we have $x = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor$.

□

Example 13.7. *Let $e = 3$, $\kappa = (0, 2)$. Then the submatrix with rows corresponding to*

Specht modules labelled by hook bipartitions is

$$
\begin{array}{c}
S_{((7),\varnothing)} \\ S_{((6),(1))} \\ S_{((5),(1^2))} \\ S_{((4),(1^3))} \\ S_{((3),(1^4))} \\ S_{((2),(1^5))} \\ S_{((1),(1^6))} \\ S_{(\varnothing,(1^7))}
\end{array}
\left(\begin{array}{cccccccccccccc|c}
1 & & & & & & & & & & & & & & \\
 & v & 1 & & & & & & & & & 0 & & & \\
 & & & v & 1 & & & & & & & & & & \\
 & & & & & v^2 & v & & & & & & & & 0 \\
 & & & & & & & v^3 & v^2 & & & & & & \\
 & & & & & & & & & v^3 & v^2 & & & & \\
 & 0 & & & & & & & & & & v^4 & v^3 & & \\
 & & & & & & & & & & & & & v^4 &
\end{array}\right).
$$

13.4 Case IV: $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$. Recall from Theorem 9.14 that $D_{\mu_{n,2m}}$, $D_{\mu_{n,2m+2}}$, $D_{\mu_{n,2m+1}}$ and $D_{\mu_{n,2m+3}}$ are the ungraded composition factors of $S_{((n-m),(1^m))}$, for $2 \leqslant m \leqslant n-2$; $S_{((n-1),(1))}$ and $S_{((1),(1^n))}$ both have three composition factors. Hence as graded $\mathscr{R}_n^\Lambda$-modules, $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n,2m}}\langle i_1\rangle$, $D_{\mu_{n,2m+2}}\langle i_2\rangle$, $D_{\mu_{n,2m+1}}\langle i_3\rangle$ and $D_{\mu_{n,2m+3}}\langle i_4\rangle$, for some $i_1, i_2, i_3, i_4 \in \mathbb{Z}$, which we now determine.

Firstly, one observes that the graded dimension of $D_{\mu_{n,2m}}$ equals the graded dimension of $D_{\mu_{n,2m+3}}$, under a grading shift.

Lemma 13.8. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$. For all $1 \leqslant m \leqslant n-2$,*

$$v^2[S_{((n-m),(1^m))} : D_{\mu_{n,2m+3}}]_v = [S_{((n-m),(1^m))} : D_{\mu_{n,2m}}]_v.$$

Proof. Follows immediately from Proposition 12.6. □

Theorem 13.9. *Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$. Then*

1. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m}}]_v = v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor + 1\right)}$, *for $1 \leqslant m \leqslant n-1$;*
2. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m+2}}]_v = v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)}$, *for $1 \leqslant m \leqslant n-2$;*
3. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m+1}}]_v = v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)}$, *for $2 \leqslant m \leqslant n-2$;*
4. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m+3}}]_v = v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor - 1\right)}$, *for $1 \leqslant m \leqslant n-2$.*

Proof. 1. We have $D_{\mu_{n,5}} \cong \ker(\gamma_2)/\operatorname{im}(\phi_2)$, $D_{\mu_{n,4}} \cong \operatorname{im}(\phi_2)$ and $D_{\mu_{n,2}} \cong \operatorname{im}(\phi_1) \cong S_{((n),\varnothing)}$ as ungraded $\mathscr{R}_n^\Lambda$-modules.

By Lemma 13.8, $v^2[S_{((n-1),(1))} : D_{\mu_{n,5}}]_v = [S_{((n-1),(1))} : D_{\mu_{n,2}}]_v$. So we determine $x, y \in \mathbb{Z}$ where

$$\operatorname{grdim}(S_{((n-1),(1))}) = v^x \operatorname{grdim}(D_{\mu_{n,2}}) + v^y \operatorname{grdim}(D_{\mu_{n,4}}) + v^{x-2} \operatorname{grdim}(D_{\mu_{n,5}}).$$

By Proposition 11.11 and Proposition 12.6, we have

$$\begin{aligned}\operatorname{grdim}(S_{((n-1),(1))}) &= \frac{n}{e}v^2 + \frac{(e-2)n}{e}v + \frac{n}{e}\\ &= v^x + v^y\left(\frac{n-e}{e}v + \frac{(e-2)n}{e} + \frac{n-e}{e}v^{-1}\right) + v^{x-2}.\end{aligned}$$

Thus, by equating terms, $y = 1 = x - 1$.

2. By Lemma 13.8, $v^2[S_{((n-m),(1^m))} : D_{\mu_{n,2m+3}}]_v = [S_{((n-m),(1^m))} : D_{\mu_{n,2m}}]_v$. holds for $2 \leqslant m \leqslant n-2$. So we determine $x, y, z \in \mathbb{Z}$ where

$$\begin{aligned}\operatorname{grdim}\left(S_{((n-m),(1^m))}\right) =& v^x \operatorname{grdim}(D_{\mu_{n,2m}}) + v^y \operatorname{grdim}(D_{\mu_{n,2m+1}})\\ &+ v^z \operatorname{grdim}(D_{\mu_{n,2m+2}}) + v^{x-2} \operatorname{grdim}(D_{\mu_{n,2m+3}}).\end{aligned}$$

(a) Firstly, let $2 \leqslant m \leqslant \frac{n}{e}$. By the first part of Proposition 11.11, the first two leading terms of $\operatorname{grdim}\left(S_{((n-m),(1^m))}\right)$ are

$$\binom{\frac{n}{e}}{m} v^{\left(m + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)} \text{ and } \frac{(e-2)n}{e}\binom{\frac{n}{e}}{m-1} v^{\left(m-1+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)},$$

respectively, and the last two trailing terms of $\operatorname{grdim}\left(S_{((n-m),(1^m))}\right)$ are

$$\frac{(e-2)n}{e}\binom{\frac{n}{e}}{m-1} v^{\left(1-m+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)} \text{ and } \binom{\frac{n}{e}}{m} v^{\left(-m+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)},$$

respectively. By Proposition 12.6, the first two leading terms of the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+2}}$ are

$$\binom{\frac{n-e}{e}}{m-1} v^{m-1}, \quad \frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{m-2} v^{m-2}$$

and

$$\binom{\frac{n-e}{e}}{m} v^{m}, \quad \frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{m-1} v^{m-1},$$

respectively, and the first two leading terms of $D_{\mu_{n,2m+1}}$ and $D_{\mu_{n,2m+3}}$ are

$$\binom{\frac{n-e}{e}}{m-2} v^{m-2}, \quad \frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{m-3} v^{m-3}$$

and

$$\binom{\frac{n-e}{e}}{m-1} v^{m-1}, \frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{m-2} v^{m-2},$$

respectively.

The graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m+2}}$ both have $2m+1$ terms, and hence $z = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor$.

Observe that the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+3}}$ both have $2m-1$

terms, so together with Lemma 13.8, $x = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor + 1$.

Clearly, $-2 \leqslant y - \lfloor \frac{m}{e} \rfloor - \lfloor \frac{m+e-1}{e} \rfloor \leqslant 2$. Now observe that the sum of the second leading coefficients in the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+2}}$ form the second leading coefficient in the graded dimension of $S_{((n-m),(1^m))}$. Also, the sum of the second trailing coefficients in the graded dimensions of $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$ form the second trailing coefficient in the graded dimension of $S_{((n-m),(1^m))}$. Hence, $y = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor$.

(b) Let $\frac{n}{e} < m < \frac{n(e-1)}{e}$. By the second part of Proposition 11.11, the leading and trailing terms of grdim $S_{((n-m),(1^m))}$ are

$$\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\left(\frac{n}{e}+\lfloor \frac{m}{e} \rfloor+\lfloor \frac{m+e-1}{e} \rfloor\right)} \text{ and } \binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\left(-\frac{n}{e}+\lfloor \frac{m}{e} \rfloor+\lfloor \frac{m+e-1}{e} \rfloor\right)},$$

respectively. By Proposition 12.6, the leading terms of grdim$(D_{\mu_{n,2m}})$ and grdim $D_{\mu_{n,2m+1}}$ are, respectively,

$$\binom{\frac{(e-2)n}{e}}{\frac{em-n}{e}} v^{\frac{n-e}{e}} \text{ and } \binom{\frac{(e-2)n}{e}}{m-\frac{n}{e}-1} v^{\frac{n-e}{e}}.$$

The graded dimension of $S_{((n-m),(1^m))}$ has $\frac{2n}{e}+1$ terms, whereas $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+3}}$ both have $\frac{2n}{e}-1$ terms. Hence $x = 1 + \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor$.

Clearly, $\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor - 1 \leqslant y, z \leqslant \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor + 1$. Now, observing that the leading coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m}}$ are equal, and that the trailing coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m+3}}$ are equal, we deduce that $y = z = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor$, as required.

(c) Let $\frac{n(e-1)}{e} \leqslant m \leqslant n-2$. By the third part of Proposition 11.11, the first two leading terms in grdim $S_{((n-m),(1^m))}$ are

$$\binom{\frac{n}{e}}{n-m} v^{\left(n-m+\lfloor \frac{m}{e} \rfloor+\lfloor \frac{m+e-1}{e} \rfloor\right)} + \frac{(e-2)n}{e} \binom{\frac{n}{e}}{n-m-1} v^{\left(n-m-1+\lfloor \frac{m}{e} \rfloor+\lfloor \frac{m+e-1}{e} \rfloor\right)},$$

respectively, and the last two trailing terms in grdim $S_{((n-m),(1^m))}$ are

$$\frac{(e-2)n}{e} \binom{\frac{n}{e}}{n-m-1} v^{\left(1-n+m+\lfloor \frac{m}{e} \rfloor+\lfloor \frac{m+e-1}{e} \rfloor\right)}, \binom{\frac{n}{e}}{n-m} v^{\left(m-n+\lfloor \frac{m}{e} \rfloor+\lfloor \frac{m+e-1}{e} \rfloor\right)}.$$

By Proposition 12.6, the first two terms in the graded dimension of $D_{\mu_{n,2m}}$ are

$$\binom{\frac{n-e}{e}}{n-m-1} v^{n-m-1} \text{ and } \frac{(e-2)n}{e} \binom{\frac{n-2}{e}}{n-m-2} v^{n-m-2},$$

respectively, and the first two terms in the graded dimension of $D_{\mu_{n,2m+1}}$ are

$$\binom{\frac{n-e}{e}}{n-m} v^{n-m} \text{ and } \frac{(e-2)n}{e}\binom{\frac{n-e}{e}}{n-m-1} v^{n-m-1},$$

respectively.

Since the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,2m+1}}$ both have $2n-2m+1$ terms, $y = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor$.

Observe that the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+3}}$ both have $2n-2m-1$ terms, so together with $v^2[S_{((n-m),(1^m))} : D_{\mu_{n,2m+3}}]_v = [S_{((n-m),(1^m))} : D_{\mu_{n,2m}}]_v$, $x = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor + 1$.

Clearly, $-2 \leqslant z - \lfloor \frac{m}{e} \rfloor - \lfloor \frac{m+e-1}{e} \rfloor \leqslant 2$. One observes that the sum of the second leading coefficients in the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ form the second leading coefficient in the graded dimension of $S_{((n-m),(1^m))}$. Similarly, the sum of the second trailing coefficients in the graded dimensions of $D_{\mu_{n,2m+1}}$ and $D_{\mu_{n,2m+3}}$ equal the second trailing coefficient in the graded dimension of $S_{((n-m),(1^m))}$. Hence $z = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor$.

3. By Lemma 8.2, we know $S_{(\varnothing,(1^n))} \cong D_\lambda$ as ungraded $\mathscr{R}_n^\Lambda$-modules, where

$$\lambda = \begin{cases} ((\{n-l\}),(1^l)) & \text{if } n \geqslant l \\ (\varnothing,(1^n)) & \text{if } n < l. \end{cases}$$

We observe that $\operatorname{grdim}(D_{\mu_{n,2n-2}}) = \operatorname{grdim}(S_{(\varnothing,(1^n))}) = 1$, where $D_{\mu_{n,2n-2}} \cong \operatorname{im}(\phi_{n-1})$ as ungraded $\mathscr{R}_n^\Lambda$-modules. We know that $\operatorname{im}(\phi_{n-1})$ is spanned by $\psi_1\psi_2\ldots\psi_m z_{((1),(1^{n-1}))}$, and $S_{(\varnothing,(1^n))}$ is spanned by $z_{((1),(1^{n-1}))}$. One finds that

$$\deg(\psi_1\psi_2\ldots\psi_m z_{((1),(1^{n-1}))}) = 2\lfloor \tfrac{m}{e} \rfloor + 2 = \deg(z_{((1),(1^{n-1}))}) + 2,$$

so

$$v^2[S_{((1),(1^{n-1}))} : S_{(\varnothing,(1^n))}]_v = [S_{((1),(1^{n-1}))} : \operatorname{im}(\phi_{n-1})]_v,$$

and hence $v^2[S_{((1),(1^{n-1}))} : D_\lambda]_v = [S_{((1),(1^{n-1}))} : D_{\mu_{n,2m-2}}]_v$. Thus we determine $x, y \in \mathbb{Z}$ where

$$\begin{aligned}&\operatorname{grdim}(S_{((1),(1^{n-1}))})\\ &= v^x \operatorname{grdim}(D_{\mu_{n,2n-2}}) + v^y \operatorname{grdim}(\ker(D_{\mu_{n,2n-1}}) + v^{x-2} \operatorname{grdim}(D_\lambda).\end{aligned}$$

By Proposition 11.11 and Proposition 12.6, we have

$$\begin{aligned}&\operatorname{grdim}(S_{((1),(1^{n-1}))}\\ &= \binom{n}{e} v^{\left(1+\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)} + \left(\frac{(e-2)n}{e}\right) v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor\right)} + \binom{n}{e} v^{\left(\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor - 1\right)}\end{aligned}$$

$$= v^x + v^y \left(\frac{n-e}{e} v + \frac{(e-2)n}{e} + \frac{n-e}{e} v^{-1} \right) + v^{x-2}.$$

Equating terms, we deduce that $y = \lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor = x - 1$.

□

Example 13.10. *Let $e = 3$, $\kappa = (0, 2)$. Then the submatrix with rows corresponding to Specht modules labelled by hook bipartitions is*

$$\begin{array}{r} S_{((6),\varnothing)} \\ S_{((5),(1))} \\ S_{((4),(1^2))} \\ S_{((3),(1^3))} \\ S_{((2),(1^4))} \\ S_{((1),(1^5))} \\ S_{(\varnothing,(1^6))} \end{array} \left(\begin{array}{cccccccccc|c} 1 & & & & & & & & & & \\ v^2 & v & 1 & & & & & & 0 & & \\ & v^2 & v & v & 1 & & & & & & \\ & & & v^3 & v^2 & v^2 & v & & & & 0 \\ & & & & & v^4 & v^3 & v^3 & v^2 & & \\ & 0 & & & & & & v^4 & v^3 & v^2 & \\ & & & & & & & & & v^4 & \end{array} \right).$$

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