Metric Number Theory: the good and the bad

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Abstract

Each aspect of this thesis is motivated by the recent paper of Beresnevich, Dickinson and Velani [BDV03]. Let ψ be a real, positive, decreasing function i.e. an approximation function. Their paper considers a general lim sup set $\Lambda(\psi)$, within a compact metric measure space (Ω, d, m), consisting of points that sit in infinitely many balls each centred at an element R_α of a countable set and of radius $\psi(\beta_\alpha)$ where β_α is a 'weight' assigned to each R_α. The classical set of ψ-well approximable numbers is the basic example. For the set $\Lambda(\psi)$, [BDV03] achieves m-measure and Hausdorff measure laws analogous to the classical theorems of Khintchine and Jarník. Our first results obtain an application of these metric laws to the set of ψ-well approximable numbers with restricted rationals, previously considered by Harman [Har88c].

Next, we consider a generalisation of the set of badly approximable numbers, **Bad**. For an approximation function ρ, a point x of a compact metric space is in a general set $\mathbf{Bad}(\rho)$ if, loosely speaking, x 'avoids' any ball centred at an element R_α of a countable set and of radius $c\,\rho(\beta_\alpha)$ for $c = c(x)$ a constant. In view of Jarník's 1928 result that $\dim \mathbf{Bad} = 1$, we aim to show the general set $\mathbf{Bad}(\rho)$ has maximal Hausdorff dimension.

Finally, we extend the theory of [BDV03] by constructing a general lim sup set dependent on two approximation functions, $\Lambda(\psi_1, \psi_2)$. We state a measure theorem for this set analogous to Khintchine's (1926a) theorem for the Lebesgue measure of the set of (ψ_1, ψ_2)-well approximable pairs in $\mathbb{R}^2$. We also remark on the set's Hausdorff dimension.

Contents

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"... A small, shy truth arrives. Arrives from without and within. Arrives and is born. Simple, steady, clear. Like a mirror, like a bell, like a flame."

Leunig, 1990.

The results in this thesis, other than those clearly marked, are the unaided work of the author.

Signed: R E Tho

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Chapter 1

Introduction

The set of rational numbers is countable and the set of irrational numbers is uncountable.

The combination of these beautifully simple facts gives way to a metric view of the approximation of irrationals by rationals. We begin by exploring this view.

* * *

Consider another basic fact. For any irrational $x \in \mathbb{R}$ and $q \in \mathbb{N}$, there exists a rational p/q such that

$$|x - p/q| < 1/q \ .$$

Thus infinitely many rationals p/q are within a distance $1/q$ of each irrational. The following less obvious fact was derived by Dirichlet in 1842 and gave birth to the modern theory of Diophantine approximation.

Dirichlet's Theorem (1842). *For every irrational $x \in \mathbb{R}$, there exist infinitely many $p/q \in \mathbb{Q}$ such that*

$$|x - p/q| < q^{-2} \ . \tag{1}$$

This theorem is a simple consequence of the pigeonhole principle [HW60, p.156]. Hurwitz showed in 1891 that the generic approximation 'factor' $1/q^2$ could only be improved for all irrationals by a constant $\varepsilon > 0$. In fact he proved that for any $\varepsilon < 1/\sqrt{5}$ there exists an irrational $x \in \mathbb{R}$ for which there exist only finitely many rationals $p/q \in \mathbb{Q}$ satisfying

$$|x - p/q| < \varepsilon\, q^{-2} .$$

In particular, for such an irrational x there exists a constant $c(x) > 0$ such that

$$|x - p/q| > c(x)\, q^{-2} \qquad \forall\, p/q \in \mathbb{Q} .$$

Irrational numbers displaying this behaviour are termed *badly approximable numbers*. The set of badly approximable numbers in $\mathbb{R}$ will be denoted by **Bad**.

It is a classical fact that an irrational is badly approximable if and only if its partial quotients associated with its continued fraction expansion are bounded [HW60, p.166]. Consequently, the set **Bad** of badly approximable numbers is uncountable. In particular, any quadratic irrational is in **Bad** since its continued fraction expansion is periodic.

Irrational numbers are also associated with the notion of ψ-well approximable. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a real, positive, decreasing function and consider the set

$$W(\psi) := \{\, x \in \mathbb{R} : |x - p/q| < \psi(q) \text{ for i.m. } p/q \in \mathbb{Q}, (p,q) = 1 \,\}.$$

Here and throughout this text 'i.m.' denotes infinitely many and the above function ψ will be refered to as an *approximation function*. An element of the set $W(\psi)$ is said to be a *ψ-well approximable number*.

In view of Hurwitz's result, $W(\varepsilon\, q^{-2}) = \mathbb{R}\setminus\mathbb{Q}$ for $\varepsilon \geq 1/\sqrt{5}$; i.e., for such an ε, all irrationals are $(\varepsilon\, q^{-2})$–well approximable. The following fundamental result provides a simple criterion for the 'size' of the set $W(\psi)$ expressed in terms of one dimensional Lebesgue measure $|\,.\,|$.

Khintchine's Theorem (1924). *Let ψ be an approximation function. Then, with respect to Lebesgue measure,*

$$W(\psi) \;\; is \;\; \begin{cases} \text{NULL} \;\; if \;\; \sum_{q\in\mathbb{N}} q\,\psi(q) \; < \infty \\ \\ \text{FULL} \;\; if \;\; \sum_{q\in\mathbb{N}} q\,\psi(q) \; = \infty \end{cases} .$$

Here 'full' means that the complement of the set under consideration is of 'zero' measure. Thus the Lebesgue measure of the set of ψ–well approximable numbers in $\mathbb{R}$ satisfies a 'null-full' law. The divergence part of the above statement constitutes the main substance of the theorem. The convergence part is a simple consequence of the convergent part of the Borel–Cantelli lemma from probability theory, stated in Chapter 2. To illustrate the subtlety of Khintchine's theorem, for $\eta \geq 0$ consider the following function

$$\psi_\eta : r \mapsto r^{-2}\,(\log r)^{-(1+\eta)} .$$

Khintchine's theorem implies that, with respect to Lebesgue measure,

$$W(\psi_\eta) \;\; \text{is} \;\; \begin{cases} \text{NULL} \;\; \text{if} \;\; \eta > 0 \\ \\ \text{FULL} \;\; \text{if} \;\; \eta = 0 \end{cases} .$$

Returning to the set of badly approximable numbers, it is easily verified that

$$\mathbf{Bad} \subset \mathbb{R}\setminus W(\psi_0) .$$

Then in view of the previous statement we have that

$$|\,\mathbf{Bad}\,| = 0 \ .$$

Thus, **Bad** is an uncountable set of Lebesgue measure zero. Nevertheless, it is a 'large' set in that it is of maximal dimension:

Jarník's Theorem (1928).

$$\dim \mathbf{Bad} = 1 = \dim \mathbb{R} \ .$$

Here $\dim X$ refers to the Hausdorff dimension of a set X. This form of dimension is derived from the Hausdorff measure of the set X. Hausdorff measure is a more delicate measure than, for example Lebesgue measure, in that it can make a distinction between sets of zero Lebesgue measure. It is therefore a very useful tool for describing the measure theoretic properties of a set and one which we shall exploit in this thesis. In a moment we survey Hausdorff measure and dimension results for ψ–well approximable numbers. First we briefly introduce these concepts and define some necessary notation.

For a non-empty set $F \subset \Omega$, we say that a countable collection $\{B_i\}_{i\in\mathbb{N}}$ of balls $B_i \subset \Omega$, with radii $r_i \leq \rho \ \forall i$, forms a *$\rho$-cover of F* if $F \subset \cup_i B_i$. Let a *dimension function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing, continuous function such that $f(r) \to 0$ as $r \to 0$. For any $\rho > 0$, define

$$\mathcal{H}^f_\rho(F) := \inf \left\{ \sum_i f(r_i) \ : \ \{B_i\} \ \text{is a} \ \rho - \text{cover of} \ F \right\}$$

where the infimum is over all ρ-covers. Then the *Hausdorff f-measure*, $\mathcal{H}^f(F)$, of F is

$$\mathcal{H}^f(F) := \lim_{\rho\to 0} \mathcal{H}^f_\rho(F) \ .$$

For $f(r) = r^s$ with $s \geq 0$, the measure is denoted $\mathcal{H}^s(F)$ and refered to as *s-dimensional Hausdorff measure.* It is easily verified that the quantity $\mathcal{H}^s(F)$ 'jumps' from infinity to zero as s increases. The value of s at this transition is defined as the *Hausdorff dimension* of the set F as follows

$$\dim F := \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

For any subset F of $\mathbb{R}^n$, the n-dimensional Hausdorff measure of F is within a constant multiple of the n-dimensional Lebesgue measure of F. This implies that, if a subset F of $\mathbb{R}^n$ has positive n-dimensional Lebesgue measure, then $\dim F = n$. This is what one would expect of any basic definition of dimension. Moreover, Hausdorff dimension has the useful property that it can assign non-integer values to 'non-regular' sets such as fractals or the set **Bad** as Jarník discovered. For a good exploration of Hausdorff measure and dimension see [Fal90].

Returning to Khitchine's theorem, if the approximation function ψ decreases sufficiently rapidly so that the 'volume' sum $\sum_{q\in\mathbb{N}} q\,\psi(q)$ converges, the corresponding set of ψ-well approximable numbers is of zero Lebesgue measure. In this case we cannot obtain any further information regarding the 'size' of $W(\psi)$ in terms of Lebesgue measure. Intuitively, the 'size' of $W(\psi)$ should decrease as the 'speed' or equivalently the 'rate' of approximation, governed by ψ, increases. To illustrate this, for $\tau > 0$ consider the approximation function $\psi : r \mapsto r^{-\tau}$ and write $W(\tau)$ for $W(\psi)$. In view of the convergence part of Khintchine's theorem we have that $|W(\tau)| = 0$ for any $\tau > 2$. However, we'd expect the set $W(8)$ to be 'smaller' than $W(4)$, for example. This is therefore a good test for demonstrating the capability of Hausdorff measure and dimension.

First we turn to the Hausdorff dimension aspect. Jarník in 1929 and independently Besicovitch in 1934, established the following precise statement regarding the size of the set $W(\tau)$ expressed in terms of its Hausdorff dimension.

Jarník–Besicovitch Theorem . *For* $\tau \geq 2, \quad \dim W(\tau) = 2/\tau$.

Thus our intuition is confirmed: the more rapid the approximation, the larger τ and so the smaller the Hausdorff dimension of the set $W(\tau)$. In particular

$$\dim W(4) \;=\; 1/2 \;>\; \dim W(8) \;=\; 1/4 \,.$$

In view of the above Theorem, it follows directly from the definition of s–dimensional Hausdorff measure $\mathcal{H}^s$ that

$$\mathcal{H}^s\left(W(\tau)\right) \;=\; \begin{cases} 0 & \text{if } \; s \;>\; 2/\tau \\ \infty & \text{if } \; s \;<\; 2/\tau \end{cases} .$$

However, we're given no information regarding the s–dimensional Hausdorff measure of $W(\tau)$ at the critical value $s = \dim W(\tau)$. The next result not only addresses this but is a natural extension of Khintchine's theorem to general Hausdorff measures $\mathcal{H}^f$.

Jarník's Theorem (1931). *Let f be a dimension function such that $r^{-1} f(r) \to \infty$ as $r \to 0$ and $r^{-1} f(r)$ is decreasing. Let ψ be an approximation function. Then*

$$\mathcal{H}^f\left(W(\psi)\right) \;=\; \begin{cases} 0 & \text{if } \; \sum_{r=1}^{\infty} r\, f\left(\psi(r)\right) \; < \infty \\[2ex] \infty & \text{if } \; \sum_{r=1}^{\infty} r\, f\left(\psi(r)\right) \; = \infty \end{cases} .$$

To be precise, in Jarník's original statement various additional hypotheses on f and ψ were assumed but they would prevent us from stating the above clear cut version which is clearly a precise Hausdorff measure analogue of Khintchine's theorem. The above form of Jarník's theorem can be found in [BDV03].

Note that in the case when $\mathcal{H}^f$ is the standard s–dimensional Hausdorff measure $\mathcal{H}^s$ (i.e. $f(r) = r^s$), it follows from the definition of Hausdorff dimension and Jarník's 1931 theorem that

$$\dim W(\psi) = \inf\{ s : \textstyle\sum_{r=1}^{\infty} r\,\psi(r)^s < \infty\} .$$

Clearly, Jarník's zero–infinity law implies the Jarník–Besicovitch theorem and moreover, implies that for $\tau > 2$

$$\mathcal{H}^{2/\tau}(W(\tau)) = \infty .$$

Furthermore, the 'zero–infinity' law allows us to discriminate between sets with the same dimension and even the same s–dimensional Hausdorff measure. This special property of Hausdorff f-measure is illuminated by the following important example. Let $\tau \geq 2$ and $0 < \epsilon_1 < \epsilon_2$ and consider the approximation functions

$$\psi_{\epsilon_i}(r) := r^{-\tau}\,(\log r)^{-\frac{\tau}{2}(1+\epsilon_i)} \qquad (i = 1, 2) .$$

It is easily verified that for any $\epsilon_i > 0$,

$$m(W(\psi_{\epsilon_i})) = 0 , \qquad \dim W(\psi_{\epsilon_i}) = 2/\tau \qquad \text{and} \qquad \mathcal{H}^{2/\tau}(W(\psi_{\epsilon_i})) = 0 .$$

However, consider the dimension function f given by

$$f(r) := r^{2/\tau}\left(\log r^{-\frac{1}{\tau}}\right)^{\epsilon_1} .$$

Then it is easily verified that

$$\sum_{r=1}^{\infty} r\, f\left(\psi_{\epsilon_i}(r)\right) \asymp \sum_{r=1}^{\infty} \frac{1}{r\,(\log r)^{1+\epsilon_i-\epsilon_1}} ,$$

where as usual the symbol $\asymp$ denotes comparability (the quotient of the associated quantities is bounded from above and below by positive, finite constants). Hence, Jarník's zero–infinity law implies that

$$\mathcal{H}^f\left(W(\psi_{\epsilon_1})\right) = \infty \qquad \text{whilst} \qquad \mathcal{H}^f\left(W(\psi_{\epsilon_2})\right) = 0 .$$

Thus the Hausdorff measure $\mathcal{H}^f$ does indeed distinguish between the 'sizes' of the sets under consideration; unlike s–dimensional Hausdorff measure.

Within this classical setup, it is apparent that Khintchine's theorem together with Jarník's zero–infinity law provide a complete measure theoretic description of the set $W(\psi)$ of ψ–well approximable numbers.

Recently, an extremely general framework has been developed by Beresnevich, Dickinson and Velani [BDV03] in which they consider a natural class of lim sup subsets of a given compact metric space. Intuitively, such lim sup sets consist of points in the space which are 'ψ–well approximated' by some countable set. The countable set plays the role of the rationals in the classical set $W(\psi)$. In this paper they obtain strong measure theoretic results for these general lim sup sets. In particular, they incorporate both Khintchine's theorem and Jarník's (1931) theorem. Moreover, their results can be applied to a wide spectra of problems both within and beyond number theory. The framework and results of [BDV03] will be discussed in Chapter 2.

The work of Beresnevich, Dickinson and Velani has been the main motivation for the problems considered in this thesis. Three main problems are addressed.

- *Is it possible to apply the results of [BDV03] to obtain a general metric theory for the problem of Diophantine approximation with restricted rationals ?*

- *Is it possible to place the set* **Bad** *of badly approximable numbers within a general framework analogous to the framework developed in [BDV03] for well approximable numbers ?*

- *Is it possible to further generalize the results of [BDV03] to incorporate approximation by convex bodies rather than simply balls within a metric space ?*

Regarding the first of these problems, in Chapter 3 we obtain precise analogues of Khintchine's theorem and Jarník's 1931 theorem for the set of ψ-well approximable numbers arising from rationals whose numerators and denominators are restricted to primes. The former re-establishes a result of Harman [Har88c] but the latter is new.

Regarding the second of these problems, in Chapter 4 we consider 'natural' classes of badly approximable subsets of a compact metric space Ω. Loosely speaking, these consist of points in Ω which 'stay clear' of some given countable set of points. The classical set **Bad** of 'badly approximable' numbers in the theory of Diophantine approximation falls within our framework as do the sets $\mathbf{Bad}(i,j)$ of simultaneously badly approximable numbers. Under various natural conditions we prove that the badly approximable subsets of Ω have full Hausdorff dimension. Applications of our general framework include those from number theory and dynamical systems.

Regarding the last of these problems, in Chapter 5 we define general lim sup subsets of a compact metric space which are dependent on two approximation functions. In doing so, we generalize the work of [BDV03] by essentially replacing 'balls' with 'rectangles'. We obtain a general analogue of Khintchine's Theorem for the measure of this lim sup set and conjecture its Hausdorff dimension.

Although Chapters 3 and 5 are both concerned with well approximable sets and so could be read consecutively, Chapter 4 on badly approximable sets appears in between. The reason for this is that it forms part of a natural progression in the global development of the subject. For example, it was the latter theorems of Chapter 4 which inspired the work of Chapter 5.

Chapter 2

lim sup sets

2.1 Basic lim sup sets

In this chapter we describe firstly the classical results on basic lim sup sets and secondly recent results on a certain class of lim sup sets motivated by the study of Diophantine approximation. These results will be used in Chapters 3 and 5.

Let (Ω, d) denote a metric space. For a sequence of sets $A_n \subset \Omega$, the set of points in infinitely many A_n is known as the lim sup set of the sequence $\{A_n\}_{n\in\mathbb{N}}$. This is equivalent to the following definition,

$$\limsup_{n\to\infty} A_n \;:=\; \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n \; .$$

If a point is in infinitely many A_n then it is in every union $\bigcup_{n=m}^{\infty} A_n$ as m increases, i.e. it's in their intersection. Conversely points in only finitely many A_n are not in the union $\bigcup_{n=N}^{\infty} A_n$ for some N large enough, and so not in the intersection.

We can apply this definition to reformulating the classical set of ψ–well approximable numbers. Let $B(c, r)$ be an open interval centred at $c \in \mathbb{R}$ and of radius $r \in \mathbb{R}^+$. For an integer $q \geq 1$, let

$$A_q(\psi) := \bigcup_{p \in \mathbb{Z}} B(\tfrac{p}{q}, \psi(q)) \ .$$

Then $W(\psi) := \{x \in \mathbb{R} : x \in A_q(\psi) \text{ for i.m. } q \in \mathbb{N}\}$ or equivalently

$$W(\psi) \ = \ \limsup_{q \to \infty} A_q(\psi) \ := \ \bigcap_{m=1}^{\infty} \bigcup_{q=m}^{\infty} A_q(\psi) \ .$$

For a measure m, let (Ω, d, m) be a metric measure space. The following fundamental result determines when a lim sup set has zero m-measure.

First Borel – Cantelli Lemma. *Let (Ω, d, m) be a measure space with $m(\Omega)$ finite and let $A_n \subset \Omega$ be a sequence of m-measurable sets, then*

$$m\left(\limsup_{n \to \infty} A_n\right) = 0 \qquad \text{if} \qquad \sum_{n=1}^{\infty} m(A_n) < \infty \ .$$

Proof. For any $N \geq 1$,

$$m\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n\right) \ \leq \ m\left(\bigcup_{n=N}^{\infty} A_n\right) \ \leq \ \sum_{n=N}^{\infty} m(A_n) \, .$$

By assumption, $\sum_{n=N}^{\infty} m(A_n) \to 0$ as $N \to \infty$. Thus, by the above definition of lim sup sets, $m\big(\limsup_{n \to \infty} A_n\big) = 0$.

□

The first Borel–Cantelli lemma therefore implies the convergence part of Khintchine's Theorem as follows. Let $\Omega := [0,1]$ and

$$A'_q(\psi) \ := \ \bigcup_{0 < p \leq q} B(\tfrac{p}{q}, \psi(q)) \ \cap \ [0,1] \ .$$

Then

$$\sum_{q=1}^{\infty} |\, A'_q(\psi) \,| \ \ll \ \sum_{q=1}^{\infty} q\,\psi(q)\ . \tag{2}$$

Thus, by the lemma, if the r.h.s. of (2) converges then

$$|\, W(\psi) \cap [0,1] \,| = 0\ .$$

The same result is true when the unit interval is replaced by any interval of unit length. This implies that $W(\psi)$ is a countable union of zero measure sets which in turn implies $|\, W(\psi) \,| = 0$.

Determining the divergence part of Khintchine's Theorem or even positive m-measure for any lim sup set is a much harder task. Considering the above lemma, a natural, but not sufficient, condition is $\sum_{n=1}^{\infty} m(A_n) = \infty$. If satisfied the following lemma can also be deployed.

Second Borel – Cantelli Lemma. *Let $(\Omega, \mathcal{A}, m)$ be a probability space and $A_n \in \mathcal{A}$ be a sequence of m-measurable sets such that $\sum_{n=1}^{\infty} m(A_n) = \infty$. Then*

$$m\left(\limsup_{n\to\infty} A_n\right) \ \geq \ \limsup_{Q\to\infty} \frac{\left(\sum_{s=1}^{Q} m(A_s)\right)^2}{\sum_{s,t=1}^{Q} m(A_s \cap A_t)}\ .$$

This result is actually a generalisation of the second Borel–Cantelli lemma. For proof of the above lemma see Lemma 5 in [Spr79]. If the right-hand side of the above inequality is positive then it obviously follows that the lim sup set has positive measure.

When considering the nature of the sum $\sum m(A_n)$, the following trivial fact turns out to be very useful.

Fact 1. Suppose that $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a real, positive, decreasing function, $\alpha \in \mathbb{R}$ and $k > 1$. Then the divergence and the convergence of

$$\sum_{n=1}^{\infty} k^{n\alpha}\, h(k^n) \quad \text{and} \quad \sum_{r=1}^{\infty} r^{\alpha-1}\, h(r) \quad \text{coincide.}$$

Proof. Fix $\alpha \in \mathbb{R}$ and $k > 1$. Then

$$\sum_{r=1}^{\infty} r^{\alpha-1}\, h(r) \;=\; \sum_{n=0}^{\infty} \sum_{r=k^n}^{k^{n+1}} r^{\alpha-1}\, h(r) \;\asymp\; \sum_{n=1}^{\infty} k^{n\alpha}\, h(k^n)$$

since h is decreasing. Since the two sums are comparable, one of the sums converges or diverges if and only if the other converges or diverges respectively.

□

2.2 General well approximable sets

Recognising the set of ψ-well approximable numbers as a lim sup set, initiates the following generalisation.

Let (Ω, d) be a compact metric space. Let $\mathcal{R} := \{R_\alpha \in \Omega : \alpha \in J\}$ be a family of points $R_\alpha \in \Omega$ indexed by an infinite countable set J. We call R_α *resonant* points. Let $\beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on J thus each R_α has a 'weight' β_α. We assume that for any $\eta > 0$ the set $\{\alpha \in J : \beta_\alpha \leq \eta\}$ is finite, thus β_α tends to infinity as α runs through J.

Given an approximation function, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, let

$$\Lambda(\psi) := \{x \in \Omega : x \in B(R_\alpha, \psi(\beta_\alpha)) \text{ for i.m. } \alpha \in J\}$$

Now let $l := \{l_n\}$ and $u := \{u_n\}$ be positive increasing sequences such that $l_n < u_n$ and $l_n \to \infty$ as $n \to \infty$. Define

$$\Delta_l^u(\psi, n) := \bigcup_{\alpha \in J_l^u(n)} B(R_\alpha, \psi(\beta_\alpha)) ,$$

where

$$J_l^u(n) := \{ \alpha \in J : l_n < \beta_\alpha \leq u_n \} .$$

By assumption $\#J_l^u(n)$ is finite and $l_n \to \infty$ as $n \to \infty$, thus we can write

$$\Lambda(\psi) = \limsup_{n\to\infty} \Delta_l^u(\psi, n) := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Delta_l^u(\psi, n) . \qquad (3)$$

We call this the *general* lim sup *set.* It is easily compared to the classical set; both are lim sup sets composed from 'balls' centred at elements of a countable set with radii dependent on an approximation function. Note that the above formulation of $\Lambda(\psi)$ is irrespective of the choice of the sequences l and u.

2.3 Recent work of Beresnevich, Dickinson and Velani

The general set up of the previous section is taken from that of [BDV03]. In this recent paper Beresnevich, Dickinson and Velani layout precise laws for the m-measure and for the Hausdorff f-measure of the general lim sup set, $\Lambda(\psi)$.

As mentioned in the introduction, their work has motivated each part of this thesis. In particular their results are employed in Chapter 3 and are extended in Chapter 5. We therefore take the rest of this section to state two key results from [BDV03]. We begin with the necessary conditions including the important notion of 'ubiquity', these are taken from [BDV03, §1.3]. The first condition concerns measure.

(A) There exist positive constants δ and r_o such that for any $x \in \Omega$ and $r \leq r_o$,

$$a\, r^\delta \;\leq\; m(B(x,r)) \;\leq\; b\, r^\delta$$

where constants a and b are independent of the ball and without loss of generality we assume $0 < a < 1 < b$.

The ubiquity condition

Recall that $\mathcal{R}$ denotes a family of resonant points R_α and that β attaches a 'weight' β_α to each resonant point $R_\alpha \in \mathcal{R}$.

Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a function with $\rho(r) \to 0$ as $r \to \infty$ and let

$$\Delta_l^u(\rho, n) := \bigcup_{\alpha \in J_l^u(n)} B(R_\alpha, \rho(u_n)).$$

The following ubiquity condition essentially ensures that within any ball centred in Ω, for n large enough, there exist 'enough' resonant points R_α with $l_n < \beta_\alpha \leq u_n$.

Local m–ubiquity Let $B = B(x,r)$ be an arbitrary ball with centre $x \in \Omega$ and radius $r \leq r_o$. Suppose there exists a function ρ, sequences l and u and an absolute constant $\kappa > 0$ such that

$$m(B \cap \Delta_l^u(\rho, n)) \;\geq\; \kappa\, m(B) \qquad \text{for } n \geq n_o(B)\,. \tag{4}$$

Then the pair $(\mathcal{R}, \beta)$ is said to be *a local m–ubiquitous system relative to* (ρ, l, u).

A weaker condition of *global m-ubiquity* is satisfied if there exists ρ and u as before and an absolute constant $\kappa > 0$ such that for $n \geq n_o$, (4) is satisfied for $B := \Omega$. In both cases ρ is referred to as the *ubiquity function.*

From [BDV03] we also include the following lemma which demonstrates how the local ubiquity condition is easily satisfied for the set $W(\psi)$. For this exercise, let $\Omega := [0,1]$, $\beta : \mathbb{Z}\times\mathbb{N} \to \mathbb{N} : (p,q) \mapsto q$ and, for $n \geq 1$,

$$\Delta_l^u(\rho, n) := \bigcup_{k^{n-1}<q\leq k^n} \ \bigcup_{0<p\leq q} B(\tfrac{p}{q}, \rho(k^n)) \, .$$

The measure m is naturally one dimensional Lebesgue measure.

Lemma 1. *There exists a constant $k > 1$ such that the pair $(\mathbb{Q}, \beta)$ is a local m-ubiquitous system relative to (ρ, l, u) where $l_{n+1} = u_n := k^n$ and $\rho : r \mapsto$ constant $\times\, r^{-2}$.*

Proof. Let $I := [a,b] \subset [0,1]$. By Dirichlet's theorem, for any $x \in I$ there are coprime integers p, q with $1 \leq q \leq k^n$ satisfying $|x - p/q| < (qk^n)^{-1}$. Clearly, $aq - 1 \leq p \leq bq + 1$. Thus, for a fixed q there are at most $m(I)q + 3$ possible values of p. Trivially, for n large

$$m\left(I \cap \bigcup_{q\leq k^{n-1}} \ \bigcup_{p} B\left(\tfrac{p}{q}, \tfrac{1}{qk^n}\right)\right) \leq 2 \sum_{q\leq k^{n-1}} \tfrac{1}{qk^n}(m(I)q + 3) \leq \tfrac{3}{k}\, m(I) \, .$$

It follows that for $k \geq 6$,

$$m\left(I \cap \bigcup_{k^{n-1}<q\leq k^n} \ \bigcup_{p} B\left(\tfrac{p}{q}, \tfrac{k}{k^{2n}}\right)\right) \geq m(I) - \tfrac{3}{k}\, m(I) \geq \tfrac{1}{2}\, m(I) \, .$$

□

Remark: The above proof uses Dirichlet's theorem. An analogy of this theorem is used to prove ubiquity for several cases of different systems $(\mathcal{R}, \beta)$.

As we shall see from the following results, Lemma 1 is sufficient to prove the divergence parts of Khintchine's Theorem and Jarnik's zero-infinity law.

The following is a regularity condition on a general function and will be applied to the approximation function ψ and/or the ubiquity function ρ.

A function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *u-regular* for a given sequence u, if there exists a positive constant $\lambda < 1$ such that for n large enough

$$h(u_{n+1}) \leq \lambda \ h(u_n) \ .$$

The constant λ is independent of n but may depend on u. For an increasing sequence u, this property implies that the function is eventually strictly decreasing.

We are now in a position to state two results of [BDV03] which shall be employed later. We do not state the first in full generality nonetheless it should still be evident that these are deep and elegant results. The first result determines positive, and even full, m-measure for the lim sup set $\Lambda(\psi)$.

Corollary BDV. *Let (Ω, d) be a compact metric space equipped with a measure m satisfying condition (A). Suppose $(\mathcal{R}, \beta)$ is a global m-ubiquitous system relative to (ρ, l, u), ψ is an approximation function and either ψ or ρ is u-regular. Then*

$$m(\Lambda(\psi)) > 0 \quad \text{if} \quad \sum_{n=1}^{\infty} \left(\frac{\psi(u_n)}{\rho(u_n)} \right)^{\delta} = \infty \ . \tag{5}$$

In addition, if any open subset of Ω is m-measurable and $(\mathcal{R}, \beta)$ is a local m-ubiquitous system relative to (ρ, l, u) then $m(\Lambda(\psi)) = 1$.

Theorem 1 of [BDV03] determines the same result in more generality by not asking for the regularity condition to be met. However, for the purposes of this thesis, the above corollary will be sufficient. The second result determines infinite Hausdorff f-measure for $\Lambda(\psi)$.

Theorem BDV. *Let (Ω, d) be a compact metric space equipped with a measure m satisfying condition (A). Suppose $(\mathcal{R}, \beta)$ is a local m-ubiquitous system relative to (ρ, l, u) and that ψ is an approximation function. Let f be a dimension function such that $r^{-\delta}f(r) \to \infty$ as $r \to 0$ and $r^{-\delta}f(r)$ is decreasing. Let g be the real, positive function given by*

$$g(r) := f(\psi(r))\rho(r)^{-\delta} \quad \text{and let} \quad G := \limsup_{n\to\infty} g(u_n)\ . \tag{6}$$

(i) Suppose that $G = 0$ and that ρ is u-regular. Then,

$$\mathcal{H}^f(\Lambda(\psi)) = \infty \quad \text{if} \quad \sum_{n=1}^{\infty} g(u_n) = \infty\ . \tag{7}$$

(ii) Suppose that $0 < G \leq \infty$. Then, $\mathcal{H}^f(\Lambda(\psi)) = \infty$.

Remarks: Applications described in [BDV03] include those to Diophantine approximation with linear forms, p-adic numbers, and manifolds. Their theory also links further afield to Kleinian groups and rational maps. It is usually the case that the convergence counterpart of the above theorems is easily derived in such applications by exploiting the lim sup nature of the sets and using the first Borel–Cantelli lemma or the definition of Hausdorff measure. Thus, complete zero-one m-measure and zero-infinity Hausdorff f-measure laws are achieved for each application.

With the information of this section the reader should not need to refer to [BDV03] to understand the theorems and proofs of this thesis. However this paper is recommended reading for seeing the bigger picture. On this point we note that the theory in [BDV03] encompasses the case when R_α are not just points but sets. We do not include this more general case in our general set up since it is not used for the well approximable sets of Chapters 3 and 5.

Chapter 3

Well approximable numbers with restricted sets

3.1 Introduction and Results

Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathbb{Z}$. Consider the following general subset of the ψ–well approximable numbers in $\mathbb{R}$.

$$W_{\mathcal{A},\mathcal{B}}(\psi) := \{ \, x \in \mathbb{R} : |x - p/q| < \psi(q) \ \text{ for i.m. } \ p/q \ \text{ with } \ p \in \mathcal{A}, q \in \mathcal{B} \, \} \, .$$

Thus $\mathcal{A} := \mathbb{Z}$ and $\mathcal{B} := \mathbb{N}$ would return us to the classical set. However there are other subsets of integers of wider number theoretic interest which may be considered for either $\mathcal{A}, \mathcal{B}$ or both. These include the sets of integers which are prime, square-free, congruent to $a \, (\text{mod } q)$ or those which can be properly represented as the sum of two squares.

In a collection of papers [Har88a], [Har88b], [Har88c], [Har89], which are summarised in Chapter 6 of [Har98], Harman considers the Lebesgue measure of $W_{\mathcal{A},\mathcal{B}}(\psi)$ where $\mathcal{A}$ and $\mathcal{B}$ are selected from the above list of subsets of $\mathbb{Z}$, in various combinations.

In this chapter we focus on the case: $\mathcal{A} = \mathcal{B} := \mathcal{P}$ where $\mathcal{P}$ is the set of prime integers. Let $\mathbb{Q}(\mathcal{P}) := \{ p/q \in \mathbb{Q} : p, q \in \mathcal{P} \}$ and let

$$W_{\mathcal{P}}(\psi) := \{x \in \mathbb{R}^+ : |x - p/q| < \psi(q) \text{ for i.m. } p/q \in \mathbb{Q}(\mathcal{P})\} .$$

The results we obtain for one-dimensional Lebesgue measure and for Hausdorff f-measure are as follows.

Theorem 1. *Let ψ be an approximation function. Then, with respect to one-dimensional Lebesgue measure,*

$$W_{\mathcal{P}}(\psi) \quad \textit{is} \quad \begin{cases} \text{NULL} & \textit{if} \quad \sum_{r=1}^{\infty} \psi(r)\, r\, (\log r)^{-2} < \infty \\ \\ \text{FULL} & \textit{if} \quad \sum_{r=1}^{\infty} \psi(r)\, r\, (\log r)^{-2} = \infty \ . \end{cases}$$

Theorem 2. *Let f be a dimension function such that $r^{-1} f(r) \to \infty$ as $r \to 0$ and $r^{-1} f(r)$ is decreasing. Let ψ be an approximation function. Then*

$$\mathcal{H}^f\left(W_{\mathcal{P}}(\psi)\right) = \begin{cases} 0 & \textit{if} \quad \sum_{r=1}^{\infty} f(\psi(r))\; r\; (\log r)^{-2} < \infty \\ \\ \infty & \textit{if} \quad \sum_{r=1}^{\infty} f(\psi(r))\; r\; (\log r)^{-2} = \infty \end{cases} .$$

By the definition of Hausdorff dimension, see Chapter 1, we derive the following corollary to Theorem 2.

Corollary 2. *For $\psi(r) = r^{-\tau}$ and $\tau > 2$*

$$\dim W_{\mathcal{P}}(\psi) \;=\; \tfrac{2}{\tau} \qquad \textit{and} \qquad \mathcal{H}^{2/\tau}(W_{\mathcal{P}}(\psi)) \;=\; 0 \ .$$

Remarks: As one might expect, the log term appearing in the sums associated with each theorem is a consequence of the prime number theorem.

Note that the corollary implies that the dimension of $W_{\mathcal{P}}(\psi)$ is the same as that of $W(\psi)$. However, the $2/\tau$-dimensional Hausdorff measure of $W_{\mathcal{P}}(\psi)$ is zero whilst that of $W(\psi)$ is infinite; recall the discussion of Jarník's theorems in Chapter 1. In conclusion, restricting to primes has no effect on dimension but drastically effects the $\mathcal{H}^{2/\tau}$ measure.

Together, the above theorems provide a complete metric description of the set $W_{\mathcal{P}}(\psi)$. Theorem 1 is not new and was first established by Harman in [Har88c]. However, the unified approach taken here, namely that of ubiquity, introduced in Chapter 2, automatically gives rise to both statements.

It is probable that our approach would give rise to a complete metric theory for $W_{\mathcal{A},\mathcal{B}}(\psi)$ in the case where $\mathcal{A}$ and $\mathcal{B}$ are the number theoretic sets mentioned in the introduction above.

3.2 Proofs of Theorems 1 & 2

We work for the time being within the interval $[1,2]$ to ensure a compact metric space and begin by showing how the intersection of $W_{\mathcal{P}}(\psi)$ and the interval $[1,2]$ can be written as a lim sup set. For $n \in \mathbb{N}$, let

$$\Delta(\psi,n) := \bigcup_{\substack{2^{n-1}<q\leq 2^n \\ q\in\mathcal{P}}} \bigcup_{\substack{q<p\leq 2q \\ p\in\mathcal{P}}} B(p/q,\psi(q))$$

where $B(c,r)$ is an interval centred at $c \in [1,2]$ and of radius $r < 1$. Then we can write $W_{\mathcal{P}}(\psi) \cap [1,2]$ as the lim sup set

$$W'_{\mathcal{P}}(\psi) := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Delta(\psi,n) .$$

Since counting primes occurs at each stage of the proof, we state the following well known result.

Prime Number Theorem. *Let $\pi(x)$ denote the number of primes not exceeding x. Then*

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1 .$$

3.2.1 The convergence part of Theorem 1

Since $W'_{\mathcal{P}}(\psi)$ can be written as the lim sup set of the sequence $\{\Delta(\psi,n)\}_{n\in\mathbb{N}}$, we can apply the first Borel–Cantelli lemma – see Chapter 2 – and say that

$$|\, W'_{\mathcal{P}}(\psi)\,| = 0 \qquad \text{if} \qquad \sum_{n=1}^{\infty} |\Delta(\psi,n)| < \infty .$$

Now by the Prime Number Theorem,

$$\begin{aligned} |\,\Delta(\psi,n)\,| \;=\; \left| \bigcup_{\substack{2^{n-1}<q\leq 2^n \\ q\in\mathcal{P}}} \; \bigcup_{\substack{q<p\leq 2q \\ p\in\mathcal{P}}} B(p/q,\psi(q)) \right| \;&\leq\; \sum_{\substack{q\leq 2^n \\ q\in\mathcal{P}}} \sum_{\substack{p\leq 2q \\ p\in\mathcal{P}}} 2\,\psi(2^{n-1}) \\ &\ll\; \frac{2^{2(n-1)}}{(\log 2^{n-1})^2}\,\psi(2^{n-1}) . \end{aligned}$$

In view of this,

$$\sum_{n=1}^{\infty} |\,\Delta(\psi,n)\,| \;\ll\; \sum_{n=1}^{\infty} \frac{2^{2n}}{(\log 2^n)^2}\,\psi(2^n) . \tag{8}$$

The r.h.s. of (8) converges since $\sum r(\log r)^{-2}\psi(r)$ converges – see Fact 1. Thus the first Borel–Cantelli lemma implies that $|W'_{\mathcal{P}}(\psi)| = 0$. The result obtained is the same when $[1,2]$ is replaced by any interval of the form $[a,a{+}1]$ $(a\in\mathbb{N})$. Also note that

$$W_{\mathcal{P}}(\psi) \;=\; \bigcup_{a\in\mathbb{N}} W_{\mathcal{P}}(\psi)\cap[a,a+1] . \tag{9}$$

Since any countable union of zero-measure sets has zero measure, we conclude that $|W_{\mathcal{P}}(\psi)| = 0$.

3.2.2 The convergence part of Theorem 2

Fix $N \in \mathbb{N}$. Note that

$$\bigcup_{n=N}^{\infty} \bigcup_{\substack{2^{n-1}<q\leq 2^n \\ q\in\mathcal{P}}} \bigcup_{\substack{q<p\leq 2q \\ p\in\mathcal{P}}} B(p/q, \psi(2^{n-1}))$$

is a $\psi(2^{N-1})$-cover of $W'_{\mathcal{P}}(\psi)$. By the definition of Hausdorff f-measure – see Chapter 1 – and the prime number theorem,

$$\begin{aligned} \mathcal{H}^f(W'_{\mathcal{P}}(\psi)) &< \lim_{N\to\infty} \left(\sum_{n=N}^{\infty} \sum_{\substack{q=2^{n-1} \\ q\in\mathcal{P}}}^{2^n} \sum_{\substack{p=q \\ p\in\mathcal{P}}}^{2q} f(\psi(2^{n-1})) \right) \\ &\ll \lim_{N\to\infty} \left(\sum_{n=N}^{\infty} f(\psi(2^n)) \frac{2^{2n}}{(\log 2^n)^2} \right). \end{aligned}$$

This limit is zero since, by assumption, $\sum_{r=1}^{\infty} f(\psi(r))r(\log r)^{-2}$ converges. Hence $\mathcal{H}^f(W'_{\mathcal{P}}(\psi)) = 0$. Finally, this result remains true when the set $W_{\mathcal{P}}(\psi)$ is restricted to any other interval of type $[a, a+1]$ $(a \in \mathbb{N})$. Again $W_{\mathcal{P}}(\psi)$ is a countable union of sets with zero Hausdorff f-measure; thus $\mathcal{H}^f(W_{\mathcal{P}}(\psi)) = 0$.

3.2.3 The divergence parts of Theorems 1 & 2

The divergence parts of both theorems are derived from the two corresponding results of [BDV03] stated in §2.3. Since the main condition of both of these results is local ubiquity we tackle the rest of the proofs of Theorems 1 and 2 together. We initially work with $W'_{\mathcal{P}}(\psi) \subset [1,2]$. This set is identified with the general notation of [BDV03] described in Chapter 2 as follows. Let

$$\Omega := [1,2]\ , \quad \mathcal{R} := \mathbb{Q}(\mathcal{P})\ , \quad J := \{\,(p,q) \in \mathcal{P} \times \mathcal{P} : q < p \leq 2q\,\}\ ,$$

$$\alpha := (p,q) \in J\ , \quad \beta : (p,q) \mapsto q\ , \quad R_\alpha := p/q\ .$$

Next we verify the conditions. The space $\Omega := [1,2]$ supports 1-dimensional Lebesgue measure and so trivially satisfies the measure condition (A) with $\delta = 1$. The following lemma shows the pair $(\mathbb{Q}(\mathcal{P}), \beta)$ is a locally ubiquitous system relative to (ρ, l, u) where l and u are sequences such that $l_{n+1} = u_n := 2^n$ ($n \in \mathbb{N}$) and the ubiquity function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$\rho : r \mapsto \rho(r) \ := \ c\,\frac{(\log r)^2}{r^2}\ .$$

Note that ρ is u-regular.

Lemma 3. *Let $I \subset [1,2]$ be an arbitrary interval. There exist absolute constants $c > 0$ and $\kappa > 0$ such that, for $n \geq n_o(I)$,*

$$\left| I \cap \bigcup_{\substack{2^{n-1} < q \leq 2^n \\ q \in \mathcal{P}}} \ \bigcup_{\substack{q < p \leq 2q \\ p \in \mathcal{P}}} B\left(p/q, \rho(2^n)\right) \right| \ \geq \ \kappa\,|I|\ . \qquad (10)$$

Then, since the measure and ubiquity conditions satisfied, Corollary BDV implies

$$|W'_{\mathcal{P}}(\psi)| = 1 \quad \text{if} \quad \sum_{n=1}^{\infty} \psi(2^n)\frac{2^{2n}}{(\log 2^n)^2} = \infty\,.$$

Accordingly, Theorem BDV implies that

$$\mathcal{H}^f(W'_{\mathcal{P}}(\psi)) = \infty \quad \text{if} \quad \sum_{n=1}^{\infty} f(\psi(2^n))\frac{2^{2n}}{(\log 2^n)^2} = \infty\,.$$

Note that, since $W'_{\mathcal{P}}(\psi) \subset W_{\mathcal{P}}(\psi) \subset \mathbb{R}^+$, if $W'_{\mathcal{P}}(\psi)$ has infinite Hausdorff f-measure then the same is true for $W_{\mathcal{P}}(\psi)$. For Lebesgue measure, it can be verified that replacing the interval $[1,2]$ with any interval of type $[a, a+1]$ for $a \in \mathbb{N}$ yields the same ubiquity result and thus the same Lebesgue measure result. Therefore, by (9), $m(\mathbb{R}^+ \backslash W_{\mathcal{P}}(\psi)) = 0$ and so $W_{\mathcal{P}}(\psi)$ has full Lebesgue measure. Hence, the divergence statements of Theorems 1 and 2 are true, if Lemma 3 is true.

3.2.4 Proof of Lemma 3

Without loss of generality and for the sake of convenience we prove the lemma for $I := (1, 1+2\theta)$ where $0 < \theta < 1/2$ is arbitrary. The proof can be easily adapted for an arbitrary interval in $[1,2]$.

Throughout the remainder of this chapter we assume p, q, s, t are all prime numbers. Note that, if we restrict p so that $q < p \leq (1+\theta)q$, then

$$\frac{p}{q} \in I^* := (1, 1+\theta] \subset I$$

and moreover, for n sufficiently large, $B(p/q, \rho(2^n)) \subset I$. In view of this,

$$\text{l.h.s. of (10)} \quad \geq \quad \left| \bigcup_{2^{n-1}<q\leq 2^n} \; \bigcup_{q<p\leq(1+\theta)q} B\left(p/q, \rho(2^n)\right) \right| . \qquad (11)$$

Let $\mathcal{R}_\theta(n)$ denote the set of rationals in the above range; that is

$$\mathcal{R}_\theta(n) := \left\{ p/q \in \mathbb{Q}(\mathcal{P}) : q < p \leq (1+\theta)q, \;\; 2^{n-1} < q \leq 2^n \right\} .$$

Also, let $\mathcal{V}_\theta(n)$ denote the subset of $\mathcal{R}_\theta(n)$ such that $p/q \in \mathcal{V}_\theta(n)$ if

$$B(p/q, \rho(2^n)) \cap B(s/t, \rho(2^n)) = \emptyset \qquad \forall \;\; s/t \; (\neq p/q) \; \in \; \mathcal{R}_\theta(n) .$$

The following two lemmas are the key to establishing Lemma 3.

Lemma 4. *For n large enough,*

$$\#\mathcal{R}_\theta(n) \;>\; \frac{\theta\, 2^{2n}}{36\, n^2}\,.$$

Lemma 5. *For $0 < \theta < 1/2$ and n large enough,*

$$\#\mathcal{V}_\theta(n) \;\geq\; \tfrac{1}{2}\#\mathcal{R}_\theta(n)\,.$$

In view of Lemmas 4 and 5, it follows that for n sufficiently large

$$\begin{aligned}
\text{r.h.s. of (11)} \;&\geq\; \left| \overset{\circ}{\bigcup_{p/q\in\mathcal{V}_\theta(n)}} B\left(p/q, \rho(2^n)\right) \right| \\
&=\; \#\mathcal{V}_\theta(n) \times 2\rho(2^n) \\
&\geq\; \tfrac{1}{2}\,\#\mathcal{R}_\theta(n) \times 2\rho(2^n) \\
&\geq\; \frac{\theta c}{36}\,(\log 2)^2 \;=\; \kappa\,|I|\,,
\end{aligned}$$

where $\kappa := \frac{c}{72}(\log 2)^2 > 0$. Thus the statement of Lemma 3 is obtained and it remains to prove Lemmas 4 and 5.

3.2.5 Proof of Lemma 4

Let $0 < \theta \leq 1$. We begin by deriving a lower bound for $\pi((1+\theta)x) - \pi(x)$, where $\pi(x)$ is the number of primes $p \leq x$. In view of the prime number theorem, for any $\varepsilon > 0$ there exists a constant $x_o(\varepsilon)$ such that, for $x > x_o(\varepsilon)$,

$$1-\varepsilon \;<\; \pi(x)\frac{\log x}{x} \;<\; 1+\varepsilon\,. \tag{12}$$

So, for $\varepsilon := \frac{\theta(3-\theta)}{6(3+2\theta)}$, take x sufficiently large that $x^{\frac{\theta}{3}} > (1+\theta)$ and (12) holds. It follows that

$$\begin{aligned} \pi((1+\theta)x) \ - \ \pi(x) & \\ & > \left(\frac{(1+\theta)}{(1+\theta/3)} \left(1 - \frac{\theta(3-\theta)}{6(3+2\theta)}\right) - \left(1 + \frac{\theta(3-\theta)}{6(3+2\theta)}\right)\right) \frac{x}{\log x} \\ & = \frac{\theta}{3} \frac{x}{\log x} . \end{aligned}$$

In turn, this implies that, for n large enough,

$$\begin{aligned} \#\mathcal{R}_\theta(n) &= \sum_{2^{n-1}<q\le 2^n} \sum_{q<p\le(1+\theta)q} 1 > \frac{2^{n-1}}{3\log 2^{n-1}} \frac{\theta\, 2^{n-1}}{3\log 2^n} \\ &> \frac{\theta\, 2^{2n}}{36\, n^2} . \end{aligned}$$

This completes the proof of the Lemma 4.

3.2.6 Proof of Lemma 5

We begin by considering the set $\mathcal{U}_\theta(n) := \mathcal{R}_\theta(n) \setminus \mathcal{V}_\theta(n)$. By definition, $p/q \in \mathcal{U}_\theta(n)$ if there exists a rational s/t $(\neq p/q) \in \mathcal{R}_\theta(n)$ such that

$$B(p/q, \rho(2^n)) \cap B(s/t, \rho(2^n)) \neq \emptyset;$$

i.e. such that

$$0 < \left|\frac{p}{q} - \frac{s}{t}\right| \le 2\rho(2^n).$$

It is easily seen that the associated 4-tuple (p,q,s,t) is an element of the set

$$\begin{aligned} \mathcal{W}_\theta(n) := \{(p,q,s,t) : 0 < |pt - sq| \ \le\ \rho(2^n)\, 2^{2n+1}, & \\ 2^{n-1} < p, s \ \le\ (1+\theta)2^n,\ 2^{n-1} < q, t \le 2^n\} . & \end{aligned}$$

Thus, $\#\mathcal{U}_\theta(n) \leq \#\mathcal{W}_\theta(n)$ and the lemma immediately follows upon showing

$$\#\mathcal{W}_\theta(n) \;<\; \tfrac{1}{2}\,\#\mathcal{R}_\theta(n)\,. \tag{13}$$

In order to establish (13) we make use of the following lemma which is a slightly simplified version of Lemma 4 from [Har88c].

Harman's Lemma (1988). *Let $\theta, A, P \in \mathbb{R}^+$ be given with $0 < \theta < 1/2$, $P \geq \theta^{-2}$ and $A \geq 1$. Let p, q, s, t be prime. Then the number of solutions to*

$$0 < |pt - sq| < A$$

with

$$P < q, t < (1+\theta)P \quad \textit{and} \quad P < p, s < (1+\theta)^2 P$$

is

$$\leq \; \frac{\omega A P^2 \theta^3}{\log^4 P} \;+\; O(AP^{39/20})$$

where ω is an absolute constant.

Now let $j \geq 0$ be the unique integer such that $(1+\theta)^j \leq 2 < (1+\theta)^{j+1}$ and for $i = 0, 1, \ldots, j$ let $P_i := 2^{n-1}(1+\theta)^i$. Furthermore, let

$$\begin{aligned}\mathcal{W}_\theta(n,i) := \{(p,q,s,t) : 0 < |pt - sq| \;&\leq\; \rho(2^n).2^{2n+1},\\ P_i < p, s \;&\leq\; P_i(1+\theta)^2,\; P_i < q, t \leq P_i(1+\theta)\}.\end{aligned}$$

Then

$$\#\mathcal{W}_\theta(n) \;\leq\; \sum_{i=0}^{j} \#\mathcal{W}_\theta(n,i) \tag{14}$$

since

$$(2^{n-1}, 2^n] \subset \bigcup_{i=0}^{j} (P_i, P_i(1+\theta)] \qquad \text{and} \qquad (2^{n-1}, 2^n(1+\theta)] \subset \bigcup_{i=0}^{j} (P_i, P_i(1+\theta)^2]\;.$$

We now invoke Harman's lemma to estimate $\#\mathcal{W}_\theta(n,i)$. With $A := 2^{2n+1}\,\rho(2^n)$ and $P := P_i$, it follows that for n sufficiently large

$$\begin{aligned} \#\mathcal{W}_\theta(n,i) &\leq \omega\, 2^{2n+1}\,\rho(2^n)\,\frac{P_i^2\,\theta^3}{\log^4 P_i} \;+\; O\left(2^{2n+1}\rho(2^n)P_i^{39/20}\right) \\ &< \omega\, c\,\theta^3\,\frac{2^{2n}}{n^2}\,(1+\theta)^{2i} \;+\; O(2^n) \\ &< 2\,\omega\, c\,\theta^3\,\frac{2^{2n}}{n^2}\,(1+\theta)^{2i}\,. \end{aligned}$$

This together with (14) implies that

$$\begin{aligned} \#\mathcal{W}_\theta(n) &\leq 2\,c\,\omega\,\theta^3\,\frac{2^{2n}}{n^2}\,\sum_{i=0}^{j}(1+\theta)^{2i} \\ &< 16\,c\,\omega\,\theta\,\frac{2^{2n}}{n^2}\,. \end{aligned}$$

By choosing $c := \frac{1}{1152\omega}$, we ensure via Lemma 4 that

$$\#\mathcal{W}_\theta(n) \;<\; \frac{1}{2}\,\frac{\theta\,2^{2n}}{36\,n^2} \;<\; \frac{1}{2}\,\#\mathcal{R}_\theta(n)\,.$$

This completes the proof of Lemma 5.

□

Chapter 4

Badly approximable sets

4.1 Introduction and basic set up

Recall from the introduction that the set of badly approximable numbers in $\mathbb{R}$ is the set

$$\mathbf{Bad} := \{x \in \mathbb{R} \ : \ \exists c(x) > 0, \ |x - p/q| > c(x)/q^2 \ \ \forall p/q \in \mathbb{Q}\}.$$

In this chapter we begin by creating a general analogue of the badly approximable numbers using a similar set up to that of [BDV03] described in §2.3. Of course, **Bad** is not a lim sup set and so the resulting general framework differs significantly to that of §2.3. This and other differences mean that for clarity we define anew the notation for this chapter.

Let (X, d) be a metric space and (Ω, d) a compact subspace of X which supports a non-atomic finite measure m. Let $\mathcal{R} = \{R_\alpha \subset X : \alpha \in J\}$ be a family of subsets R_α of X indexed by an infinite, countable set J. The sets R_α will be referred to as *resonant sets*. Next, let $\beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on J. To avoid pathological situations within our framework, we shall assume that for any constant $\eta > 0$, the set

$$\{\alpha \in J : \beta_\alpha \leq \eta\} \qquad \text{is finite.}$$

Thus β_α tends to infinity as α runs through J.

Given a real, positive function $\rho : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : r \mapsto \rho(r)$ such that $\rho(r) \longrightarrow 0$ as $r \longrightarrow \infty$ and that ρ is decreasing for r large enough, consider the set

$$\mathbf{Bad}(\rho) := \{x \in \Omega : \exists\, c(x) > 0 \text{ s.t. } d(x, R_\alpha) \geq c(x)\rho(\beta_\alpha) \ \forall \alpha \in J\} ,$$

where $d(x, R_\alpha) := \inf_{a \in R_\alpha} d(x, a)$. In the case that the resonant sets are points, loosely speaking, $\mathbf{Bad}(\rho)$ consists of points in Ω which 'avoid' 'ρ-balls' centred at resonant points. In this case, $\mathbf{Bad}(\rho)$ can be easily compared to the classical set of badly approximable numbers. Notice that since ρ is eventually decreasing to zero and, for any $\eta > 0$, the set $\{\alpha \in J : \beta_\alpha \leq \eta\}$ is finite, it follows that the number of $\alpha \in J$ with $\rho(\beta_\alpha) \geq \varepsilon$, for any $\varepsilon > 0$, is also finite. Therefore, $\max_{\alpha \in J} \rho(\beta_\alpha)$ exists and is finite.

The set $\mathbf{Bad}(\rho)$ is also easily identified with the set $\mathbf{Bad}(\mathrm{N})$ of badly approximable points in $\mathbb{R}^{\mathrm{N}}$. For $\mathrm{N} \geq 1$, a point $x := (x_1, \ldots, x_{\mathrm{N}})$ is an element of $\mathbf{Bad}(\mathrm{N})$, if there exists a positive constant $c(x)$ such that for all $q \in \mathbb{N}$,

$$\max\{\, ||q x_1|| \,, \ldots, \, ||q x_{\mathrm{N}}||\,\} \ > \ c(x)\, q^{-1/\mathrm{N}} \quad ,$$

where $||\,.\,||$ denotes the distance of a real number to the nearest integer. Schmidt [Sch69] proved that

$$\dim \mathbf{Bad}(\mathrm{N}) \ = \ \mathrm{N}\ .$$

Thus, as with Jarník's 1928 result, this badly approximable set has maximal dimension. In view of these results, the initial aim of this chapter is to find a suitably general framework which allows us to conclude that $\dim \mathbf{Bad}(\rho) = \dim \Omega$; that is to say that the set of badly approximable points in Ω is of maximal dimension.

The work presented in this chapter is joint work with S. Kristensen and S. L. Velani. This chapter constitutes the main body of [KTV] which has been recently submitted for publication.

4.1.1 The conditions on the basic set up

Throughout, a ball $B(c,r)$ with centre c and radius r is defined to be the set $\{x \in X : d(c,x) \leq r\}$. Thus a ball is closed and is a subset of X, unless stated otherwise. The following conditions on the measure m and the function ρ highlight the properties of Ω and of $\mathbf{Bad}(\rho) \subset \Omega$ which are used to set a good environment for finding a maximal dimension result.

(A) There exist strictly positive constants δ and r_o such that for any $c \in \Omega$ and $r \leq r_o$

$$a\, r^{\delta} \leq m(B(c,r)) \leq b\, r^{\delta} \,,$$

where $0 < a \leq 1 \leq b$ are constants independent of the ball.

If the measure m supported on Ω is of type (A) then $\dim \Omega = \delta$ - this statement is the subject of Proposition 8 which we state and prove in §4.3. Trivially, this implies that $\dim X \geq \delta$.

(B) For $k > 1$ sufficiently large and any integer $n \geq 1$,

$$\lambda^l(k) \leq \frac{\rho(k^n)}{\rho(k^{n+1})} \leq \lambda^u(k)$$

where λ^l and λ^u are lower and upper bounds independent of n such that $\lambda^l(k) \to \infty$ as $k \to \infty$.

Note that this condition on ρ is satisfied by any function satisfying the following 'regularity' condition. There exists a constant $k > 1$ such that for r

sufficiently large

$$\lambda^l \leq \frac{\rho(r)}{\rho(k\,r)} \leq \lambda^u ,$$

where $1 < \lambda^l \leq \lambda^u$ are constants independent of r but may depend on k.

4.1.2 The basic result

Let $k \in \mathbb{R}^+$ and, for any integer $n \geq 1$, let $B_n := \{x \in \Omega : d(c,x) \leq \rho(k^n)\}$ denote a generic closed ball of radius $\rho(k^n)$ with centre c in Ω. For $\theta \in \mathbb{R}^+$, let $\theta B_n := \{x \in \Omega : d(c,x) \leq \theta\rho(k^n)\}$ denote the ball B_n scaled by θ. Notice, that by definition any generic ball B_n is a subset of Ω. Also, for $n \geq 1$, let $J(n) := \{\alpha \in J : k^{n-1} \leq \beta_\alpha < k^n\}$, a finite subset of J.

Theorem 3. *Let (X,d) be a metric space and (Ω,d,m) a compact measure subspace of X. Let the measure m and the function ρ satisfy conditions (A) and (B) respectively. For $k \geq k_o > 1$, suppose there exists some $\theta \in \mathbb{R}^+$ so that for $n \geq 1$ and any ball B_n there exists a collection $C(\theta B_n)$ of disjoint balls $2\theta B_{n+1}$ contained within θB_n satisfying*

$$\#C(\theta B_n) \geq \kappa_1 \left(\frac{\rho(k^n)}{\rho(k^{n+1})}\right)^\delta \tag{15}$$

and

$$\#\left\{2\theta B_{n+1} \subset C(\theta B_n) : \min_{\alpha\in J(n+1)} d(c,R_\alpha) \leq 2\theta\rho(k^{n+1})\right\} \leq \kappa_2 \left(\frac{\rho(k^n)}{\rho(k^{n+1})}\right)^\delta , \tag{16}$$

where $0 < \kappa_2 < \kappa_1$ are absolute constants independent of k and n. Furthermore, suppose $\dim(\cup_{\alpha\in J}R_\alpha) < \delta$. Then

$$\dim \mathbf{Bad}(\rho) = \delta .$$

Remarks: It is worth noting that this result says more than just that the set $\mathbf{Bad}(\rho)$ has maximal dimension. In fact, it implies that $\mathbf{Bad}(\rho)$ is *thick* i.e. it has maximal dimension everywhere in the sense that for any ball $B = B(c,r)$ with centre $c \in \Omega$ and radius $r < r_o$,

$$\dim(\mathbf{Bad}(\rho) \cap B) = \delta \ .$$

This is easily seen by restricting the measure on Ω to the ball so condition (A) can be satisfied.

In applications, the 'scaling factor' θ can be shrunk to help (16) to be met and is usually dependent on k – see the basic example below. For k sufficiently large, it is always possible to find the collection $\mathcal{C}(\theta B_n)$ satisfying condition (15) – see Lemma 6 in §4.3 for the details. Note that in the case where the resonant sets are points $\dim(\cup_{\alpha\in J} R_\alpha) = 0$ and the hypothesis that $\dim(\cup_{\alpha\in J} R_\alpha) < \delta$ is trivially satisfied. This follows from the fact that the indexing set J is countable.

It is not necessary for θB_n to be as arbitrary as the generic ball in the theorem. In fact, in the context of the proof, the concentric ball of θB_n of twice the radius is free of R_α for all $\alpha \in J(n)$.

Finally, it is worth pointing out that if we had defined the generic ball B_n to be a subset of X with centre c still in Ω then the theorem remains unchanged and condition (16) is equivalent to:

$$\#\left\{2\theta B_{n+1} \subset \mathcal{C}(\theta B_n) : 2\theta B_{n+1} \cap \bigcup_{\alpha\in J(n+1)} R_\alpha \neq \emptyset\right\} \leq \kappa_2 \left(\frac{\rho(k^n)}{\rho(k^{n+1})}\right)^\delta \quad (17)$$

4.1.3 The basic example: Bad

Let $I = [0,1]$ and consider the set of badly approximable numbers restricted to the unit interval

$$\mathbf{Bad}_I := \{x \in [0,1] : |x - p/q| > c(x)/q^2 \text{ for all rationals } p/q \ \ (q > 0)\} .$$

Clearly, it can be expressed in the form $\mathbf{Bad}(\rho)$ with $\rho(r) := r^{-2}$ and

$$X = \Omega := [0,1] , \quad J := \{(p,q) \in \mathbb{N} \times \mathbb{N}\backslash\{0\} : p \leq q\} ,$$

$$\alpha := (p,q) \in J , \quad \beta_\alpha := q , \quad R_\alpha := p/q .$$

The metric d is of course the standard Euclidean metric; $d(x,y) := |x-y|$. Thus in this basic example, the resonant sets R_α are simply rational points p/q and the function ρ satisfies condition (B) with $\lambda^l = \lambda^u = k^2$. Let the measure m be one–dimensional Lebesgue measure on I. Thus, $\delta = 1$ and m clearly satisfies condition (A).

We show that the hypotheses (15) and (16) of Theorem 3 are satisfied for this basic example. The existence of the collection $C(\theta B_n)$, where B_n is an arbitrary closed interval of length $2\,k^{-2n}$ follows immediately from the following simple observation. For any two distinct rationals p/q and p'/q' with $k^n \leq q, q' < k^{n+1}$ we have that

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| \geq \frac{1}{qq'} > k^{-2n-2} .$$

Thus, any interval θB_n with $\theta := \frac{1}{2}k^{-2}$ contains at most one rational p/q with $k^n \leq q < k^{n+1}$. Let $C(\theta B_n)$ denote the collection of intervals $2\theta B_{n+1}$ obtained by subdividing θB_n into intervals of length $2k^{-2n-4}$ starting from the left hand side of θB_n. Clearly

$$\#\mathcal{C}(\theta B_n) \ \geq \ [k^2/2] \ > \ k^2/4 \ = \ \text{r.h.s. of (15) with } \kappa_1 := 1/4 \ .$$

Also, in view of the above observation, for k sufficiently large

$$\text{l.h.s. of (16)} \ \leq \ 1 \ < \ k^2/8 \ = \ \text{r.h.s. of (16) with } \kappa_2 := 1/8 \ .$$

The upshot of this is that Theorem 3 implies that

$$\dim \ \mathbf{Bad}_I = \ 1 \ .$$

In turn, since **Bad** is a subset of $\mathbb{R}$, this implies that $\dim \mathbf{Bad} = 1$ – Jarník's (1928) result. As we shall see in §4.5, Theorem 3 has other much less trivial consequences.

4.2 A more general framework

It is clear that the basic set up encompasses the set $\mathbf{Bad}(N)$ of badly approximable points in $\mathbb{R}^N$, but there are other well known badly approximable sets in $\mathbb{R}^N$ which are not covered by this set up. These include the set of (i,j)-badly approximable pairs in $\mathbb{R}^2$ denoted as $\mathbf{Bad}(i,j)$. For $i,j \geq 0$ with $i+j = 1$, (x_1,x_2) is an element of $\mathbf{Bad}(i,j)$ if there exists a positive constant $c(x_1,x_2)$ such that for all $q \in \mathbb{N}$

$$\max\{ \ ||qx_1||^{1/i} \ , \ ||qx_2||^{1/j} \ \} \ > \ c(x_1,x_2) \ q^{-1} \quad .$$

In the case $i = j = 1/2$, the set $\mathbf{Bad}(\frac{1}{2},\frac{1}{2})$ is simply the standard set of badly approximable pairs $\mathbf{Bad}(2)$, defined in §4.1. The set $\mathbf{Bad}(i,j)$ was recently shown by Pollington and Velani [PV02] to have maximal dimension, i.e. $\dim \mathbf{Bad}(i,j) = 2$: a result which generalized Schmidt's result that $\dim \mathbf{Bad}(2) = 2$.

The aim of this section is to encompass the set $\mathbf{Bad}(i,j)$ into our generalisation and make a fruitful extension of Theorem 3. To do this we now consider a more general framework in which the 'badly approximable' set consists of points avoiding 'rectangular' neighborhoods of resonant sets. This framework will allow us to further the work of Pollington and Velani and consider an enticing conjecture of W. M. Schmidt [Sch83]. His conjecture states that, for any two pairs (i,j) and (i',j') of numbers such that $0 < i,j,i',j' < 1$ and that the sum of each pair is one,

$$\mathbf{Bad}(i,j) \cap \mathbf{Bad}(i',j') \neq \emptyset .$$

The dimension result of Pollington and Velani implies that the above intersection is of two sets of maximal dimension. This adds to the evidence that the conjecture may be true. However, if this conjecture was false then it would be even more interesting since it would imply that the famous Littlewood's conjecture of Diophantine approximation is true. Littlewood's conjecture states that for any (x_1, x_2) in $\mathbb{R}^2$,

$$\liminf_{q \to \infty} q \, ||q\, x_1|| \, ||q\, x_2|| = 0 .$$

We discuss Schmidt's conjecture again in §4.5.4, in light of the results of this section.

4.2.1 The general set up

Let (X, d) be the product space of t metric spaces (X_i, d_i) and let (Ω, d) be a compact subspace of X which supports a non-atomic finite measure m. As before, let $\mathcal{R} = \{R_\alpha \subset X : \alpha \in J\}$ be a family of subsets R_α of X indexed by an infinite, countable set J. Thus, each resonant set R_α can be split into its t components $R_{\alpha,i} \subset (X_i, d_i)$. As before, let $\beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on J and assume that, for any $\eta > 0$, the set

$$\{\alpha \in J : \beta_\alpha \leq \eta\} \qquad \text{is finite.}$$

For each $1 \leq i \leq t$, let $\rho_i : \mathbb{R}^+ \to \mathbb{R}^+ : r \mapsto \rho_i(r)$ be a real, positive function such that $\rho_i(r) \to 0$ as $r \to \infty$ and that ρ_i is decreasing for r large enough. Furthermore, assume that $\rho_1(r) \geq \rho_2(r) \geq \cdots \geq \rho_t(r)$ for all r large enough – the ordering is irrelevant. Given a resonant set R_α, let

$$F_\alpha(\rho_1, \ldots, \rho_t) := \{x \in X \ : \ d_i(x_i, R_{\alpha,i}) \leq \rho_i(\beta_\alpha) \quad \forall\, 1 \leq i \leq t\} \ ,$$

denote the 'rectangular' $(\rho_1, \ldots, \rho_t)$–neighbourhood of R_α. Consider the set

$$\mathbf{Bad}(\rho_1, \ldots, \rho_t) := \{x \in \Omega : \exists\, c(x) > 0 \ \text{ s.t. } x \notin c(x) F_\alpha(\rho_1, \ldots, \rho_t) \ \forall\, \alpha \in J\} \, .$$

Thus, $x \in \mathbf{Bad}(\rho_1, \ldots, \rho_t)$ if there exists a constant $c(x) > 0$ such that

$$d_i(x_i, R_{\alpha,i}) \ \geq \ c(x)\, \rho_i(\beta_\alpha) \quad \text{for any } i = 1, \ldots, t, \quad \forall\, \alpha \in J \ .$$

Clearly, $\mathbf{Bad}(\rho_1, \ldots \rho_t)$ is precisely the set $\mathbf{Bad}(\rho)$ of §4.1 in the case $t = 1$. The overall aim of this section is to find a suitably general framework which gives a lower bound for the Hausdorff dimension of $\mathbf{Bad}(\rho_1, \ldots, \rho_t)$.

For each i, notice that since ρ_i is eventually decreasing to zero and, for any $\eta > 0$, the set $\{\alpha \in J : \beta_\alpha \leq \eta\}$ is finite, it follows that the number of

$\alpha \in J$ with $\rho_i(\beta_\alpha) \geq \varepsilon$, for any $\varepsilon > 0$, is also finite by hypothesis. Therefore, $\max_{\alpha\in J} \rho_i(\beta_\alpha)$ exists and is finite.

4.2.2 The conditions on the general set up

Given $l_1, \ldots, l_t \in \mathbb{R}^+$ and $c \in \Omega$ let

$$F(c; l_1, \ldots, l_t) \ := \ \{x \in X \ : \ d_i(x_i, c_i) \leq l_i \quad \forall\, 1 \leq i \leq t\} \ ,$$

denote the closed *rectangle* centred at c with *sidelengths* determined by $l_1, \ldots, l_t$. Also, for any $k > 1$ and $n \in \mathbb{N}$, let F_n denote a generic rectangle $F(c; \rho_1(k^n), \ldots, \rho_t(k^n)) \cap \Omega$ centred at a point c in Ω. As before, $B(c, r)$ is a closed ball with centre c and radius r. The following conditions on the measure m and the functions ρ_i will set a suitable environment for deducing Hausdorff dimension. The first two are reminiscent of conditions (A) and (B) of §4.1.1.

(A*) There exists a strictly positive constant δ such that for any $c \in \Omega$

$$\liminf_{r\to 0} \frac{\log m(B(c,r))}{\log r} \ = \ \delta \ .$$

It can be verified that if the measure m supported on Ω is of type (A*) then $\dim \Omega \geq \delta$ [Fal90, Proposition 4.9] and so $\dim X \geq \delta$. Clearly condition (A) of §4.1.1 implies (A*).

(B*) For $k > 1$ sufficiently large and any integer $n \geq 1$,

$$\lambda_i^l(k) \leq \frac{\rho_i(k^n)}{\rho_i(k^{n+1})} \leq \lambda_i^u(k) \qquad \forall\, 1 \leq i \leq t \ ,$$

where λ_i^l and λ_i^u are lower and upper bounds independent of n such that $\lambda_i^l(k) \to \infty$ as $k \to \infty$.

Clearly, this is just condition (B) of §4.1.1 imposed on each function ρ_i.

(C*) There exist constants $0 < a \leq 1 \leq b$ and $l_0 > 0$ such that

$$a \leq \frac{m(F(c; l_1, \ldots, l_t))}{m(F(c'; l_1, \ldots, l_t))} \leq b \qquad \forall\, c, c' \in \Omega \qquad \forall\ l_1, \ldots, l_t \leq l_0 \, .$$

This condition implies that rectangles of the same size centred at points of Ω have comparable m measure.

(D*) There exist strictly positive constants D and l_0 such that

$$\frac{m(2\, F(c; l_1, \ldots, l_t))}{m(F(c; l_1, \ldots, l_t))} \leq D \qquad \forall\, c \in \Omega \qquad \forall\ l_1, \ldots, l_t \leq l_0 \, .$$

This condition simply says that the measure m is 'doubling' with respect to rectangles. In terms of achieving the aim of obtaining a lower bound for $\dim \mathbf{Bad}(\rho_1, \ldots \rho_t)$, the above four conditions are rather natural. The following final condition is in some sense the only genuine technical condition yet is not particularly restrictive.

(E*) For $k > 1$ sufficiently large and any integer $n \geq 1$

$$\frac{m(F_n)}{m(F_{n+1})} \geq \lambda(k) \, ,$$

where λ is a constant such that $\lambda(k) \to \infty$ as $k \to \infty$.

4.2.3 The general result

Recall, that $F_n := \{x \in \Omega : \ d_i(x_i, c_i) \leq \rho_i(k^n) \ \ \forall\ 1 \leq i \leq t\}$ is a generic rectangle with centre c in Ω and sidelengths determined by $\rho_i(k^n)$ and for $\theta \in \mathbb{R}^+$, θF_n is the rectangle F_n scaled by θ. Also, for $n \geq 1$, let

$$J(n) := \{\alpha \in J : k^{n-1} \leq \beta_\alpha < k^n\} \, .$$

Theorem 4. *Let (X,d) be the product space of the metric spaces (X_1,d_1), ..., (X_t,d_t) and let (Ω,d,m) be a compact measure subspace of X. Let the measure m and the functions ρ_i satisfy conditions (A*) to (E*). For $k \geq k_0 > 1$, suppose there exists some $\theta \in \mathbb{R}^+$ so that for $n \geq 1$ and any rectangle F_n there exists a disjoint collection $C(\theta F_n)$ of rectangles $2\theta F_{n+1}$ contained within θF_n satisfying*

$$\#C(\theta F_n) \;\geq\; \kappa_1 \, \frac{m(\theta F_n)}{m(\theta F_{n+1})} \tag{18}$$

and

$$\#\left\{2\theta F_{n+1} \subset C(\theta F_n) : \min_{\alpha \in J(n+1)} d_i(c_i, R_{\alpha,i}) \leq 2\theta\rho_i(k^{n+1}) \;\; \forall\, 1 \leq i \leq t\right\} \leq \; \kappa_2 \, \frac{m(\theta F_n)}{m(\theta F_{n+1})} \,. \tag{19}$$

where $0 < \kappa_2 < \kappa_1$ are absolute constants independent of k and n. Furthermore, suppose $\dim(\cup_{\alpha\in J} R_\alpha) < \delta$. *Then*

$$\dim \mathbf{Bad}(\rho_1, \ldots \rho_t) \geq \delta \;\; .$$

Remarks: For k sufficiently large, it is always possible to find a collection $C(\theta F_n)$ satisfying condition (18) – see Lemma 6 in §4.3 for details. Clearly, the lower bound result for $\dim \mathbf{Bad}(\rho)$ of Theorem 3 is an immediate consequence of Theorem 4. To see this, simply note that if $t = 1$ then the rectangles F_n are balls B_n and if conditions (A) and (B) are satisfied then trivially so are the conditions (A*) to (E*). In fact, if condition (A*) is replaced by the stronger condition (A) in the above theorem, then we are able to conclude that $\dim \mathbf{Bad}(\rho_1, \ldots \rho_t) = \delta$ – see below.

We now consider an extremely useful specialization of the above general framework in which the space Ω is a product space equipped with a product measure.

Theorem 5. *For $1 \leq i \leq t$, let (X_i, d_i) be a metric space and (Ω_i, d_i, m_i) be a compact measure subspace of X_i where the measure m_i satisfies condition (A) with exponent δ_i. Let (X, d) be the product space of the spaces (X_i, d_i) and let (Ω, d, m) be the product measure space of the measure spaces (Ω_i, d_i, m_i). Let the functions ρ_i satisfy condition (B*). For $k \geq k_0 > 1$, suppose there exists some $\theta \in \mathbb{R}^+$ so that for $n \geq 1$ and any rectangle F_n there exists a disjoint collection $C(\theta F_n)$ of rectangles $2\theta F_{n+1}$ contained within θF_n satisfying*

$$\#C(\theta F_n) \ \geq \ \kappa_1 \prod_{i=1}^{t} \left(\frac{\rho_i(k^n)}{\rho_i(k^{n+1})} \right)^{\delta_i} \tag{20}$$

and

$$\#\left\{ 2\theta F_{n+1} \subset C(\theta F_n) : \min_{\alpha \in J(n+1)} d_i(c_i, R_{\alpha,i}) \leq 2\theta \rho_i(k^{n+1}) \ \ \forall \ 1 \leq i \leq t \right\}$$
$$\leq \ \kappa_2 \prod_{i=1}^{t} \left(\frac{\rho_i(k^n)}{\rho_i(k^{n+1})} \right)^{\delta_i} , \tag{21}$$

where $0 < \kappa_2 < \kappa_1$ are absolute constants independent of k and n. Furthermore, suppose $\dim(\cup_{\alpha \in J} R_\alpha) < \sum_{i=1}^{t} \delta_i$. Then

$$\dim \mathbf{Bad}(\rho_1, \ldots \rho_t) = \sum_{i=1}^{t} \delta_i \ .$$

The deduction of Theorem 5 from Theorem 4 is relatively straightforward and hinges on the following simple observation. Since m is the product measure of the measures m_i and the latter satisfy condition (A) with exponents

δ_i $(1 \le i \le t)$, we have that

$$a^t \le \frac{m(F(c; l_1, \ldots, l_t))}{\prod_{i=1}^t l_i^{\delta_i}} \le b^t \qquad \forall\, c \in \Omega \quad \forall\ l_1, \ldots, l_t \le l_0 \,. \qquad (22)$$

It follows that conditions (C*) and (D*) are trivially satisfied as is condition (A) with $\delta := \sum_{i=1}^t \delta_i$. Recall, that (A) implies (A*). Also, (22) together with (B*) implies that condition (E*) is satisfied. Thus, Theorem 4 implies the desired lower bound estimate for $\dim \mathbf{Bad}(\rho_1, \ldots \rho_t)$. The complementary upper bound estimate is a simple consequence of the fact that m satisfies (A). If m satisfies (A), then $\dim \Omega = \delta$ (see §4.3) and since $\mathbf{Bad}(\rho_1, \ldots, \rho_t) \subseteq \Omega$ the upper bound follows.

4.2.4 The general basic example: $\mathbf{Bad}(i, j)$

We are now in a position to reprove the result of [PV02] that $\dim \mathbf{Bad}(i, j) = 2$, by showing it is a simple consequence of Theorem 5.

Let $\mathbf{Bad}_{I^2}(i, j) := \mathbf{Bad}(i, j) \cap I^2$ where $I^2 := [0, 1] \times [0, 1]$. Without loss of generality assume that $i \le j$. Clearly, it can be expressed in the form $\mathbf{Bad}(\rho_1, \rho_2)$ with $\rho_1(r) := r^{-(1+i)}$, $\rho_2(r) := r^{-(1+j)}$ and

$$X = \Omega := I^2 \ , \quad J := \{((p_1, p_2), q) \in \mathbb{N}^2 \times \mathbb{N}\backslash\{0\} : p_1, p_2 \le q\} \ ,$$

$$\alpha := ((p_1, p_2), q) \in J \ , \quad \beta_\alpha := q \ , \quad R_\alpha := (p_1/q, p_2/q) \ .$$

Furthermore, $d_1 = d_2$ is the standard Euclidean metric on I and $m_1 = m_2$ is one–dimensional Lebesgue measure on I. By definition, the metric d on I^2 is the product metric $d_1 \times d_1$ and the measure $m := m_1 \times m_1$ is simply two–dimensional Lebesgue measure on I^2.

We show that the conditions of Theorem 5 are satisfied for this basic example. Clearly the functions ρ_1, ρ_2 satisfy condition (B*) and the measures m_1, m_2 satisfy condition (A) with $\delta_1 = \delta_2 = 1$. We now need to establish the existence of the collection $\mathcal{C}(\theta F_n)$, where F_n is an arbitrary closed rectangle of size $2k^{-n(1+i)} \times 2k^{-n(1+j)}$. To start with, note that $m(\theta F_n) = 4\theta^2 k^{-3n}$. Now assume there are at least three rational points $(p_1/q, p_2/q), (p_1'/q', p_2'/q')$ and $(p_1''/q'', p_2''/q'')$ with

$$k^n \leq q, q', q'' < k^{n+1}$$

lying within θF_n. Suppose for the moment that they do not lie on a line and form the triangle Δ sub-tended by them. Twice the area of the triangle Δ is equal to the absolute value of the determinant

$$\det := \begin{vmatrix} 1 & p_1/q & p_2/q \\ 1 & p_1'/q' & p_2'/q' \\ 1 & p_1''/q'' & p_2''/q'' \end{vmatrix} .$$

Then, in view of the denominator constraint, it follows that

$$2 \times m(\Delta) \geq \frac{1}{qq'q''} > k^{-3(n+1)} .$$

Now put

$$\theta := 2^{-1}(2k^3)^{-1/2}.$$

Then $m(\Delta) > m(\theta F_n)$ and this is impossible since $\Delta \subset \theta F_n$. The upshot of this is that the triangle in question cannot exist. Thus, if there are two or more rational points with $k^n \leq q < k^{n+1}$ lying within θF_n then they must lie on a line $\mathcal{L}$.

Starting from a 'corner' of the rectangle θF_n, partition θF_n into rectangles $2\theta F_{n+1}$ of size $4\theta k^{-(n+1)(1+i)} \times 4\theta k^{-(n+1)(1+j)}$ and denote by $\mathcal{C}(\theta F_n)$ the collection of rectangles $2\theta F_{n+1}$ obtained. Trivially

$$\#\mathcal{C}(\theta F_n) \;\geq\; \left[\frac{2\theta k^{-n(1+i)}}{4\theta k^{-(n+1)(1+i)}}\right]\left[\frac{2\theta k^{-n(1+j)}}{4\theta k^{-(n+1)(1+j)}}\right] \geq \frac{k^3}{16}\,.$$

Here $[x]$ denotes the integer part of x. In view of the above 'triangle' argument we have that

$$\#\left\{2\theta F_{n+1}\subset\mathcal{C}(\theta F_n):\min_{\alpha\in J(n+1)} d_i(c_i,R_{\alpha,i})\leq 2\theta\rho_i(k^{n+1})\;\;\forall\;1\leq i\leq t\right\}$$
$$\leq\#\left\{2\theta F_{n+1}\subset\mathcal{C}(\theta F_n):2\theta F_{n+1}\cap\mathcal{L}\neq\emptyset\right\}\;,$$

where $\mathcal{L}$ is any line passing through θF_n. Recall, that we are assuming that $i\leq j$. A simple geometric argument ensures that for k sufficiently large

$$\#\left\{2\theta F_{n+1}\subset\mathcal{C}(\theta F_n):2\theta F_{n+1}\cap\mathcal{L}\neq\emptyset\right\}\;\leq\;\left[\frac{2\theta k^{-n(1+j)}}{4\theta k^{-(n+1)(1+j)}}\right]=\left[\frac{k^{1+j}}{2}\right]$$
$$\leq\;k^{1+j}\;\leq\;k^3/32\,.$$

The result of this is that the collection $\mathcal{C}(\theta F_n)$ satisfies the required conditions and Theorem 5 implies that

$$\dim\mathbf{Bad}_{I^2}(i,j)\;=\;2\,.$$

In turn, since $\mathbf{Bad}(i,j)$ is a subset of $\mathbb{R}^2$, this implies that $\dim\mathbf{Bad}(i,j)=2$. In [PV02], the stronger result that

$$\dim\mathbf{Bad}(i,j)\cap\mathbf{Bad}(1,0)\cap\mathbf{Bad}(0,1)=2$$

is established; i.e. the set of pairs (x_1,x_2) with x_1 and x_2 both badly approximable numbers and an (i,j)–badly approximable pair has full dimension. Consider the following extension of $\mathbf{Bad}(i,j)$ into higher dimensions.

For any N–tuple of real numbers $i_1,\dots,i_{\rm N}\geq 0$ such that $\sum i_r=1$, denote by $\mathbf{Bad}(i_1,\dots,i_{\rm N})$ the set of points $(x_1,\dots,x_{\rm N})\in\mathbb{R}^{\rm N}$ for which there exists a positive constant $c(x_1, ..., x_N)$ such that

$$\max\{ ||qx_1||^{1/i_1}, ..., ||qx_N||^{1/i_N} \} > c(x_1, ..., x_N)\, q^{-1} \quad \forall \quad q \in \mathbb{N}.$$

Clearly, the above argument can easily be modified to show that

$$\dim \mathbf{Bad}(i_1, ..., i_N) = \mathrm{N} .$$

In §4.5, by considering the intersection of sets with the set $\mathbf{Bad}(i_1, ..., i_N)$, we obtain a much more general result than the above result of [PV02]. We also remark on the implications of this result for Schmidt's conjecture.

4.3 Preliminaries

In this section we introduce the following powerful tools which are fundamental within the proceeding proof of Theorem 4.

Covering Lemma. *Let (Ω, d) be the product space of the metric spaces $(\Omega_1, d_1), \dots, (\Omega_t, d_t)$ and $\mathcal{F}$ be a finite collection of rectangles $F := F(c; l_1, \dots, l_t)$ with $c \in \Omega$ and $l_1, \dots, l_t$ fixed. Then there exists a disjoint sub-collection $\{F_m\}$ such that*

$$\bigcup_{F \in \mathcal{F}} F \subset \bigcup_m 3F_m .$$

Proof. Let S denote the set of centres c of the rectangles in $\mathcal{F}$. Choose $c(1) \in S$ and for $k \geq 1$, choose

$$c(k+1) \in S \setminus \bigcup_{m=1}^{k} 2\, F(c(m); l_1, \dots, l_t)$$

as long as $S \setminus \bigcup_{m=1}^{k} 2\,F(c(m); l_1, \ldots, l_t) \neq \emptyset$. Since $\#S$ is finite, there exists $k_1 \leq \#S$ such that

$$S \subset \bigcup_{m=1}^{k_1} 2\,F(c(m); l_1, \ldots, l_t)\ .$$

By construction, any rectangle $F(c; l_1, \ldots, l_t)$ in the original collection $\mathcal{F}$ is contained in some rectangle $3\,F(c(m); l_1, \ldots, l_t)$ and since $d_i(c_i(m), c_i(n)) > 2l_i$ for each $1 \leq i \leq t$ the chosen rectangles $F(c(m); l_1, \ldots, l_t)$ are clearly disjoint.

□

Next we make use of the covering lemma in order to establish the following assertion made in §4.2.3. The result is extremely useful when it comes to applying our theorems – see §4.5. With reference to Theorem 4, it guarantees the existence of a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles with the necessary cardinality.

Lemma 6. *Let (X, d) be the product space of the metric spaces $(X_1, d_1), \ldots, (X_t, d_t)$ and let (Ω, d, m) be a compact measure subspace of X. Let the measure m and the functions ρ_i satisfy conditions (B*) to (D*). Let k be sufficiently large. Then for any $\theta \in \mathbb{R}^+$ and for any rectangle F_n $(n \geq 1)$ there exists a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1}$ contained within θF_n satisfying (18) of Theorem 4.*

Proof. Begin by choosing k large enough so that

$$\frac{\rho_i(k^n)}{\rho_i(k^{n+1})} \geq 4 \qquad \forall\ 1 \leq i \leq t\ . \tag{23}$$

That this is possible follows from the fact that $\lambda_i^l(k) \to \infty$ as $k \to \infty$ (condition (B*)). Take an arbitrary rectangle F_n and let $l_i(n) := \theta\rho_i(k^n)$.

Thus $\theta F_n := F(c; l_1(n), \ldots, l_t(n))$. Consider the rectangle $T_n \subset \theta F_n$ where

$$T_n := F(c; l_1(n) - 2l_1(n+1), \ldots, l_t(n) - 2l_t(n+1)) \ .$$

Note that in view of (23) we have that $T_n \supset \frac{1}{2}\theta F_n$. Now, cover T_n by rectangles $2\theta F_{n+1}$ with centres in $\Omega \cap T_n$. By construction, these rectangles are contained in θF_n. In view of the covering lemma, there exists a disjoint sub-collection $C(\theta F_n)$ such that

$$T_n \subset \bigcup_{2\theta F_{n+1} \subset C(\theta F_n)} 6\theta F_{n+1} \ .$$

Since rectangles of the same size centred at points of Ω have comparable m measure (condition (C*)), it follows that

$$a\, m(\tfrac{1}{2}\theta F_n) \ \leq\ m(T_n) \ \leq\ \#C(\theta F_n)\, b\, m(6\theta F_{n+1}) \ .$$

Using that fact that the measure m is doubling on rectangles (condition (D*)), it follows that there exists a positive constant D, such that

$$\#C(\theta F_n) \ \geq\ \frac{a}{bD^4}\ \frac{m(\theta F_n)}{m(\theta F_{n+1})} \ .$$

□

Remark. Clearly, with reference to Theorem 3, the above lemma guarantees the existence of the collection $C(\theta B_n)$ satisfying (15).

The following well known result is used in the proof to determine the lower bound of dimension.

Lemma 7. (Mass Distribution Principle) *Let μ be a probability measure supported on a subset F of (Ω, d). Suppose there are positive constants c and*

r_o such that

$$\mu(B) \le cf(r),$$

for any ball B centred in Ω with radius $r \le r_o$. If X is a subset of F with $\mu(X) = \lambda > 0$ then

$$\mathcal{H}^f(X) \ge \lambda / c.$$

Proof. If $\{B_i\}$ is a ρ-cover of X with $\rho \le r_o$ then

$$\lambda = \mu(X) = \mu(\cup_i B_i) \le \sum_i \mu(B_i) \le c \sum_i f(r_i).$$

It follows that $\mathcal{H}^f_\rho(X) \ge \lambda/c$ for any $\rho \le r_o$. On letting $\rho \to 0$, the quantity $\mathcal{H}^f_\rho(X)$ increases and so we obtain the required result. □

Using the covering lemma and the mass distribution principle, we now prove a result already refered to earlier in this chapter.

Proposition 8. *Suppose (Ω, d) is a compact, metric space and Ω supports a non-atomic probability measure m satisfying condition (A) with exponent $\delta > 0$. Then*

$$\dim \Omega \ = \ \delta\ .$$

Proof. By the definition of Hausdorff dimension, it suffices to prove

$$0 \ < \ \mathcal{H}^\delta(\Omega) \ < \ \infty\ .$$

Condition (A) on the measure m, implies that, for any ball B of radius $0 < \rho \le r_o$,

$$m(B) \ \le \ b\,\rho^\delta$$

where b and δ are positive constants. It then follows directly from the mass distribution principle that $m(\Omega) = 1$ implies $\mathcal{H}^\delta(\Omega) \ \ge \ b^{-1} > 0$. Since the space Ω is compact, there exists a cover of Ω by a finite collection $\mathcal{B}$ of balls B centred in Ω and of radius $\rho > 0$. By the covering lemma, there exists a disjoint sub-collection $\{B_i(\rho)\}$ of $\mathcal{B}$ such that

$$\bigcup_i^{\circ} B_i(\rho) \subset \bigcup_{B \in \mathcal{B}} B \subset \bigcup_i B_i(3\rho) .$$

Thus the collection $\{B_i(3\rho)\}$ is a 3ρ-cover of Ω. Hence,

$$\mathcal{H}^{\delta}_{3\rho}(\Omega) \leq \sum_i (3\rho)^{\delta} \ll \sum_i m(B_i(\rho)) = m\left(\bigcup_i^{\circ} B_i(\rho)\right) \leq m(\Omega) = 1 .$$

On letting $\rho \to 0$, we obtain $\mathcal{H}^{\delta}(\Omega) \ll 1$ as required.

□

4.4 Proof of Theorem 4

The overall strategy is as follows. For any k sufficiently large we construct a Cantor-type set $\mathbf{K}_{c(k)}$ such that $\mathbf{K}_{c(k)}$ with at most a finite number of points removed is a subset of $\mathbf{Bad}(\rho_1, \ldots, \rho_t)$. Next, we construct a measure μ supported on $\mathbf{K}_{c(k)}$ with the property that for any ball A with radius $r(A)$ sufficiently small

$$\mu(A) \ll r(A)^{\delta - \varepsilon(k)} ;$$

where $\varepsilon(k) \to 0$ as $k \to \infty$. Hence, by construction and the mass distribution principle we have that

$$\dim \mathbf{Bad}(\rho_1, \ldots, \rho_t) \geq \dim \mathbf{K}_{c(k)} \geq \delta - \varepsilon(k) .$$

Now suppose that $\dim \mathbf{Bad}(\rho_1, \ldots, \rho_t) < \delta$. Then, $\dim \mathbf{Bad}(\rho_1, \ldots, \rho_t) = \delta - \eta$ for some $\eta > 0$. However, by choosing k large enough so that $\varepsilon(k) < \eta$ we obtain a contradiction and thereby the lower bound result follows.

4.4.1 The Cantor-type set $\mathbf{K}_{c(k)}$

Choose k_o sufficiently large so that for $k \geq k_o$, $\rho_i(k)$ $(1 \leq i \leq t)$ is decreasing and the hypotheses of the theorem are valid. Now fix some $k \geq k_o$ and suppose that

$$\{\alpha \in J : \beta_\alpha < k\} \;=\; \emptyset \,. \tag{24}$$

Define $\mathcal{F}_1$ to be any rectangle θF_1 with sidelengths $\theta\rho_i(k)$ and centre c in Ω. The idea is to establish, by induction on n, the existence of a collection $\mathcal{F}_n$ of disjoint rectangles θF_n such that $\mathcal{F}_n$ is nested in $\mathcal{F}_{n-1}$; that is, each rectangle θF_n in $\mathcal{F}_n$ is contained in some rectangle θF_{n-1} of $\mathcal{F}_{n-1}$. Also, any θF_n in $\mathcal{F}_n$ will have the property that for all points $x \in \theta F_n$ and for all $\alpha \in J$ with $\beta_\alpha \;<\; k^n$,

$$d_i(x, R_{\alpha,i}) \;\geq\; c(k)\,\rho_i(\beta_\alpha) \qquad \text{for some } i:\; 1 \leq i \leq t \;, \tag{25}$$

where the constant

$$c(k) \;:=\; \min_{1\leq i\leq t}(\theta/\lambda_i^u(k))$$

is dependent on k but is independent of n. Then, since the rectangles θF_n of $\mathcal{F}_n$ are closed, nested and the space Ω is compact, any limit point in θF_n will satisfy (25) for all α in J with $\beta_\alpha \geq k$. In particular, we put

$$\mathbf{K}_{c(k)} \;:=\; \bigcap_{n=1}^{\infty} \mathcal{F}_n \,.$$

By construction, we have that $\mathbf{K}_{c(k)}$ is a subset of $\mathbf{Bad}(\rho_1, \ldots, \rho_t)$ under the assumption (24).

The induction. For $n = 1$, (25) is trivially satisfied for $\mathcal{F}_1 = \theta F_1$ since we are assuming (24). Given $\mathcal{F}_n$ satisfying (25) we wish to construct a nested collection $\mathcal{F}_{n+1}$ for which (25) is satisfied for $n+1$. Consider any rectangle $\theta F_n \subset \mathcal{F}_n$. We construct a 'local' collection $\mathcal{F}_{n+1}(\theta F_n)$ of disjoint rectangles θF_{n+1} contained in θF_n so that for any point $x \in \theta F_{n+1}$ the condition given by (25) is satisfied for $n+1$. Given that any rectangle θF_{n+1} of $\mathcal{F}_{n+1}(\theta F_n)$ is to be nested in θF_n, it is enough to show that for any point $x \in \theta F_{n+1}$ at least one of the inequalities

$$d_i(x_i, R_{\alpha,i}) \;\geq\; c(k)\,\rho_i(\beta_\alpha) \qquad\qquad (1 \leq i \leq t)$$

is satisfied for $\alpha \in J$ with $k^n \leq \beta_\alpha < k^{n+1}$; i.e. with $\alpha \in J(n+1)$.

For k sufficiently large, by the hypotheses of the theorem, there exists a disjoint sub-collection $\mathcal{G}(\theta F_n)$ of $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1} \subset \theta F_n$ with

$$\#\mathcal{G}(\theta F_n) \;=\; \left[\kappa\,\frac{m(\theta F_n)}{m(\theta F_{n+1})}\right] \qquad\qquad \kappa := \min\{1,\, \tfrac{1}{2}(\kappa_1 - \kappa_2)\}\;, \qquad (26)$$

and such that for any rectangle $2\theta F_{n+1} \subset \mathcal{G}(\theta F_n)$ with centre c

$$\min_{\alpha \in J(n+1)} d_i(c_i, R_{\alpha,i}) \;\geq\; 2\,\theta\,\rho_i(k^{n+1})\;.$$

Clearly, by choosing k large enough we can ensure that $\#\mathcal{G}(\theta F_n) > 1$ – this makes use of conditions (D*) and (E*). Now let

$$\mathcal{F}_{n+1}(\theta F_n) \;:=\; \{\theta F_{n+1} : 2\theta F_{n+1} \subset \mathcal{G}(\theta F_n)\}\;.$$

Thus the rectangles of $\mathcal{F}_{n+1}(\theta F_n)$ are precisely those of $\mathcal{G}(\theta F_n)$ but scaled by a factor 1/2. Then, by construction for any $x \in \theta F_{n+1} \subset \mathcal{F}_{n+1}(\theta F_n)$ and for at least one i with $1 \leq i \leq t$

$$d_i(x_i, R_{\alpha,i}) \;\geq\; \theta\rho_i(k^{n+1}) \;=\; \theta\rho_i(k^n)\,\frac{\rho_i(k^{n+1})}{\rho_i(k^n)} \geq \frac{\theta}{\lambda_i^u(k)}\rho_i(\beta_\alpha) \;\geq\; c(k)\,\rho_i(\beta_\alpha).$$

Here we have made use of condition (B*) and the fact that $\rho_i(k)$ is decreasing for $k \geq k_o$ and that $\alpha \in J(n+1)$. Finally let

$$\mathcal{F}_{n+1} := \bigcup_{\theta F_n \in \mathcal{F}_n} \mathcal{F}_{n+1}(\theta F_n)\;.$$

This completes the proof of the induction step and so the construction of the Cantor-type set

$$\mathbf{K}_{c(k)} := \bigcap_{n=1}^{\infty} \mathcal{F}_n ,$$

where $c(k) := \min_{1\le i\le t}(\theta/\lambda_i^u(k))$ and k is sufficiently large.

Note, that in view of (26) we have that for $n \geq 2$

$$\begin{aligned} \#\mathcal{F}_n &= \#\mathcal{F}_{n-1} \times \#\mathcal{F}_n(\theta F_{n-1}) = \prod_{m=2}^{n} \#\mathcal{F}_m(\theta F_{m-1}) \\ &\geq \prod_{m=2}^{n} \frac{\kappa}{2}\, \frac{m(\theta F_{m-1})}{m(\theta F_m)} = \left(\frac{\kappa}{2}\right)^{n-1} \frac{m(\theta F_1)}{m(\theta F_n)}. \end{aligned} \tag{27}$$

4.4.2 The measure μ on $\mathbf{K}_{c(k)}$

We now describe a probability measure μ supported on the Cantor-type set $\mathbf{K}_{c(k)}$ constructed in the previous subsection. For any rectangle θF_n in $\mathcal{F}_n$ we attach a weight $\mu(\theta F_n)$ which is defined recursively as follows: for $n = 1$,

$$\mu(\theta F_1) := \frac{1}{\#\mathcal{F}_1} = 1$$

and for $n \geq 2$,

$$\mu(\theta F_n) := \frac{1}{\#\mathcal{F}_n(\theta F_{n-1})}\, \mu(\theta F_{n-1}) \qquad (F_n \subset F_{n-1}) .$$

This procedure thus defines inductively a mass on any rectangle used in the construction of $\mathbf{K}_{c(k)}$. In fact a lot more is true — μ can be further extended to all Borel subsets A of Ω to determine $\mu(A)$ so that μ constructed as above actually defines a measure supported on $\mathbf{K}_{c(k)}$; see [Fal90, Proposition 1.7]. We state this formally as a

Fact. The probability measure μ constructed above is supported on $\mathbf{K}_{c(k)}$ and for any Borel subset A of Ω

$$\mu(A) \; = \; \inf \sum_{F \in \mathcal{F}} \mu(F)$$

where the infimum is taken over all coverings $\mathcal{F}$ of A by rectangles $F \in \{\mathcal{F}_n : n \geq 1\}$.

Notice that, in view of (27), we simply have that

$$\mu(\theta F_n) = \frac{1}{\#\mathcal{F}_n} \qquad (n \geq 1) \; .$$

4.4.3 A lower bound for dim $\mathbf{K}_{c(k)}$

Let A be an arbitrary ball with centre a not necessarily in Ω and of radius $r(A) < \theta\rho_*(k^{n_o})$ where $\rho_*(r) := \max_{1 \leq i \leq t} \rho_i(r)$ and n_o is to be determined later. We now determine an upper bound for $\mu(A)$ in terms of its radius. Choose $n \geq n_o$ so that

$$\theta\rho_*(k^{n+1}) \; < \; r(A) \; \leq \; \theta\rho_*(k^n) \; .$$

Without loss of generality, assume that $A \cap \mathbf{K}_{c(k)} \neq \emptyset$ since otherwise there is nothing to prove. Clearly

$$\mu(A) \; \leq \; \mathcal{N}_{n+1}(A) \times \mu(\theta F_{n+1})$$

where

$$\mathcal{N}_{n+1}(A) \; := \; \#\{\theta F_{n+1} \subset \mathcal{F}_{n+1} : \theta F_{n+1} \cap A \neq \emptyset\} \; .$$

If $\theta F_{n+1} \cap A \neq \emptyset$, then $\theta F_{n+1} \subset 3\,A$ since $r(A) \geq \theta\rho_i(k^{n+1})$ for $1 \leq i \leq t$. The balls in $\mathcal{F}_{n+1}$ are disjoint and have comparable m measure (condition (C*)), thus

$$\mathcal{N}_{n+1}(A) \; \leq \; \frac{m(3A)}{a\, m(\theta F_{n+1})} \; .$$

It follows by (27), that

$$\mu(A) \le \frac{m(3A)}{a\, m(\theta F_{n+1})} \times \frac{1}{\# \mathcal{F}_{n+1}} \le \frac{m(3A)}{a\, m(\theta F_1)} \left(\frac{2}{\kappa}\right)^n .$$

Using the fact that $\rho_*(k^n) \le \lambda_*^l(k)^{-(n-1)} \rho_*(k)$, it is easily verified that

$$\frac{1}{a\, m(\theta F_1)} \left(\frac{2}{\kappa}\right)^n < \left(\frac{1}{\theta \rho_*(k^n)}\right)^{\varepsilon(k)}$$

for

$$n \ge n_1 := \left[4 + \frac{\log \frac{(\theta \rho_*(k))^{\varepsilon(k)}}{a\, m(\theta F_1)}}{\log \frac{2}{\kappa}} \right] \qquad \text{and} \qquad \varepsilon(k) := \frac{4 \log \frac{2}{\kappa}}{\log \lambda_*^l(k)} .$$

Note $\varepsilon(k) > 0$ since $\kappa \le 1$. Hence,

$$\mu(A) \le m(3A) \times (\theta \rho_*(k^n))^{-\varepsilon(k)} .$$

Since $A \cap \mathbf{K}_{c(k)} \neq \emptyset$, there exists some point $x \in A \cap \Omega$. Moreover, $3A \subset B(x, 4\, r(A))$ which together with condition (A*) implies that

$$m(3A) \le m(B(x, 4\, r(A))) \le r(A)^{\delta - \varepsilon(k)}$$

for $r(A) \le r_0 := r_0(\varepsilon(k))$. Now $\rho_*(r) \to 0$ as $r \to \infty$, so $\theta \rho_*(k^n) < r_0$ for $n \ge n_2$. Thus, for $n \ge n_o := \max\{n_1, n_2\}$

$$\mu(A) \le r(A)^{\delta - \varepsilon(k)} \times (\theta \rho_*(k^n))^{-\varepsilon(k)} .$$

On using the fact that $r(A) \le \theta \rho_*(k^n)$, we obtain that

$$\mu(A) \le r(A)^{\delta - 2\varepsilon(k)} .$$

This together with the mass distribution principle implies that

$$\dim \mathbf{K}_{c(k)} \ge \delta - 2\varepsilon(k) .$$

Note that since $\varepsilon(k) \to 0$ as $k \to \infty$ we have that $\dim \mathbf{K}_{c(k)} \to \delta$ as $k \to \infty$.

4.4.4 Completion of proof

Recall, that $\dim(\cup_{\alpha \in J} R_\alpha) < \delta$. Now suppose that $\dim \mathbf{Bad}(\rho_1, \ldots, \rho_t) < \delta$. It follows that, $\max\{\dim \mathbf{Bad}(\rho_1, \ldots, \rho_t), \dim(\cup_{\alpha \in J} R_\alpha)\} = \delta - \eta$ for some $\eta > 0$. Fix some k sufficiently large so that $2\,\epsilon(k) < \eta$. Then,

$$\dim \mathbf{K}_{c(k)} \geq \delta - 2\varepsilon(k) > \delta - \eta .$$

By construction, for any point $x \in \mathbf{K}_{c(k)}$ we have that

$$d_i(x_i, R_{\alpha,i}) \geq c(k)\, \rho_i(\beta_\alpha) \qquad \forall\, \alpha \in J \text{ with } \beta_\alpha \geq k \qquad (1 \leq i \leq t) .$$

Now let $J_k := \{\alpha \in J : \beta_\alpha < k\}$. If (24) is true for our fixed k then $J_k = \emptyset$ and clearly $\mathbf{K}_{c(k)} \subseteq \mathbf{Bad}(\rho_1, \ldots, \rho_t)$. In turn, $\dim \mathbf{Bad}(\rho_1, \ldots, \rho_t) \geq \dim \mathbf{K}_{\mathbf{c(k)}} > \delta - \eta$ and we have a contradiction. So suppose, $J_k \neq \emptyset$ and let $\mathcal{R}_k := \{R_\alpha : \alpha \in J_k\}$. For any fixed k the number of elements in J_k is finite. So, if $x \notin R_k$ then there exists a constant $c'(x) > 0$ such that

$$d_i(x_i, R_{\alpha,i}) \geq c'(k)\, \rho_i(\beta_\alpha) \qquad \forall\, \alpha \in J_k \quad (1 \leq i \leq t) .$$

Thus, for $x \in \mathbf{K}_{c(k)} \backslash \mathcal{R}_k$

$$d_i(x_i, R_{\alpha,i}) \geq c^*(k)\, \rho_i(\beta_\alpha) \qquad \forall\, \alpha \in J \quad (1 \leq i \leq t) ,$$

where $c^*(x) := \min\{c(k), c'(x)\}$. It follows that $\mathbf{Bad}(\rho_1, \ldots, \rho_t) \supseteq \mathbf{K}_{c(k)} \backslash \mathcal{R}_k$ and since $\dim \mathcal{R}_k < \dim \mathbf{K}_{c(k)}$ we have that

$$\dim \mathbf{Bad}(\rho_1, \ldots, \rho_t) \geq \dim(\mathbf{K}_{c(k)} \backslash \mathcal{R}_k) = \dim \mathbf{K}_{c(k)} \geq \delta - 2\varepsilon(k) > \delta - \eta .$$

This is a contradiction and completes the proof of Theorem 4.

4.5 Applications

In this section we look at applications of the theorems of this chapter. We begin in the field of real Diophantine approximation. For $N \geq 1$, let $\Omega \subset \mathbb{R}^N$ be a compact metric space which supports a non-atomic finite measure m. We consider the set

$$\mathbf{Bad}_\Omega(i_1, ..., i_N) \ := \ \Omega \cap \mathbf{Bad}(i_1, ..., i_N).$$

where the set $\mathbf{Bad}(i_1, ..., i_N)$ of $(i_1, ..., i_N)$–badly approximable points in $\mathbb{R}^N$ was defined in §4.2.4. Initially this section considers the more specific set

$$\mathbf{Bad}_\Omega(N) := \mathbf{Bad}_\Omega(i_1, ..., i_N) \qquad \text{for} \quad i_1 = ... = i_N := 1/N$$

where Ω supports an 'absolutely α–decaying' measure.

4.5.1 Bad(N) and 'absolutely α–decaying' measures

To demonstrate the versatility of our results, primarily Theorem 3, we look at the case where the measure m is 'absolutely α–decaying'. This type of measure has been recently studied and, although introduced in [KLW03], we use the following definition from [PV03].

Let $\mathcal{L}$ denote a generic $(N-1)$–dimensional hyperplane of $\mathbb{R}^N$ and let $\mathcal{L}^{(\epsilon)}$ denote its ϵ-neighborhood. A measure m is *absolutely α–decaying* if there exist positive constants C, α, r_o such that for any hyperplane $\mathcal{L}$, any $\epsilon > 0$ and any open ball $B(x,r)$ in $\mathbb{R}^N$,

$$m\left(B(x,r) \cap \mathcal{L}^{(\epsilon)}\right) \ \leq \ C \left(\frac{\epsilon}{r}\right)^\alpha m(B(x,r)) \qquad \forall\, x \in \Omega \ \ \forall\, r < r_o \,.$$

It is easily seen that N-dimensional Lebesgue measure is absolutely 1–decaying. In the case $\Omega \subset \mathbb{R}$, note that if the measure m satisfies condition (A) with exponent δ then m is automatically absolutely δ–decaying.

Theorem 6. *Let Ω be a compact subset of $\mathbb{R}^{\mathrm{N}}$ which supports an absolutely α–decaying measure m satisfying condition (A) . Then*

$$\dim \mathbf{Bad}_{\Omega}(\mathrm{N}) = \dim \Omega \ .$$

Note that, in the case $\Omega := \mathbb{R}^{\mathrm{N}}$, the conditions are trivially satisfied and thus the above theorem implies Schmidt's result that $\dim \mathbf{Bad}(\mathrm{N}) = \mathrm{N}$, as mentioned in the introduction of this chapter.

Proof. With reference to the basic framework of §4.1, the set $\mathbf{Bad}_{\Omega}(\mathrm{N})$ can be expressed in the form $\mathbf{Bad}(\rho)$ with $\rho(r) := r^{-(1+\frac{1}{\mathrm{N}})}$ and

$$X = (\mathbb{R}^{\mathrm{N}}, d) \ , \quad J := \{((p_1, \ldots, p_{\mathrm{N}}), q) \in \mathbb{N}^{\mathrm{N}} \times \mathbb{N}\backslash\{0\}\} \ ,$$

$$\alpha := ((p_1, \ldots, p_{\mathrm{N}}), q) \in J \ , \quad \beta_\alpha := q \ , \quad R_\alpha := (p_1/q, \ldots, p_{\mathrm{N}}/q) \ .$$

Here d is the standard sup metric on $\mathbb{R}^{\mathrm{N}}$; $d(x, y) := \max\{d(x_1, y_1), \ldots, d(x_{\mathrm{N}}, y_{\mathrm{N}})\}$. Thus balls $B(c, r)$ in $\mathbb{R}^{\mathrm{N}}$ are genuinely cubes of sidelength $2r$.

We show that the conditions of Theorem 3 are satisfied. Clearly the function ρ satisfies condition (B) and we are given that the measure m supported on Ω satisfies condition (A). Also, since the resonant sets are points the condition that $\dim(\cup_{\alpha\in J} R_\alpha) < \delta$ is satisfied. We need to establish the existence of the disjoint collection $C(\theta B_n)$ of balls (cubes) $2\theta B_{n+1}$ where B_n

is an arbitrary ball of radius $k^{-n(1+\frac{1}{k})}$ with centre in Ω. In view of Lemma 6, there exists a disjoint collection $\mathcal{C}(\theta B_n)$ such that

$$\#\mathcal{C}(\theta B_n) \ \geq \ \kappa_1 \, k^{(1+\frac{1}{k})\delta} \ ; \tag{28}$$

i.e. (15) of Theorem 3 holds. We now verify that (16) is satisfied for any such collection. We consider two cases.

Case 1: N $= 1$. The trivial argument of §4.1.3 shows that any interval θB_n with $\theta := \frac{1}{2}k^{-2}$ contains at most one rational p/q with $k^n \leq q < k^{n+1}$; i.e. $\alpha \in J(n+1)$. Thus, for k sufficiently large

$$\text{l.h.s. of (16)} \ \leq \ 1 \ < \ \tfrac{1}{2} \ \times \ \text{r.h.s. of (28)} \ .$$

Hence (16) is trivially satisfied and Theorem 3 implies the desired result.

Case 2: N ≥ 2. We shall prove the theorem in the case that N $= 2$. There are no difficulties and no new ideas are required in extending the proof to higher dimensions.

Suppose that there are three or more rational points $(p_1/q, p_2/q)$ with $k^n \leq q < k^{n+1}$ lying within the ball/square θB_n. Now put $\theta := 2^{-1}(2k^3)^{-1/2}$. Then the 'triangle' argument of §4.2.4 (where m is Lebesgue measure) implies that the rational points must lie on a line $\mathcal{L}$ passing through θB_n. Since the balls $2\theta B_{n+1}$ are disjoint and m is absolutely α–decaying, it follows that, for

$\epsilon := 8\theta k^{-(n+1)\frac{3}{2}}$,

$$
\begin{aligned}
\text{l.h.s. of (16)} &\leq \#\{2\theta B_{n+1} \subset \mathcal{C}(\theta B_n) : 2\theta B_{n+1} \cap \mathcal{L} \neq \emptyset\} \\
&\leq \#\left\{2\theta B_{n+1} \subset \mathcal{C}(\theta B_n) : 2\theta B_{n+1} \subset \mathcal{L}^{(\epsilon)}\right\} \\
&\leq \frac{m(\theta B_n \cap \mathcal{L}^{(\epsilon)})}{m(2\theta B_{n+1})} \\
&\leq a^{-1} b C\, 8^{\alpha}\, 2^{-\delta}\, k^{\frac{3}{2}(\delta-\alpha)} \\
&< \tfrac{1}{2} \times \text{r.h.s. of (28)} \qquad \text{for } k \text{ sufficiently large.}
\end{aligned}
$$

Hence (16) is satisfied and Theorem 3 implies the desired result.

□

4.5.2 Intersecting Bad(N) with self-similar sets

We deviate for a moment to consider the construction of some fractals. The purpose is to show a dimension result using an example of an absolutely α–decaying measure which is naturally supported on these fractals.

A function $S : \mathbb{R}^N \to \mathbb{R}^N$ is a *similarity map* if it can be written as

$$S(x) = c\Theta(x) + a$$

where $c \in \mathbb{R}^+$, $a \in \mathbb{R}^N$ and $\Theta \in O(N, \mathbb{R})$, the orthogonal group. The similarity map S is *contracting* if $c < 1$. For any finite family $\{S_1, \ldots, S_k\}$ of contracting similarity maps there exists a unique set K, know as the *attractor*, for which

$$K = \bigcup_{i=1}^{k} S_i(K).$$

In other words K is invariant under the family of maps. Such a family of maps is an example of an 'iterated function scheme' - see [Fal90, §9.1] for more information.

A finite family of contracting similarities $\{S_1, \ldots, S_k\}$ is *irreducible* if there does not exist a finite collection of proper affine subspaces of $\mathbb{R}^N$ which is invariant under $\{S_1, \ldots, S_k\}$. The *open set condition* is satisfied for the above family if there exists an open set $U \subset \mathbb{R}^N$ such that

$$S_i(U) \subset U \qquad \forall\, i = 1, \ldots, k,$$

and

$$i \neq j \implies S_i(U) \cap S_j(U) = \emptyset.$$

The *similarity dimension* $s > 0$ of the family $\{S_1, \ldots, S_k\}$ is the unique solution of $\sum_i c_i^s = 1$. Hutchinson [Hut81] proved that if S_i satisfy the open set condition then

$$0 \; < \; \mathcal{H}^s(K) \; < \; \infty.$$

This trivially implies $\dim K = s$. Moreover it shows that K supports a non-atomic, finite measure. Thus the attractor K can be seen as the compact metric space Ω from our general framework.

This type of attractor was studied recently by Kleinbock, Lindenstrauss and Weiss. The following theorem combines Theorems 2.2 and 8.1 of [KLW03].

Theorem KLW *Let $\{S_1, \ldots, S_k\}$ be an irreducible family of contracting self similarity maps of $\mathbb{R}^N$ satisfying the open set condition and let m be the restriction of $\mathcal{H}^\delta$ to its attractor K where $\delta := \dim K$. Then m is absolutely α-decaying and satisfies condition (A).*

The above theorem implies the following application of Theorem 6. It has also been independently established by Kleinbock and Weiss [KLW03, Theorem 10.3], [KW03].

Corollary 9. *Let $\{S_1, \ldots, S_k\}$ be an irreducible family of contracting self similarity maps of $\mathbb{R}^N$ satisfying the open set condition and let m be the restriction of $\mathcal{H}^\delta$ to its attractor K where $\delta := \dim K$. Then*

$$\dim(K \cap \mathbf{Bad}(N)) = \dim K \ .$$

A simple example of such an attractor is the middle third Cantor set, $C \subset [0,1]$. If $S_1 : x \mapsto \frac{x}{3}$ and $S_2 : x \mapsto \frac{x}{3} + \frac{2}{3}$ we can see that

$$C = S_1(C) \cup S_2(C) \quad \text{and} \quad s = \frac{\log 2}{\log 3} = \dim C$$

is the unique solution to $(1/3)^s + (1/3)^s = 1$. Other examples of attractors which work with the above results are the von Koch curve and the Sierpiński gasket.

4.5.3 Intersecting $\mathbf{Bad}(i_1, ..., i_N)$ with product spaces

Naturally we consider next the condition of absolute α–decay in relation to the more general set $\mathbf{Bad}_\Omega(i_1, ..., i_N)$. We state two theorems in this section both with illuminating corollaries. However it turns out that the first result does not require the 'decay' condition. Note that this is an extension of the discussion on the 2-dimensional set $\mathbf{Bad}(i, j)$ from §4.2.4.

Theorem 7. *For $1 \leq j \leq \mathrm{N}$, let Ω_j be a compact subset of $\mathbb{R}$ which supports a measure m_j satisfying condition (A) with exponent δ_j. Let Ω denote the product set $\Omega_1 \times \ldots \times \Omega_N$. Then, for any* N*-tuple $(i_1, ..., i_N)$ with $i_j \geq 0$ and $\sum_{j=1}^{\mathrm{N}} i_j = 1$,*

$$\dim \mathbf{Bad}_\Omega(i_1, ..., i_N) = \dim \Omega \ .$$

A simple application of the above theorem leads to the following result.

Corollary 10. *Let K_1 and K_2 be regular Cantor subsets of $\mathbb{R}$. Then*

$$\dim((K_1 \times K_2) \cap \mathbf{Bad}(i,j)) \ = \ \dim(K_1 \times K_2) \ = \ \dim K_1 \ + \ \dim K_2 \ .$$

Proof of Theorem 7. Without loss of generality assume that $\mathrm{N} \geq 2$. The case that $\mathrm{N} = 1$ is covered by Theorem 6. For the sake of clarity, as with the proof of Theorem 6, we shall restrict our attention to the case $\mathrm{N} = 2$.

Recall that since $\Omega_j \subset \mathbb{R}$ and m_j satisfies (A), then m_j is automatically absolutely δ_j–decaying. A relatively straightforward argument shows that $m := m_1 \times m_2$ is absolutely α–decaying on Ω with $\alpha := \min\{\delta_1, \delta_2\}$. In fact the following more general fact is true - see [KLW03, §9].

Fact: For $2 \leq j \leq N$, if each m_j is absolutely α_j–decaying on Ω_j, then $m := m_1 \times \ldots \times m_N$ is absolutely α–decaying on $\Omega = \Omega_1 \times \ldots \times \Omega_N$ with $\alpha = \min\{\alpha_1, \ldots, \alpha_N\}$.

Now let us write $\mathbf{Bad}_\Omega(i,j)$ for $\mathbf{Bad}_\Omega(i_1, i_2)$ and without loss of generality assume that $i < j$. The case $i = j$ is already covered by Theorem 6 since

m is absolutely α–decaying on Ω and clearly satisfies condition (A). The set $\mathbf{Bad}_\Omega(i,j)$ can be expressed in the form $\mathbf{Bad}(\rho_1,\rho_2)$ with $\rho_1(r) = r^{-(1+i)}$, $\rho_2(r) = r^{-(1+j)}$ and

$$X = \mathbb{R}^2 \ , \quad \Omega := \Omega_1 \times \Omega_2 \ , \quad J := \{((p_1,p_2),q) \in \mathbb{N}^2 \times \mathbb{N}\backslash\{0\}\} \ ,$$

$$\alpha := ((p_1,p_2),q) \in J \ , \quad \beta_\alpha := q \ , \quad R_\alpha := (p_1/q, p_2/q) \ .$$

With reference to Theorem 5, the functions ρ_1, ρ_2 satisfy condition (B*) and the measures m_1, m_2 satisfy condition (A). Also note that $\dim(\cup_{\alpha\in J} R_\alpha) = 0$ since the union in question is countable. We need to establish the existence of the collection $\mathcal{C}(\theta F_n)$, where F_n is an arbitrary closed rectangle of size $2k^{-n(1+i)} \times 2k^{-n(1+j)}$ with centre c in Ω. In view of Lemma 6, there exists a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1} \subset \theta F_n$ such that

$$\#\mathcal{C}(\theta F_n) \ \geq \ \kappa_1 \, k^{(1+i)\delta_1} k^{(1+j)\delta_2} \, ; \tag{29}$$

i.e. (20) of Theorem 5 is satisfied. We now verify that (21) is satisfied for any such collection. With $\theta = 2^{-1}(2k^3)^{-1/2}$, the 'triangle' argument of §4.2.4 implies that

$$\text{l.h.s. of } \ (21) \ \leq \ \#\{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \cap \mathcal{L} \neq \emptyset\} \ , \tag{30}$$

where $\mathcal{L}$ is a line passing through θF_n. Consider the thickening $T(\mathcal{L})$ of $\mathcal{L}$ obtained by placing rectangles $4\theta F_{n+1}$ centred at points of $\mathcal{L}$; that is, by 'sliding' a rectangle $4\theta F_{n+1}$, centred at a point of $\mathcal{L}$, along $\mathcal{L}$. Then, since the rectangles $2\theta F_{n+1} \subset \mathcal{C}(\theta F_n)$ are disjoint,

$$\begin{aligned} &\#\{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \cap \mathcal{L} \neq \emptyset\} \\ &\qquad\qquad \leq \ \#\{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \subset T(\mathcal{L})\} \\ &\qquad\qquad \leq \ \frac{m(T(L) \cap \theta F_n)}{m(2\theta F_{n+1})} \ . \end{aligned} \tag{31}$$

Without loss of generality we can assume that $\mathcal{L}$ passes through the centre of θF_n. To see this, suppose that $m(T(\mathcal{L}) \cap \theta F_n) \neq 0$ since otherwise there is nothing to prove. Then, there exists a point $x \in T(\mathcal{L}) \cap \theta F_n \cap \Omega$ such that

$$T(\mathcal{L}) \cap \theta F_n \subset 2\theta F_n' \cap T'(\mathcal{L}') .$$

Here F_n' is the rectangle of size $k^{-n(1+i)} \times k^{-n(1+j)}$ centred at x, $\mathcal{L}'$ is the line parallel to $\mathcal{L}$ passing through x and $T'(\mathcal{L}')$ is the thickening obtained by 'sliding' a rectangle $8\theta F_{n+1}$ centred at x, along $\mathcal{L}'$. Then the following argument works just as well on $2\theta F_n' \cap T'(\mathcal{L}')$.

Let Δ denote the slope of the line $\mathcal{L}$ and assume that $\Delta \geq 0$. The case $\Delta < 0$ can be dealt with similarly. By moving the rectangle θF_n to the origin, straightforward geometric considerations lead to the following facts:

(F1) $$T(\mathcal{L}) = \mathcal{L}^{(\epsilon)} \quad \text{where} \quad \epsilon := \frac{4\theta\left(k^{-(n+1)(1+j)} + \Delta k^{-(n+1)(1+i)}\right)}{\sqrt{1+\Delta^2}} ,$$

(F2) $T(\mathcal{L}) \cap \theta F_n \subset F(c; l_1, l_2)$ where $F(c; l_1, l_2)$ is the rectangle with the same centre c as F_n and of size $2l_1 \times 2l_2$ with

$$l_1 := \frac{\theta}{\Delta}\left(k^{-n(1+j)} + 4k^{-(n+1)(1+j)} + \Delta k^{-(n+1)(1+i)}\right) \quad \text{and} \quad l_2 := \theta k^{-n(1+j)} .$$

We now estimate the right hand side of (31) by considering two cases. Throughout, let a_i, b_i denote the constants associated with the measure m_i and condition (A) and let

$$\varpi := 3\left(\frac{4b_1 b_2}{\kappa_1\, a_1 a_2 2^{\delta_1+\delta_2}}\right)^{1/\delta_1} .$$

The following two cases consider whether the line $\mathcal{L}$ travels through the long or the short side of θF_n, i.e. they compare the slope Δ of $\mathcal{L}$ with the slope of the diagonal which links opposite corners of θF_n.

Case (i): $\Delta \geq \varpi k^{-n(1+j)}/k^{-n(1+i)}$. In view of (F2) above, we trivially have that

$$m(\theta F_n \cap T(\mathcal{L})) \leq m(F(c; l_1, l_2)) \leq b_1 b_2 l_1^{\delta_1} l_2^{\delta_2} .$$

It follows that

$$\frac{m(T(L) \cap \theta F_n)}{m(2\theta F_{n+1})} \leq \frac{b_1 b_2 l_1^{\delta_1} l_2^{\delta_2}}{a_1 a_2 (2\theta)^{\delta_1+\delta_2} k^{-(n+1)(1+j)\delta_1} k^{-(n+1)(1+i)\delta_2}}$$

$$\leq \frac{b_1 b_2}{a_1 a_2 2^{\delta_1+\delta_2}} \left(\frac{1}{\varpi} + \frac{1}{\varpi k^{1+j}} + \frac{1}{k^{1+i}}\right)^{\delta_1} k^{(1+j)\delta_1} k^{(1+i)\delta_2}$$

$$\leq \frac{b_1 b_2}{a_1 a_2 2^{\delta_1+\delta_2}} \left(\frac{3}{\varpi}\right)^{\delta_1} k^{(1+j)\delta_1} k^{(1+i)\delta_2} = \frac{\kappa_1}{4} k^{(1+j)\delta_1} k^{(1+i)\delta_2} .$$

Case (ii): $0 \leq \Delta < \varpi k^{-n(1+j)}/k^{-n(1+i)}$. By the covering lemma of §4.3, there exists a collection $\mathcal{B}_n$ of disjoint balls B_n with centres in $\theta F_n \cap \Omega$ and radii $\theta k^{-n(1+j)}$ such that

$$\theta F_n \cap \Omega \subset \bigcup_{B_n \in \mathcal{B}_n} 3B_n .$$

Since $i < j$, it is easily verified that the disjoint collection $\mathcal{B}_n$ is contained in $2\theta F_n$ and thus $\#\mathcal{B}_n \leq m(2\theta F_n)/m(B_n)$. It follows that

$$m(\theta F_n \cap T(\mathcal{L})) \leq m\left(\cup_{B_n \in \mathcal{B}_n} 3B_n \cap T(\mathcal{L})\right)$$

$$\leq \#\mathcal{B}_n \, m(3B_n \cap T(\mathcal{L}))$$

$$\leq \frac{m(2\theta F_n)}{m(B_n)} m\left(3B_n \cap \mathcal{L}^{(\epsilon)}\right) \qquad \text{by (F1) above}$$

$$\leq m(2\theta F_n) \frac{m(3B_n)}{m(B_n)} \left(\frac{\epsilon}{3\theta k^{-n(i+j)}}\right)^{\alpha}$$

since m is absolutely α–decaying. Now notice that for this case

$$\frac{\epsilon}{3\theta k^{-n(i+j)}} \leq \frac{4}{3}\left(k^{-(1+j)} + \varpi k^{-(1+i)}\right) .$$

Hence, for k sufficiently large we have that

$$\frac{m(T(\mathcal{L}) \cap \theta F_n)}{m(2\theta F_{n+1})} \leq \frac{\kappa_1}{4}\, k^{(1+j)\delta_1} k^{(1+i)\delta_2} .$$

On combining the above two cases, we have that

$$\text{l.h.s. of (21)} \leq \frac{m(T(\mathcal{L}) \cap \theta F_n)}{m(2\theta F_{n+1})} \leq \frac{\kappa_1}{4}\, k^{(1+j)\delta_1} k^{(1+i)\delta_2} = \frac{1}{4} \times \text{ l.h.s. of (29).}$$

Hence (21) is satisfied and Theorem 5 implies the desired result.

□

The argument used to establish Theorem 7 can be adapted to prove a slightly more general result.

Theorem 8. *For $1 \leq j \leq \mathrm{N}$, let Ω_j be a compact subset of $\mathbb{R}^{d_j}$ which supports an absolutely α_j–decaying measure m_j satisfying condition (A) with exponent δ_j. Let Ω denote the product set $\Omega_1 \times \ldots \times \Omega_N$. Then, for any* N*-tuple $(i_1, ..., i_N)$ with $i_j \geq 0$ and $\sum_{j=1}^{\mathrm{N}} d_j\, i_j = 1$,*

$$\dim \mathbf{Bad}_\Omega(\underbrace{i_1, \ldots, i_1}_{d_1 \text{ times}};\ \underbrace{i_2, \ldots, i_2}_{d_2 \text{ times}};\ \ldots;\ \underbrace{i_N, \ldots, i_N}_{d_\mathrm{N} \text{ times}}) = \dim \Omega = \sum_{j=1}^{N} \delta_j .$$

The following is a simple consequence of Theorem KLW and Theorem 8.

Corollary 11. *For $1 \leq j \leq \mathrm{N}$, let K_j be the attractor of a finite irreducible family of contracting self similarity maps of $\mathbb{R}^{d_j}$ satisfying the open set condition. Let m_j be the restriction of $\mathcal{H}^{\delta_j}$ to K_j where $\delta_j = \dim K_j$. Let K denote* *the 'product attractor'* $K_1 \times \ldots \times K_N$. *Then, for any* N*-tuple* $(i_1, \ldots, i_N)$ *with* $i_j \geq 0$ *and* $\sum_{j=1}^{N} d_j\, i_j = 1$,

$$\dim(K \cap \mathbf{Bad}(\underbrace{i_1, \ldots, i_1}_{d_1 \text{ times}};\ \underbrace{i_2, \ldots, i_2}_{d_2 \text{ times}};\ \ldots;\ \underbrace{i_N, \ldots, i_N}_{d_N \text{ times}})) = \dim K\ .$$

As an application of Corollary 11 we obtain the following statement which to some extent is more illuminating – even this special case appears to be new.

Corollary 12. *Let* $V \subset \mathbb{R}^2$ *be the von Koch curve and* $K \subset \mathbb{R}$ *be the middle third Cantor set. Then, for any positive* i *and* j *with* $2\,i + j = 1$

$$\dim\left((V \times K) \cap \mathbf{Bad}(i, i, j)\right) = \dim(V \times K) = \frac{\log 8}{\log 3}\ .$$

4.5.4 Remarks related to Schmidt's conjecture.

In §4.2.4, we mentioned the result of [PV02] that

$$\dim(\mathbf{Bad}(i, j) \cap \mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1)) = 2\ .$$

This can easily be obtained via Theorem 7. To see this, first of all notice that $\mathbf{Bad} \times \mathbf{Bad} = \mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1)$. For $N \geq 2$, let $F_N := \{x \in [0, 1] : x := [a_1, a_2, \ldots] \text{ with } a_i \leq N \text{ for all } i\}$. Thus F_N is the set of real numbers in the unit interval with partial quotients bounded above by N. By definition F_N is a compact subset of **Bad** and moreover it is well known that F_N supports a measure m_N which satisfies condition (A) with exponent δ_N with $\delta_N \to 1$ as $N \to \infty$. Now let $\Omega := F_N \times F_N$, then Theorem 7 implies that

$$\dim(\mathbf{Bad}(i, j) \cap \mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1)) \geq \dim(\mathbf{Bad}_\Omega(i, j)) = 2\delta_N\ .$$

On letting $N \to \infty$, we obtain that $\dim(\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1,0) \cap \mathbf{Bad}(0,1)) \geq 2$. The complementary upper bound result is trivial since the set in question is a subset of $\mathbb{R}^2$.

Recall, that Schmidt's conjecture states that $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(i',j') \neq \emptyset$. In order to illustrate a possible approach towards the conjecture via the results of this chapter we consider the special case of $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2)$. A straightforward application of Theorem 3 together with the 'triangle' argument of §4.2.4 leads to the following enticing statement:

If there exists a compact subset Ω of $\mathbf{Bad}(i,j)$ which supports a measure m satisfying condition (A) with exponent $\delta > 1$, then

$$\dim(\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2)) \geq \delta \,.$$

Clearly, this would imply that $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2) \neq \emptyset$. Regarding the above statement, it is not particularly difficult to prove the existence of a compact subset Ω supporting a measure m satisfying condition (A) with $\delta < 1$. However, from this we are not able to deduce that $\dim(\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2)) \geq \delta$ or even that $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2) \neq \emptyset$.

4.5.5 Rational maps

In this section we consider the 'badly approximable' analogue of the 'shrinking target' problem introduced in [HV95] for expanding rational maps. Let T be an expanding rational map (degree ≥ 2) of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $J(T)$ be its Julia set. The *Julia set* of a rational map T is the closure of the set of repelling periodic points of T. Note that $J(T)$ is non-empty, compact and invariant with respect to T. See [Fal90, §14] for a basic introduction to rational maps. For any $z_o \in J(T)$ consider the set

$$\mathbf{Bad}_{z_o}(J) := \{z \in J(T) \ : \exists \, c(z) > 0 \ \text{s.t.} \ T^n(z) \notin B\left(z_o, c(z)\right) \ \forall \ n \in \mathbb{N}\} \ .$$

Clearly, the forward orbit of points in $\mathbf{Bad}_{z_o}(J)$ are not dense in $J(T)$. Now let m be Sullivan measure and $\delta = \dim J(T)$. Thus m is a non-atomic, δ–conformal probability measure supported on $J(T)$ and since T is expanding it satisfies condition (A) [Sul81, Theorem 4]. Moreover, m is equivalent to δ–dimensional Hausdorff measure $\mathcal{H}^\delta$ - see [HV95, §2.3.4]. In view of the 'Khintchine type' result for expanding rational maps (see, for example [BDV03, §8.4]) it is easily verified that $\mathcal{H}^\delta(\mathbf{Bad}_{z_o}(J)) = 0 = m(\mathbf{Bad}_{z_o}(J))$. Nevertheless, the set $\mathbf{Bad}_{z_o}(J)$ is large in that it is of maximal dimension.

Theorem 9.

$$\dim \mathbf{Bad}_{z_o}(J) \ = \ \delta \ .$$

This result is not new and has been established by numerous people, for example [AN97]. However, we give a short proof which indicates the versatility and generality of our framework and results.

Proof of Theorem 9. In view of the bounded distortion property for expanding maps , [HV95, Proposition 1], we can rewrite $\mathbf{Bad}_{z_o}(J)$ in terms of points in the Julia set which 'stay clear' of balls centred around the backward orbit of the selected point z_o. Thus, the set $\mathbf{Bad}_{z_o}(J)$ is equivalent to the set

$$\{ \ z \in J(T) \ : \ \exists \, c(z)>0 \ \text{s.t.} \ z \notin B\left(y, \, c(z)|(T^q)'(y)|^{-1}\right) \ \ \forall \, (y,q) \in I \ \} \ ,$$

where $I := \{(y,q) : q \in \mathbb{N} \text{ with } T^q(y) = z_o\}$. Also, since T is expanding, $\dim J(T) < 2$ [Sul81, Theorem 4]. So we assume that $J(T)$ is a subset of $\mathbb{C}$ and we use the usual metric on $\mathbb{C}$. It is now clear that $\mathbf{Bad}_{z_o}(J)$ can be expressed in the form $\mathbf{Bad}(\rho)$ with $\rho(r) := r^{-1}$ and

$$X = \Omega := J(T)\ ,\quad J := I\ ,\quad \alpha := (y,q) \in I\ ,\quad \beta_\alpha := |(T^q)'(y)|\ ,\quad R_\alpha := y\ .$$

With reference to Theorem 3, the Sullivan measure m and the function ρ satisfy conditions (A) and (B) respectively. To deduce Theorem 9 from Theorem 3 we need to establish the existence of the disjoint collection $\mathcal{C}(\theta B_n)$ of balls $2\theta B_{n+1}$ where B_n is an arbitrary ball of radius k^{-n} with centre in Ω. In view of Lemma 6, for k sufficiently large, there exists a disjoint collection $\mathcal{C}(\theta B_n)$ such that

$$\#\mathcal{C}(\theta B_n) \ \geq\ \kappa_1\, k^{\delta}\ ; \tag{32}$$

i.e. (15) of Theorem 3 holds. We now verify that (16) is satisfied for any such collection. First we recall a key result which is the second part of the statement of Lemma 8 in [HV97]. For ease of reference we keep the same numbering of constants as in [HV97].

- *Constant Multiplicity:* For $X \in \mathbb{R}^+$, let $P(X)$ denote the set of pairs $(y,q) \in I$ such that $f_q(y) - C_8 \leq X \leq f_{q+1}(y) + C_8$, where $f_q(y) := \log|(T^q)'(y)|$. Let $z \in J(T)$. Then there are no more than C_9 pairs $(y,q) \in P(X)$ such that $z \in B\left(y, C_{10}\,|(T^q)'(y)|^{-1}\right)$.

We are now in the position to verify (16) of Theorem 3. By definition $J(n+1) := \{(y,q) \in I : k^{n-1} \leq |(T^q)'(y)| < k^n\}$ and let $\theta := C_{10}k^{-1}$. It follows that

$$\begin{aligned} \text{l.h.s. of (16)} \ &\leq\ \#\{\, y \in \theta B_n : (y,q) \in J(n+1)\,\} \\ &\leq\ \#\{\, y \in B(c, C_{10}\,|(T^q)'(y)|^{-1}) \ :\ (y,q) \in J(n+1)\,\}, \end{aligned} \tag{33}$$

where c is the centre of θB_n. Without loss of generality, assume that $|T'(z_0)| > 1$. Otherwise, since T is expanding we simply work with some higher iterate T^q of T for which $|(T^q)'(z_o)| > 1$. Next, note that the chain rule implies

$$P(X) = \{ (y,q) \in I : |T'(z_o)|^{-1} e^{X-C_8} \leq |(T^q)'(y)| \leq e^{X+C_8} \} .$$

Then, by the constant multiplicity statement,

$$\#\{ (y,q) \in P(X) : y \in B(c, C_{10} |(T^q)'(y)|^{-1}) \} \leq C_9 .$$

Now let $\Delta := |T'(z_o)|$ and take $X = X_j := \log(\Delta^j k^n) + C_8(2j-1)$ then

$$[k^n, k^{n+1}) \subset \bigcup_{j=1}^{t} [\Delta^{-1} e^{X_j - C_8}, e^{X_j + C_8}]$$

where t is the unique integer such that $(\Delta e^{2C_8})^{t-1} < k \leq (\Delta e^{2C_8})^t$. This implies that the r.h.s. of (33) is $\ll C_9 \log k$, for k large enough. Hence, for k sufficiently large

$$\text{l.h.s. of (16)} \leq \tfrac{1}{2} \times \text{r.h.s. of (32)} .$$

Thus, (16) is easily satisfied and Theorem 3 implies Theorem 9.

□

Remark: It is worth mentioning that our framework also yields (just as easily) the analogue of Theorem 9 within the Kleinian group setup. Briefly, let G be either a geometrically finite Kleinian group of the first kind or a convex co-compact group and let $\Lambda(G)$ denote its limit set. For these groups, Patterson measure supported on $\Lambda(G)$ satisfies condition (A) and plays the role of Sullivan measure. Then, it is not difficult to obtain the Kleinian group analogue of Theorem 9 via Theorem 3; i.e. the set of 'badly approximable' limit points is of full dimension - $\dim \Lambda(G)$.

Chapter 5

General lim sup sets of two approximation functions

5.1 Introduction

As described in Chapter 2, Beresnevich, Dickinson and Velani recently proved laws for the m-measure and Hausdorff f-measure of the following general lim sup set dependent on an approximation function ψ,

$$\Lambda(\psi) := \{ x \in \Omega : x \in B(R_\alpha, \psi(\beta_\alpha)) \text{ for i.m. } \alpha \in J \} .$$

Recall that R_α are resonant points in a compact metric space Ω and are indexed by a countable set J with the function $\beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ assigning a 'weight' β_α to each R_α. Also recall that an approximation function is a positive, real, decreasing function.

Although the general framework of [BDV03] is broad ranging in application, it does not cover the case of a lim sup set dependent on two or more approximation functions. Consider the classical example for approximation functions $\psi_1, \dots, \psi_n$. The set of $(\psi_1, \dots, \psi_n)$–well approximable points in $\mathbb{R}^n$ is defined as the set $W(\psi_1, \dots, \psi_n)$ of points $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ such that

$$\left| x_t - \tfrac{p_t}{q} \right| < \psi_t(q) \quad (1 \le t \le n) \qquad \text{for i.m. } \left(\tfrac{p_1}{q}, \dots, \tfrac{p_n}{q} \right).$$

The n-dimensional Lebesgue measure law of $W(\psi_1, \ldots, \psi_n)$ is obtained in the following theorem [Sch80, Theorem 3A].

Khintchine's Theorem (1926a). *For $1 \leq t \leq n$, let $\psi_t : \mathbb{R}^+ \to \mathbb{R}^+$ be a real, positive function such that $0 < \psi_t(q) \leq 1/q$. Suppose that $\prod_{t=1}^{n} q^n \psi_t(q)$ is non-increasing. Then, with respect to n-dimensional Lebesgue measure,*

$$W(\psi_1, \ldots, \psi_n) \ \text{is} \ \begin{cases} \text{NULL} & \text{if} \ \sum_{q=1}^{\infty} q^n \psi_1(q) \ldots \psi_n(q) < \infty \\ \\ \text{FULL} & \text{if} \ \sum_{q=1}^{\infty} q^n \psi_1(q) \ldots \psi_n(q) = \infty \end{cases} .$$

Thus the Lebesgue measure of the set of $(\psi_1, \ldots, \psi_n)$–well approximable numbers in $\mathbb{R}^n$ satisfies a 'null-full' law. The divergence part of the above statement constitutes the main substance of the proof. The convergence part is a simple consequence of the first Borel–Cantelli lemma – see Chapter 2.

The aim of this chapter is to extend the general set up of [BDV03] and to obtain a statement analogous to the above theorem of Khintchine. For simplicity we focus on the case of a lim sup set dependent on two approximation functions. Thus, at a basic level, we shall be replacing balls in the set up of [BDV03] with 'rectangles'. In this respect we are taking a similar approach to that of §4.2 in which we extended the basic set up of the general badly approximable set $\mathbf{Bad}(\rho)$ to a more general framework for the set $\mathbf{Bad}(\rho_1, \ldots, \rho_t)$.

In the future, we hope to establish a complete measure theoretic description of lim sup sets dependent on two or more approximation functions. Unfortunately time has only allowed us to make the following initial study.

5.2 The set up

Let (Ω_1, d_1) and (Ω_2, d_2) be compact metric spaces and $\Omega := \Omega_1 \times \Omega_2$. Suppose each Ω_t supports a non-atomic, probability measure m_t and that the product measure $m := m_1 \times m_2$ is supported on Ω. As in Chapter 2, let $\mathcal{R} = \{R_\alpha \in \Omega : \alpha \in J\}$ be a collection of resonant points $R_\alpha := (R_{\alpha,1}, R_{\alpha,2})$ in Ω indexed by an infinite, countable set J. As usual, let $\beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on J and assume that for any $\eta > 0$, the set $\{\alpha \in J : \beta_\alpha \leq \eta\}$ is finite. Define

$$F(c; l_1, l_2) \ := \ \{x \in \Omega \ : \ d_t(x_t, c_t) \leq l_t \ \ (t = 1, 2)\} \ ,$$

as the rectangle centred at $c := (c_1, c_2) \in \Omega$ with sidelengths $l_1, l_2 \in \mathbb{R}^+$. Then, given approximation functions ψ_1, ψ_2, consider the set

$$\Lambda(\psi_1, \psi_2) := \{\, x \in \Omega \ : \ x \in \ F(R_\alpha; \psi_1(\beta_\alpha), \psi_2(\beta_\alpha)) \ \text{ for i.m. } \alpha \in J \,\} \ .$$

As in the general set up of §2.2, let $l := \{l_n\}$ and $u := \{u_n\}$ be positive increasing sequences such that $l_n < u_n$ and $l_n \to \infty$ as $n \to \infty$. Define

$$\Delta_l^u(\psi_1, \psi_2; n) := \bigcup_{\alpha \in J_l^u(n)} F(R_\alpha; \psi_1(\beta_\alpha), \psi_2(\beta_\alpha)) \ ,$$

where

$$J_l^u(n) := \{\, \alpha \in J : l_n < \beta_\alpha \leq u_n \,\} \ .$$

Then, using the assumptions that $\# J_l^u(n) < \infty$ and $l_n \to \infty$ as $n \to \infty$, we can write

$$\Lambda(\psi_1, \psi_2) \ = \ \limsup_{n \to \infty} \Delta_l^u(\psi_1, \psi_2; n) \ := \ \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Delta_l^u(\psi_1, \psi_2; n) \ . \qquad (34)$$

Therefore we call $\Lambda(\psi_1, \psi_2)$ the general lim sup set with respect to two functions. As an analogue to Corollary BDV, we aim to determine the conditions under which the m-measure of $\Lambda(\psi_1, \psi_2)$ is positive or even full. We also aim to make some remarks on the Hausdorff dimension of $\Lambda(\psi_1, \psi_2)$. Since this is an extension of the theory of [BDV03], we begin by setting out similar but adapted conditions.

5.3 Conditions

Measure : Suppose the measures m_1 and m_2 are each of type (A) as defined in §2.3. Then there exist positive constants δ_1, δ_2, r_o such that, for any ball $B(x_t, r_t) \subset \Omega_t$ centred at $x_t \in \Omega_t$ and of radius $r_t \leq r_o$,

$$a_t\, r_t^{\delta_t} \leq m_t(B(x_t, r_t)) \leq b_t\, r_t^{\delta_t} \qquad (t = 1, 2)\ .$$

The constants a_t, b_t are independent of the ball and without loss of generality $0 < a_t < 1 < b_t$. Since $F(x; r_1, r_2) = B(x_1, r_1) \times B(x_2, r_2)$, it follows that

$$a\, r_1^{\delta_1} r_2^{\delta_2} \leq m(F(x; r_1, r_2)) \leq b\, r_1^{\delta_1} r_2^{\delta_2}$$

where $a := a_1 a_2$ and $b := b_1 b_2$. Trivially, this implies that m is of type (A) with exponent $\delta := \delta_1 + \delta_2$. Furthermore, by Proposition 8 of §4.3, condition (A) implies that

$$\dim \Omega_1 = \delta_1, \qquad \dim \Omega_2 = \delta_2 \qquad \text{and} \qquad \dim \Omega = \delta > \delta_t \ \ (t = 1, 2)\ .$$

Ubiquity with rectangles : We now extend the notion of ubiquity in a natural way to involve two ubiquity functions. Recall from §2.3 that $(\mathcal{R}, \beta)$ can be said to be a global or local m-ubiquitous system relative to (ρ, l, u) where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is the ubiquity function with the property that $\rho(r) \to 0$ as $r \to \infty$.

Let $\rho_i, \rho_j : \mathbb{R}^+ \to \mathbb{R}^+$ be real, positive functions with $\rho_i(r), \rho_j(r) \to 0$ as $r \to \infty$. Define

$$\Delta_l^u(\rho_i, \rho_j; n) := \bigcup_{\alpha \in J_l^u(n)} F(R_\alpha; \rho_i(u_n), \rho_j(u_n)) \, .$$

(U) There exists a collection Φ of pairs of functions (ρ_i, ρ_j) and a positive constant r_u such that for each pair

(U1) $\qquad \rho_i(r)^{\delta_1} \, \rho_j(r)^{\delta_2} = \rho(r)^\delta \qquad \forall \, r > r_u \; ; \qquad$ and

(U2) there exists an absolute constant $\kappa > 0$ such that for a ball $B \subset \Omega$,

$$m(B \cap \Delta_l^u(\rho_i, \rho_j; n)) \;\geq\; \kappa \, m(B) \qquad \text{for } \; n \geq n_o(B) \, . \qquad (35)$$

If the system $(\mathcal{R}, \beta)$ satisfies condition (U), with (U2) satisfied for $B = \Omega$, then $(\mathcal{R}, \beta)$ is said to be a *global m-ubiquitous system relative to* (Φ, l, u). If (U2) is satisfied for any ball B with radius sufficiently small, centred in Ω, then $(\mathcal{R}, \beta)$ is said to be a *local m-ubiquitous system relative to* (Φ, l, u).

The key outcome of this ubiquity condition is that for any pair (ρ_i, ρ_j) from Φ we know something about the measure of $\Delta_l^u(\rho_i, \rho_j; n)$. By comparing (ψ_1, ψ_2) to a pair (ρ_i, ρ_j) we obtain information on the measure of $\Lambda(\psi_1, \psi_2)$. This will become clear during the course of the proof. In view of this method, there should always be an appropriate pair (ρ_i, ρ_j) with which a 'useful' comparison can be made. However, for some 'extreme' pairs (ψ_1, ψ_2) there may not be any useful pair (ρ_i, ρ_j). In these rare circumstances, we are not able to say anything about the measure of $\Lambda(\psi_1, \psi_2)$. In order to ensure the pair (ψ_1, ψ_2) is not one of these 'extreme' pairs, we introduce the following properties.

Properties P_1 and P_2 : The pair (ψ_1, ψ_2) has the property P_1 if either

1. there exists a pair $(\rho_i, \rho_j) \in \Phi$ such that

$$\limsup_{n\to\infty} \frac{\psi_1(u_n)}{\rho_i(u_n)} < \infty \qquad \text{or} \qquad \limsup_{n\to\infty} \frac{\psi_2(u_n)}{\rho_j(u_n)} \geq 1 ,$$

or

2. there does not exist such a pair but there exists a pair (ρ_i, ρ_j) such that ρ_i is u-regular,

$$\liminf_{n\to\infty} \frac{\psi_1(u_n)}{\rho_i(u_n)} > 0 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left(\frac{\psi_2(u_n)}{\rho_j(u_n)}\right)^{\delta_2} = \infty .$$

The pair (ψ_1, ψ_2) has the property P_2 if either

1. there exists a pair $(\rho_i, \rho_j) \in \Phi$ such that

$$\limsup_{n\to\infty} \frac{\psi_1(u_n)}{\rho_i(u_n)} \geq 1 \qquad \text{or} \qquad \limsup_{n\to\infty} \frac{\psi_2(u_n)}{\rho_j(u_n)} < \infty ,$$

or

2. there does not exist such a pair but there exists a pair (ρ_i, ρ_j) such that ρ_j is u-regular,

$$\liminf_{n\to\infty} \frac{\psi_2(u_n)}{\rho_j(u_n)} > 0 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left(\frac{\psi_1(u_n)}{\rho_i(u_n)}\right)^{\delta_1} = \infty .$$

The property P_1 avoids pairs (ψ_1, ψ_2) such that, for all (ρ_i, ρ_j) from Φ, the ratio $\psi_1(u_n)/\rho_i(u_n)$ oscillates between two sequences with one sequence tending to zero and the other to infinity as n tends to infinity. The same can be said for property P_2 and the ratio $\psi_2(u_n)/\rho_j(u_n)$. However, the property P1 or P2 only disallows this oscillation when, for all n large enough, $\psi_2(u_n) < \rho_j(u_n)$, or $\psi_1(u_n) < \rho_i(u_n)$, respectively.

5.4 Basic example

We demonstrate how these conditions are satisfied for the two-dimensional case of the classical set defined in the introduction, i.e. we consider the set of (ψ_1, ψ_2)–well approximable pairs in $\mathbb{R}^2$, denoted $W(\psi_1, \psi_2)$. For $n \geq 1$, let

$$\Delta_l^u(\psi_1, \psi_2; n) := \bigcup_{k^{n-1} < q \leq k^n} \bigcup_{p_1, p_2 \leq q} F\left(\left(\tfrac{p_1}{q}, \tfrac{p_2}{q}\right); \psi_1(q), \psi_2(q)\right) \cap [0,1]^2$$

where the sequences l, u are such that $l_{n+1} = u_n = k^n$, for some $k > 1$. Thus we can write

$$W(\psi_1, \psi_2) \cap [0,1]^2 = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Delta_l^u(\psi_1, \psi_2; n) .$$

To identify this set with the general framework let

$$\Omega_1 = \Omega_2 := [0,1], \quad \Omega := [0,1]^2 , \qquad J := \{(p_1, p_2, q) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : p_1, p_2 \leq q\} ,$$

$$\alpha := (p_1, p_2, q) \in J , \qquad \beta : (p_1, p_2, q) \mapsto q , \qquad R_\alpha := \left(\tfrac{p_1}{q}, \tfrac{p_2}{q}\right) .$$

Let the measure m be two-dimensional Lebesgue measure. Naturally, let m_1, m_2 be one-dimensional Lebesgue measure then trivially each satisfies the measure condition (A). Thus $\delta_1 = \delta_2 = 1$ and $\delta = 2$.

Proposition 13. *The pair $(\mathcal{R}, \beta)$ is a local m-ubiquitous system relative to (Φ, l, u) where Φ is the collection of pairs (ρ_i, ρ_j) such that, for a constant $k > 1$,*

$$\rho_i(r) := k\, r^{-(1+i)}, \quad \rho_j(r) := k\, r^{-(1+j)}$$

for $0 \leq i, j \leq 1$ and $i + j = 1$.

Proof. The pair $(\mathcal{R}, \beta)$ is known to be a local m-ubiquitous system relative to (ρ, l, u) where $\rho(r) :=$ constant $\times\ r^{-3/2}$ – see [BDV03, §12.1]. It follows immediately that for any pair (ρ_i, ρ_j),

$$\rho_i(r)\,\rho_j(r) = \rho(r)^2$$

and so (U1) is satisfied. We now show (U2).

Let $B = [a,b]\times[a,b] \subset [0,1]^2$. Fix i, j such that $0 \leq i, j \leq 1$ and $i+j = 1$. It can be verified by the pigeonhole principle that for any $x = (x_1, x_2)$ in B, there exist integers p_1, p_2, q with $(p_1, q) = (p_2, q) = 1$ and $1 \leq q \leq k^n$ satisfying

$$\left|x_1 - \tfrac{p_1}{q}\right| < \tfrac{1}{qk^{in}} \quad \text{and} \quad \left|x_2 - \tfrac{p_2}{q}\right| < \tfrac{1}{qk^{jn}} .$$

Clearly, $aq - 1 \leq p_1, p_2 \leq bq + 1$. Thus for a fixed integer q there exist at most $((b-a)q+3)^2$ many pairs (p_1, p_2). Thus, for n large enough,

$$m\left(B \cap \bigcup_{q \leq k^{n-1}} \bigcup_{p_1, p_2 \leq q} F\left(\left(\tfrac{p_1}{q}, \tfrac{p_2}{q}\right); \tfrac{1}{qk^{in}}, \tfrac{1}{qk^{jn}}\right)\right)$$

$$\leq \; 2 \sum_{q \leq k^{n-1}} ((b-a)q+3)^2 \tfrac{1}{q^2 k^n} \quad \leq \quad \tfrac{3}{k} m(B) .$$

It follows that for $k \geq 6$,

$$m(B \cap \Delta_l^u(\rho_i, \rho_j; n))$$

$$\geq \; m\left(B \cap \bigcup_{k^{n-1} < q \leq k^n} \bigcup_{p_1, p_2 \leq q} F\left(\left(\tfrac{p_1}{q}, \tfrac{p_2}{q}\right); \tfrac{k}{k^{n(1+i)}}, \tfrac{k}{k^{n(1+j)}}\right)\right)$$

$$\geq \; m\left(B \cap \bigcup_{k^{n-1} < q \leq k^n} \bigcup_{p_1, p_2 \leq q} F\left(\left(\tfrac{p_1}{q}, \tfrac{p_2}{q}\right); \tfrac{1}{qk^{in}}, \tfrac{1}{qk^{jn}}\right)\right)$$

$$\geq \; m(B) \; - \; \tfrac{3}{k} m(B) \; \geq \; \tfrac{1}{2} m(B) .$$

□

We now consider properties P_1 and P_2 for the basic example. Note that, in this set up, ρ_i, ρ_j have a 'maximum function' $\rho_{\max}$ and a 'minimum function' $\rho_{\min}$ in the sense that for any $(\rho_i, \rho_j) \in \Phi$, and any $r \geq 1$,

$$\rho_{\min}(r) := r^{-2} \leq \rho_i(r),\ \rho_j(r) \leq r^{-1} =: \rho_{\max}(r) .$$

Thus the pair (ψ_1, ψ_2) satisfies property P_1 if either (1)

$$\limsup_{n\to\infty} \psi_1(k^n)\, k^n < \infty \quad \text{or} \quad \limsup \psi_2(k^n) k^{2n} \geq 1 , \tag{36}$$

or (2) neither of the conditions of (36) are satisfied but there exists a pair (i, j) such that

$$\liminf_{n\to\infty} \psi_1(k^n)\, k^{n(1+i)} > 0 \quad \text{and} \quad \sum_{q=1}^{\infty} \psi_2(q)\, q^j = \infty .$$

Likewise, the pair (ψ_1, ψ_2) satisfies property P_2 if either (1)

$$\limsup_{n\to\infty} \psi_2(k^n)\, k^n < \infty \quad \text{or} \quad \limsup \psi_1(k^n) k^{2n} \geq 1 , \tag{37}$$

or (2) neither of the conditions of (37) are satisfied but there exists a pair (i, j) such that

$$\liminf_{n\to\infty} \psi_2(k^n)\, k^{n(1+j)} > 0 \quad \text{and} \quad \sum_{q=1}^{\infty} \psi_1(q)\, q^i = \infty .$$

With this in mind, we could consider the following simplification. Properties P_1 and P_2 are satisfied if

$$\limsup_{n\to\infty} \psi_t(k^n)\, k^n < \infty \quad (t = 1, 2) ,$$

i.e. there exists a constant c such that, for all q large enough, $\psi_t(q) \leq c/q$ $(t = 1, 2)$. Note that this is a slightly stronger condition than properties P_1 and P_2.

5.5 Results

Theorem 10. *Suppose m is a non-atomic probability measure of type (A), $(\mathcal{R}, \beta)$ is a global m-ubiquitous system relative to (Φ, l, u) and ψ_1, ψ_2 are approximation functions such that the pair (ψ_1, ψ_2) satisfies the properties P_1 and P_2.*

Next, suppose either

$$\limsup_{n\to\infty} \frac{\psi_1(u_n)^{\delta_1}\psi_2(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} > 0 \; ; \tag{38}$$

$$or \quad \sum_{n=1}^{\infty} \frac{\psi_1(u_n)^{\delta_1}\psi_2(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} = \infty \quad and \tag{39}$$

$$either\ \rho_i,\ \rho_j\ are\ each\ u\text{-}regular, \quad \forall(\rho_i, \rho_j) \in \Phi; \tag{40}$$

$$or\ \psi_1,\ \psi_2\ are\ each\ u\text{-}regular. \tag{41}$$

Then $m(\Lambda(\psi_1, \psi_2)) > 0$. In addition, suppose that $(\mathcal{R}, \beta)$ is a local m-ubiquitous system relative to (Φ, l, u) and that any open subset of Ω is m-measurable. Then $m(\Lambda(\psi_1, \psi_2)) = 1$.

Remarks: We can make a comparison between this theorem and Corollary BDV stated in §2.3. One can see that, if $\Omega_1 = \Omega_2$, $\psi := \psi_1 = \psi_2$ and $\Phi := (\rho, \rho)$, then properties P_1 and P_2 hold and the statement of this theorem is the same as that of Corollary BDV. Properties P_1 and P_2 are the only main addition to the theorem for two approximation functions. These properties are necessary to handle the extra cases which occur in the proof as a result of this set up.

Theorem 10 verifies the divergence part of the following classical result which we prove in §5.9.

Corollary 14. *Suppose* (ψ_1, ψ_2) *is a pair of approximation functions satisfying properties* P_1 *and* P_2*. Then, with respect to Lebesgue measure,*

$$W(\psi_1, \psi_2) \text{ is } \begin{cases} \text{NULL} & \text{if } \sum_{q=1}^{\infty} q^2\psi_1(q)\psi_2(q) < \infty \\ \\ \text{FULL} & \text{if } \sum_{q=1}^{\infty} q^2\psi_1(q)\psi_2(q) = \infty \end{cases}.$$

Remarks: This corollary differs in two ways from Khintchine's Theorem (1926a), as stated earlier in this chapter, for the case $n = 2$. The above corollary asks that ψ_1 and ψ_2 are decreasing which implies the condition of Khintchine: $q^2\psi_1(q)\psi_2(q)$ is non-increasing. However, the condition that the pair (ψ_1, ψ_2) satisfies properties P_1 and P_2 is weaker than Khintchine's condition that $\psi_1(q), \psi_2(q) \leq 1/q$. Therefore the above corollary is stronger than Khintchine's theorem in this regard. The meaning of properties P_1 and P_2 for the classical case is discussed in §5.4. Within this discussion, we state the following alternative condition which is simpler than properties P_1 and P_2: there exists a constant c such that, for all q large enough, $\psi_t(q) \leq c/q \quad (t = 1, 2)$. Even this stronger condition is weaker than Khintchine's condition.

Slightly more interestingly, Theorem 10 leads to the following result for the set $W(\psi_1, \psi_1, \psi_2)$ contained in $\mathbb{R}^3$, defined at the beginning of the chapter.

Corollary 15. *Suppose* (ψ_1, ψ_2) *are pair of approximation functions satisfying properties* P_1 *and* P_2*. Then, with respect to Lebesgue measure,*

$$W(\psi_1, \psi_1, \psi_2) \text{ is } \begin{cases} \text{NULL} & \text{if } \sum_{q=1}^{\infty} q^3\psi_1(q)^2\psi_2(q) < \infty \\ \\ \text{FULL} & \text{if } \sum_{q=1}^{\infty} q^3\psi_1(q)^2\psi_2(q) = \infty \end{cases}.$$

These corollaries are proved at the end of this chapter.

5.6 Hausdorff dimension

Theorem BDV stated in §2.3, obtains a Hausdorff f-measure statement for the general lim sup set, $\Lambda(\psi)$. In view of Theorem 10 it should be possible to extend Theorem BDV in order to to make a similar statement for the Hausdorff f-measure of $\Lambda(\psi_1, \psi_2)$. Theorem 10 and some of the sets constructed in its proof are the basis from which a Hausdorff measure theorem could be established. However, even by glancing at the paper [BDV03], one can see that obtaining the Hausdorff f-measure of a lim sup set requires considerably more work than obtaining its m-measure.

Some headway has been made with the easier task of deriving the Hausdorff dimension result for $\Lambda(\psi_1, \psi_2)$. Within [BDV03] this is the case that $G = \infty$. In view of work on this task to date, we make the following conjecture.

Conjecture. *Suppose m is a non-atomic probability measure of type (A), $(\mathcal{R}, \beta)$ is a local m-ubiquitous system relative to (Φ, l, u) and ψ_1, ψ_2 are approximation functions such that $\psi_2(r) \leq \psi_1(r)$ for r large enough. If*

$$\limsup_{n\to\infty} \frac{\psi_1(u_n)^{\delta_1}}{\rho(u_n)^{\delta}} = \infty \tag{42}$$

then

$$\dim \Lambda(\psi_1, \psi_2) \geq \min\{\delta, \sigma\}$$

where

$$\sigma := \limsup_{n\to\infty} \left(\frac{\log\left(\psi_1(u_n)^{\delta_1} \rho(u_n)^{-\delta} \psi_2(u_n)^{-\delta_1}\right)}{\log \psi_2(u_n)^{-1}} \right) .$$

Remarks: We state the above result as a conjecture although we have all the structure of a proof. The hypothesis (42) implies that $\sigma > \delta_1$. Therefore the conjecture would only obtain a result if the dimension is greater than δ_1. In the case that the dimension is less than δ_1, we are less sure of the exact condition to replace (42). Nonetheless, we conjecture that whatever the condition the result for this case should show

$$\dim \Lambda(\psi_1, \psi_2) \;\geq\; \min\left\{ \delta_1,\ \limsup_{n\to\infty} \left(\frac{\log \rho(u_n)^{-\delta}}{\log \psi_1(u_n)^{-1}} \right) \right\} .$$

This pair of results would verify Rynne's result [Ryn98] for the following set. Let $\tau_1 \leq \tau_2$ be two positive parameters then consider the following set

$$W(\tau_1, \tau_2) := \left\{ x \in \mathbb{R}^2 : \max\left\{ ||qx_1||^{1/\tau_1},\ ||qx_2||^{1/\tau_2} \right\} \;<\; q^{-1} \quad \text{for i.m. } q \in \mathbb{N} \right\} .$$

In the case of this set, Rynne's result is as follows.

Theorem (Rynne, 1998). *Suppose* $\tau_1 + \tau_2 \geq 1$ *then*

$$\dim W(\tau_1, \tau_2) \;=\; \begin{cases} \frac{3}{1+\tau_1} & \text{if } \tau_1 \geq 2 \\ \\ \frac{3+\tau_2-\tau_1}{1+\tau_2} & \text{if } \tau_1 < 2 \end{cases} .$$

5.7 Preliminaries

The proof of Theorem 10 employs the following classical measure results which we have taken from [BDV03, §8] where they are neatly written with proofs.

Proposition BDV1. *Let* (Ω, d) *be a metric space and let* m *be a finite, doubling measure on* Ω *such that any open set is measurable. Let* E *be a*

Borel subset of Ω. Assume that there are constants $r_o, c > 0$ such that for any ball B of radius $r(B) < r_o$ and centre in Ω we have that

$$m(E \cap B) \ \geq \ c\, m(B) \ .$$

Then E has full measure in Ω, i.e. $m(\Omega \setminus E) = 0$.

Note that any measure of type (A) is automatically doubling. We will also use the second Borel–Cantelli Lemma from Chapter 2, which features in [BDV03] as Proposition 2. This lemma leads to the conclusion of the next result.

Proposition BDV3. *Let $(\Omega, \mathcal{A}, m)$ be a probability space, $F \in \mathcal{A}$ and $A_n \in \mathcal{A}$ a sequence of m-measurable sets. Suppose there exists a constant $c > 0$ such that*

$$\limsup_{n\to\infty} m(F \cap A_n) \ \geq \ c\, m(F) \ .$$

Then

$$m\left(F \cap \limsup_{n\to\infty} A_n\right) \ \geq \ c^2\, m(F) \ .$$

5.8 Proof of Theorem 10

Let B be an arbitrary ball centred at a point in Ω. The aim of the proof is to show that

$$m(\Lambda(\psi_1, \psi_2) \cap B) \ \geq \ m(B)/C \ , \qquad (43)$$

where $C > 0$ is a constant independent of B. Under the global ubiquity hypothesis, Theorem 10 follows on establishing (43) with $B := \Omega$ – the space Ω can be thought of as a ball since it is compact. In the case of local ubiquity (43) will be established for balls B with sufficiently small radii so that the conditions of local ubiquity and BDV Proposition 1 are fulfilled. Then (43) together with BDV Proposition 1 implies Theorem 10 for local ubiquity. Since in the local case we appeal to BDV Proposition 1, the extra hypothesis that any open subset of Ω is m-measurable is necessary.

The method for obtaining (43) splits into cases. The cases depend on the pair (ψ_1, ψ_2) and its relationship with the collection Φ. Specifically, each case depends upon the comparison of $\psi_1(u_n)$ with $\rho_i(u_n)$ and $\psi_2(u_n)$ with $\rho_j(u_n)$ for the pairs (ρ_i, ρ_j) from Φ. Therefore, we have the following four cases.

Case A1 : (ψ_1, ψ_2) for which there exists a pair (ρ_i, ρ_j) from Φ such that

$$\psi_1(u_n) \geq \rho_i(u_n) \quad \text{and} \quad \psi_2(u_n) \geq \rho_j(u_n) \quad \text{for infinitely many } n.$$

Case A2 : (ψ_1, ψ_2), not in Case A1, for which there exists a pair (ρ_i, ρ_j) from Φ such that

$$\psi_1(u_n) < \rho_i(u_n) \quad \text{and} \quad \psi_2(u_n) < \rho_j(u_n) \quad \text{for all } n \text{ sufficiently large.}$$

Case A3 : (ψ_1, ψ_2) such that for all pairs (ρ_i, ρ_j) from Φ

$$\begin{aligned} \psi_1(u_n) &\geq \rho_i(u_n) \quad \text{for infinitely many } n \\ \text{and} \quad \psi_2(u_n) &< \rho_j(u_n) \quad \text{for all } n \text{ sufficiently large}. \end{aligned}$$

Case A4 : (ψ_1, ψ_2) such that for all pairs (ρ_i, ρ_j) from Φ

$$\begin{aligned} \psi_1(u_n) &< \rho_i(u_n) \quad \text{for all } n \text{ sufficiently large} \\ \text{and} \quad \psi_2(u_n) &\geq \rho_j(u_n) \quad \text{for infinitely many } n. \end{aligned}$$

Note that Cases A3 and A4 are the contrapositive of the combination of Cases A1 and A2. We proceed to prove (43) for each case.

5.8.1 Case A1

In this case we begin by fixing a pair (ρ_i, ρ_j) for which

$$\psi_1(u_n) \ \geq \ \rho_i(u_n) \quad \text{and} \quad \psi_2(u_n) \ \geq \ \rho_j(u_n)$$

for infinitely many n. For these n, by the fact that each of ψ_1, ψ_2 is decreasing, and by (U2), we have

$$m(\Delta_l^u(\psi_1, \psi_2; n) \cap B) \ \geq \ m(\Delta_l^u(\rho_i, \rho_j; n) \cap B) \ > \ \kappa\, m(B) \ .$$

Thus, $\limsup_{n\to\infty} m(\Delta_l^u(\psi_1, \psi_2; n)) \ > \ \kappa\, m(B)$ and, by Proposition BDV3,

$$m(\Lambda(\psi_1, \psi_2) \cap B) \ \geq \ \kappa^2\, m(B) \ .$$

Thus (43) is satisfied for Case A1. The argument is true regardless of whether $B = B(x, r) \subset \Omega$ or $B = \Omega$.

Before proving (43) for Case A2, we show how Cases A3 and A4 each reduce to Case A2.

5.8.2 Case A3

For this case we fix a pair (ρ_i, ρ_j) from Φ such that

$$\psi_1(u_n) \ \geq \ \rho_i(u_n) \quad \text{for infinitely many } n$$
$$\text{and} \quad \psi_2(u_n) \ < \ \rho_j(u_n) \quad \text{for all } n \text{ sufficiently large}$$

and such that either

$$\limsup_{n\to\infty} \frac{\psi_1(u_n)}{\rho_i(u_n)} < \infty \quad \text{or} \quad \liminf_{n\to\infty} \frac{\psi_1(u_n)}{\rho_i(u_n)} > 0 \ . \tag{44}$$

Such a pair (ρ_i, ρ_j) exists due to property P_1 and the assumption of this case. We begin by choosing a pair (ρ_i, ρ_j) such that the lim sup condition of (44) holds; thus there exists a constant $1 < k < \infty$ such that, for all n sufficiently large,

$$\psi_1(u_n) \; < \; k\,\rho_i(u_n) \; .$$

Let $\psi_1'(u_n) := \frac{1}{k}\psi_1(u_n)$. For this new function the following are true.

- $\Lambda(\psi_1', \psi_2) \; \subset \; \Lambda(\psi_1, \psi_2)$;
- $\psi_1'(u_n) \; < \; \rho_i(u_n)$ for all n sufficiently large ;
- by hypothesis (39),

$$\sum_{n=1}^{\infty} \frac{\psi_1'(u_n)^{\delta_1}\psi_2(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} = \infty \; ; \qquad \text{and}$$

- ψ_1' is u-regular if ψ is u-regular.

Thus, under the lim sup assumption of (44), Case A3 reduces to Case A2. We can now assume the lim inf condition of (44). In view of this and of property P_1, we choose a pair (ρ_i, ρ_j) for which the following hold:

$$\limsup_{n\to\infty} \frac{\psi_1(u_n)}{\rho_i(u_n)} = \infty \; , \qquad \liminf_{n\to\infty} \frac{\psi_1(u_n)}{\rho_i(u_n)} > 0$$

$$\text{and} \qquad \sum_{n=1}^{\infty} \left(\frac{\psi_2(u_n)}{\rho_j(u_n)}\right)^{\delta_2} \; = \; \infty \; . \tag{45}$$

Therefore, there exists some constant $1 < k' < \infty$ such that

$$\psi_1(u_n) \geq \tfrac{1}{k'}\,\rho_i(u_n) \qquad \text{for all } n \text{ large enough.}$$

Let $\psi_1''(u_n) := \frac{1}{k'}\rho_i(u_n)$. Then the following hold.

- $\Lambda(\psi_1'', \psi_2) \subset \Lambda(\psi_1, \psi_2)$;
- $\psi_1''(u_n) < \rho_i(u_n)$ for all n sufficiently large;
- by (45) and (U1),

$$\sum_{n=1}^{\infty} \frac{\psi_1''(u_n)^{\delta_1} \psi_2(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} = \infty \ ; \quad \text{and}$$

- ψ_1'' is u-regular since ρ_i is u-regular by P_1.

Thus Case A3 reduces to Case A2.

5.8.3 Case A4

In this case, we choose a pair (ρ_i, ρ_j) from the collection Φ such that

$$\begin{aligned} \psi_1(u_n) &< \rho_i(u_n) \quad \text{for all } n \text{ sufficiently large} \\ \text{and} \quad \psi_2(u_n) &\geq \rho_j(u_n) \quad \text{for infinitely many } n \end{aligned}$$

and such that either

$$\limsup_{n\to\infty} \frac{\psi_2(u_n)}{\rho_j(u_n)} < \infty \quad \text{or} \quad \liminf_{n\to\infty} \frac{\psi_2(u_n)}{\rho_j(u_n)} > 0 \ . \tag{46}$$

Such a pair (ρ_i, ρ_j) exists due to property P_2 and the assumption of this case. We begin by choosing a pair (ρ_i, ρ_j) such that the lim sup condition of (46) holds; thus there exists a constant $1 < k'' < \infty$ such that

$$\psi_2(u_n) < k'' \, \rho_j(u_n) \qquad \text{for all } n \text{ large enough.}$$

Thus, for $\psi_2'(u_n) := \frac{1}{k''}\psi_2(u_n)$, the following are true.

- $\Lambda(\psi_1, \psi_2') \subset \Lambda(\psi_1, \psi_2)$;

- $\psi_2'(u_n) \ < \ \rho_j(u_n)$ for all n sufficiently large;
- by hypothesis (39),

$$\sum_{n=1}^{\infty} \frac{\psi_1(u_n)^{\delta_1}\psi_2'(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} = \infty \; ; \qquad \text{and}$$

- ψ_2' is u-regular if ψ_2 is u-regular.

Thus, under the lim sup condition of (46), Case A4 reduces to Case A2. We now assume the lim inf condition of (46). In view of this and of property P_2, we choose a pair (ρ_i, ρ_j) for which the following hold

$$\limsup_{n\to\infty} \frac{\psi_2(u_n)}{\rho_j(u_n)} = \infty \, , \qquad\qquad \liminf_{n\to\infty} \frac{\psi_2(u_n)}{\rho_j(u_n)} > 0$$

$$\text{and} \qquad \sum_{n=1}^{\infty} \left(\frac{\psi_1(u_n)}{\rho_i(u_n)}\right)^{\delta_1} \; = \; \infty \, . \tag{47}$$

Then there exists a constant $1 < k''' < \infty$ such that

$$\psi_2(u_n) \geq \tfrac{1}{k'''}\, \rho_j(u_n) \qquad \text{for all } n \text{ large enough.}$$

Thus, for $\psi_2''(u_n) := \frac{1}{k'''}\rho_j(u_n)$ the following are true.

- $\Lambda(\psi_1, \psi_2'') \ \subset \ \Lambda(\psi_1, \psi_2)$;
- $\psi_2''(u_n) \ < \ \rho_j(u_n)$ for all n sufficiently large;
- by (47) and (U1),

$$\sum_{n=1}^{\infty} \frac{\psi_1(u_n)^{\delta_1}\psi_2''(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} = \infty \; ; \qquad \text{and}$$

- ψ_2'' is u-regular since ρ_j is u-regular.

Thus Case A4 reduces to Case A2.

5.8.4 Case A2

For this case we fix a pair (ρ_i, ρ_j) for which

$$\psi_1(u_n) \;<\; \rho_i(u_n) \quad \text{and} \quad \psi_2(u_n) \;<\; \rho_j(u_n) \tag{48}$$

for all n sufficiently large. In order to reach our goal of establishing (43) we begin by constructing a 'good' subset $A(\psi_1, \psi_2; B)$ of $\Lambda(\psi_1, \psi_2) \cap B$. Since $\#J_l^u(n)$ is finite, the collection of rectangles $F(R_\alpha; 3\rho_i(u_n), 3\rho_j(u_n))$ centred at $R_\alpha \in J_l^u(n)$ is trivially a finite cover of $\Delta_l^u(\rho_i, \rho_j; n)$. By the covering lemma stated in §4.3, there exists a disjoint subcollection of rectangles $F(R_\alpha; 3\rho_i(u_n), 3\rho_j(u_n))$ such that

$$\overset{\circ}{\bigcup_{R_\alpha \in G_\Omega(n)}} F(R_\alpha; \rho_i(u_n), \rho_j(u_n))$$

$$\subset \; \Delta_l^u(\rho_i, \rho_j; n) \;\subset\; \bigcup_{R_\alpha \in G_\Omega(n)} F(R_\alpha; 9\rho_i(u_n), 9\rho_j(u_n)) \;, \tag{49}$$

where $G_\Omega(n)$ is the set consisting of the centres of rectangles in the subcollection. With regard to the discussion at the beginning, let $B(x, r)$ be a ball, centred at $x \in \Omega$, with radius r sufficiently small that, for $B = B(x, r)$, (35) and Proposition BDV1 are satisfied. Now choose n sufficiently large so that $36\max\{\rho_i(u_n), \rho_j(u_n)\} < r$; by definition $\rho_i(u_n), \rho_j(u_n) \to 0$ as $n \to \infty$. We next construct a subset $G_B(n)$ of $G_\Omega(n)$ for $B = B(x, r)$. Let

$$G_B(n) \;:=\; \{\, R_\alpha \in G_\Omega(n) \;:\; R_\alpha \in \tfrac{1}{2}B \,\} \,.$$

Thus, for R_α in $G_B(n)$, the rectangle $F(R_\alpha; 9\rho_i(u_n), 9\rho_j(u_n))$ is contained in B. In view of this, and of (49),

$$\overset{\circ}{\bigcup_{R_\alpha \in G_B(n)}} F(R_\alpha; \rho_i(u_n), \rho_j(u_n)) \;\subset\; \Delta_l^u(\rho_i, \rho_j; n) \cap B \tag{50}$$

and

$$\Delta_l^u(\rho_i,\rho_j;n) \cap \tfrac{1}{4}B \subset \bigcup_{R_\alpha \in G_B(n)} F(R_\alpha; 9\rho_i(u_n), 9\rho_j(u_n)) \ . \qquad (51)$$

To see that the latter is true observe that, if R_α is not in $\frac{1}{2}B$, then $F(R_\alpha; 9\rho_i(u_n), 9\rho_j(u_n))$ does not intersect $\frac{1}{4}B$. We now estimate the cardinality of $G_B(n)$. By (35), (51) and the fact that the measure m is of type (A), for n sufficiently large we have

$$\begin{aligned}
\#G_B(n)\, m(F_n(\rho_i,\rho_j)) &\gg m\left(\bigcup_{R_\alpha \in G_B(n)} F(R_\alpha; 9\rho_i(u_n), 9\rho_j(u_n))\right) \\
&\geq m\left(\Delta_l^u(\rho_i,\rho_j;n) \cap \tfrac{1}{4}B\right) \\
&\geq \kappa\, m(\tfrac{1}{4}B) \gg m(B)
\end{aligned}$$

where $F_n(\rho_i,\rho_j)$ is a generic rectangle centred at a point R_α in $G_\Omega(n)$ with side lengths $\rho_i(u_n), \rho_j(u_n)$. It also follows from (50) that

$$\begin{aligned}
m(B) &\geq m\left(\mathring{\bigcup_{R_\alpha \in G_B(n)}} F(R_\alpha; \rho_i(u_n), \rho_j(u_n))\right) \\
&\gg \#G_B(n)\, m(F_n(\rho_i,\rho_j)) \ .
\end{aligned}$$

In view of the above statements and (U1),

$$\#G_B(n) \asymp \frac{m(B)}{m(F_n(\rho_i,\rho_j))} \asymp \frac{m(B)}{\rho_i(u_n)^{\delta_1}\rho_j(u_n)^{\delta_2}} = \frac{m(B)}{\rho(u_n)^{\delta}} \qquad (52)$$

where the implied constants are dependent only on the constants a and b from the measure condition (A). In the case of $B = \Omega$, (52) is satisfied with $m(B)$ replaced by $m(\Omega) = 1$. We are now in a position to prove the theorem under the lim sup hypothesis (38). By the hypothesis of Case A2, the fact that ψ_1, ψ_2 are decreasing and the construction of $G_B(n)$,

$$\begin{aligned} m(\Delta_l^u(\psi_1,\psi_2;n) \cap B) &\geq m\left(\bigcup_{R_\alpha \in G_B(n)}^{\circ} F(R_\alpha;\psi_1(u_n),\psi_2(u_n)) \right) \\ &\gg \#G_B(n)\, m(F_n(\psi_1,\psi_2)) \\ &\gg m(B)\ \frac{\psi_1(u_n)^{\delta_1}\psi_2(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} . \end{aligned}$$

Note that the same is true for $B = \Omega$. From hypothesis (38), there exists a constant $k > 0$ such that

$$\limsup_{n\to\infty} \frac{\psi_1(u_n)^{\delta_1}\psi_2(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} \geq k .$$

Since $m(B)$ is independent of n,

$$\limsup_{n\to\infty} m(\Delta_l^u(\psi_1,\psi_2;n) \cap B) \gg k\, m(B) .$$

We now invoke Proposition BDV3 with $F = B$ to conclude that

$$m(\Lambda(\psi_1,\psi_2) \cap B) \gg m(B) .$$

This proves Theorem 10 for Case A2 under the lim sup hypothesis (38).

We now turn our attention to proving (43) for Case A2 under the hypotheses (39) to (41). Without loss of generality we can assume that

$$\limsup_{n\to\infty} \frac{\psi_1(u_n)^{\delta_1}\psi_2(u_n)^{\delta_2}}{\rho(u_n)^{\delta}} = 0 .$$

Let

$$A_n(\psi_1,\psi_2;B) := \bigcup_{R_\alpha \in G_B(n)} F(R_\alpha;\psi_1(u_n),\psi_2(u_n)) .$$

Note that the above union is disjoint for n large enough by the assumption of Case A2 and by the construction of $G_B(n)$. Hence, for n large enough,

$$m(A_n(\psi_1,\psi_2;B)) \asymp \#G_B(n)\, m(F_n(\psi_1,\psi_2))$$

and in view of (52)

$$m(A_n(\psi_1,\psi_2;B)) \asymp m(B)\,\frac{\psi_1(u_n)^{\delta_1}\psi_2(u_n)^{\delta_2}}{\rho(u_n)^{\delta}}\,. \tag{53}$$

Next, let

$$A(\psi_1,\psi_2;B) := \limsup_{n\to\infty} A_n(\psi_1,\psi_2;B) = \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n(\psi_1,\psi_2;B)\,.$$

By construction, and since ψ_1,ψ_2 are each decreasing,

$$A(\psi_1,\psi_2;B) \subset \Lambda(\psi_1,\psi_2) \cap B\,.$$

In view of (43), the theorem will follow on showing

$$m(A(\psi_1,\psi_2;B)) \geq m(B)/C\,. \tag{54}$$

Notice that the estimate (53) together with the divergent sum hypothesis (39) imply that

$$\sum_{n=1}^{\infty} m(A_n(\psi_1,\psi_2;B)) = \infty\,. \tag{55}$$

The divergence of this sum is significant since it shows that the set $A(\psi_1,\psi_2;B)$ is 'useful' in the sense that, if the sum converged, then the first Borel–Cantelli lemma would imply $m(A(\psi_1,\psi_2;B)) = 0$. This would obviously not tell us anything about $m(\Lambda(\psi_1,\psi_2) \cap B)$. In addition to (55) we also require the following result on independence.

Lemma 16 (Quasi-independence on average for rectangles). *If, for either of the pairs (ψ_1,ψ_2) or (ρ_i,ρ_j), both functions are u-regular, then there exists a constant $C \geq 1$ such that, for Q large enough,*

$$\sum_{s,t=1}^{Q} m(A_s(\psi_1,\psi_2;B)\cap A_t(\psi_1,\psi_2;B)) \leq \frac{C}{m(B)}\left(\sum_{s=1}^{Q} m(A_s(\psi_1,\psi_2;B))\right)^2\,.$$

Therefore, the second Borel–Cantelli lemma, as stated in §2.1 together with the divergent sum (55), Lemma 16 and hypothesis (40) or (41) imply (54) as required. The above argument also holds for $B = \Omega$. Thus, assuming Lemma 16 is true, the proof of Case A2 is complete and we have proved the statement of the theorem in all possible cases. We now proceed to verify Lemma 16.

5.8.5 Proof of Lemma 16

Let $A_t := A_t(\psi_1, \psi_2; B)$. Fix the ball B and a positive integer s. Let $t > s$ and note that

$$\begin{aligned} m\,(A_s \cap A_t) &= m\left(\mathop{\dot{\bigcup}}_{R_\alpha \in G_B(s)} F(R_\alpha; \psi_1(u_s), \psi_2(u_s)) \,\cap\, A_t \right) \\ &\asymp \#G_B(s)\ \, m\,(\,F_s(\psi_1, \psi_2) \,\cap\, A_t\,) \\ &\asymp \frac{m(B)}{\rho(u_s)^\delta}\ m\,(\,F_s(\psi_1, \psi_2) \cap A_t\,)\ . \end{aligned} \tag{56}$$

We aim to find an upper bound for (56) where $F_s(\psi_1, \psi_2)$ is any rectangle of A_s, as previously defined. Also note that

$$\begin{aligned} m\,(F_s(\psi_1, \psi_2) \cap A_t\,) &:= m\left(F_s(\psi_1, \psi_2) \cap \mathop{\dot{\bigcup}}_{R_\alpha \in G_B(t)} F(R_\alpha; \psi_1(u_t), \psi_2(u_t)) \right) \\ &= \sum_{R_\alpha \in G_B(t)} m\,(\,F_s(\psi_1, \psi_2) \,\cap\, F(R_\alpha; \psi_1(u_t), \psi_2(u_t))\,)\ . \end{aligned} \tag{57}$$

In order to find an upper bound for the above sum, we consider four cases depending upon the size of $\psi_1(u_s)$ compared with $\rho_i(u_t)$ and the size of $\psi_2(u_s)$ compared with $\rho_j(u_t)$.

Case B1 : $t > s$ such that $2\psi_1(u_s) < \rho_i(u_t)$ and $2\psi_2(u_s) < \rho_j(u_t)$.

Suppose there are two points $R_\alpha, R_{\alpha'} \in G_B(t)$ such that

$$F_s(\psi_1,\psi_2) \,\cap\, F(R_{\alpha^*};\rho_i(u_t),\rho_j(u_t)) \neq \emptyset \qquad (\alpha^* = \alpha, \alpha')\ .$$

Then

$$d_1(R_\alpha, R_{\alpha'}) \leq 2\psi_1(u_s) + 2\rho_i(u_t) < 3\rho_i(u_t)$$

and

$$d_2(R_\alpha, R_{\alpha'}) \leq 2\psi_2(u_s) + 2\rho_j(u_t) < 3\rho_j(u_t)\ .$$

However, by construction the rectangles $F(R_{\alpha^*};3\rho_i(u_t),3\rho_j(u_t))$ are disjoint and so

$$d_1(R_\alpha, R_{\alpha'}) \geq 3\rho_i(u_t) \quad \text{and} \quad d_2(R_\alpha, R_{\alpha'}) \geq 3\rho_j(u_t)\ .$$

Hence, at most one rectangle $F(R_\alpha;\rho_i(u_t),\rho_j(u_t))$ with R_α in $G_B(t)$ can possibly intersect $F_s(\psi_1,\psi_2)$. This fact together with (48) and (57) implies $m\,(F_s(\psi_1,\psi_2)\cap A_t\,) \ll m(F_t(\psi_1,\psi_2))$. In view of (56),

$$m\,(A_s \cap A_t) \ll m(B)\,\frac{\psi_1(u_t)^{\delta_1}\psi_2(u_t)^{\delta_2}}{\rho(u_s)^{\delta}}\ .$$

Case B2 : $t > s$ such that $2\psi_1(u_s) < \rho_i(u_t)$ and $2\psi_2(u_s) \geq \rho_j(u_t)$.

Let N_2 be the number of rectangles $F(R_\alpha;\rho_i(u_t),\rho_j(u_t))$, with $R_\alpha \in G_B(t)$, for which

$$F_s(\psi_1,\psi_2) \,\cap\, F(R_\alpha;\rho_i(u_t),\rho_j(u_t)) \neq \emptyset\ .$$

In view of the fact that rectangles $F(R_\alpha;3\rho_i(u_t),3\rho_j(u_t))$ with R_α in $G_B(t)$ are disjoint, and by the set up of Case B2, when we restrict our view to the Ω_1 'plane' we see that at most one of these disjoint rectangles can intersect $F_s(\psi_1,\psi_2)\cap\Omega_1$. Therefore it is only necessary to count N_2 from the Ω_2 perspective. This means that N_2 is no more than the number of balls $B(c,\rho_j(u_t)) \subset \Omega_2$ which intersect a fixed ball $B(c',\psi_2(u_s)) \subset \Omega_2$ where $c,c' \in \Omega_2$. This implies that

$$N_2 \ll \frac{m_2(B(c',\psi_2(u_s)))}{m_2(B(c,\rho_j(u_t)))} \ll \left(\frac{\psi_2(u_s)}{\rho_j(u_t)}\right)^{\delta_2} . \tag{58}$$

Using the fact that ψ_1 and ψ_2 are each decreasing, together with (57), we have

$$m\,(F_s(\psi_1,\psi_2) \cap A_t\,) \ll N_2\, m(F_t(\psi_1,\psi_2)) . \tag{59}$$

Then (57), (56), (58) and (59) together give

$$m\,(A_s \cap A_t) \ll m(B) \left(\frac{\psi_2(u_s)}{\rho_j(u_t)}\right)^{\delta_2} \frac{\psi_1(u_t)^{\delta_1}\psi_2(u_t)^{\delta_2}}{\rho(u_s)^{\delta}} .$$

Case B3 : $t > s$ such that $2\,\psi_1(u_s) \geq \rho_i(u_t)$ and $2\,\psi_2(u_s) < \rho_j(u_t)$.

Let N_3 be the number of rectangles $F(R_\alpha;\rho_i(u_t),\rho_j(u_t))$, with $R_\alpha \in G_B(t)$, for which

$$F_s(\psi_1,\psi_2) \,\cap\, F(R_\alpha;\rho_i(u_t),\rho_j(u_t)) \neq \emptyset .$$

We can then follow an equivalent argument to that of Case B2 to say that N_3 is no more than the number of balls $B(c,\rho_i(u_t)) \subset \Omega_1$ which intersect a fixed ball $B(c',\psi_1(u_s)) \subset \Omega_1$ where $c,c' \in \Omega_1$. This argument implies that

$$N_3 \ll \frac{m_1(B(c',\psi_1(u_s)))}{m_1(B(c,\rho_i(u_t)))} \ll \left(\frac{\psi_1(u_s)}{\rho_i(u_t)}\right)^{\delta_1} . \tag{60}$$

Since ψ_1 and ψ_2 are each decreasing, we have

$$m\,(F_s(\psi_1,\psi_2) \cap A_t\,) \ll N_3\, m(F_t(\psi_1,\psi_2)) . \tag{61}$$

Then (57), (56), (60) and (61) together give

$$m\,(A_s \cap A_t) \ll m(B) \left(\frac{\psi_1(u_s)}{\rho_i(u_t)}\right)^{\delta_1} \frac{\psi_1(u_t)^{\delta_1}\psi_2(u_t)^{\delta_2}}{\rho(u_s)^{\delta}} .$$

Case B4 : $t > s$ such that $2\psi_1(u_s) \geq \rho_i(u_t)$ and $2\psi_2(u_s) \geq \rho_j(u_t)$.

In this case, if $R_\alpha \in G_B(t)$ and

$$F_s(\psi_1, \psi_2) \cap F(R_\alpha; \rho_i(u_t), \rho_j(u_t)) \neq \emptyset , \tag{62}$$

then $F(R_\alpha; \rho_i(u_t), \rho_j(u_t)) \subset F_s(5\psi_1, 5\psi_2)$ where $F_s(5\psi_1, 5\psi_2)$ naturally denotes the rectangle with sidelengths $5\psi_1(u_s), 5\psi_2(u_s)$. Let N_4 be the number of rectangles $F(R_\alpha; \rho_i(u_t), \rho_j(u_t))$ with R_α in $G_B(t)$ satisfying (62). Then

$$N_4 \ll \frac{m(F_s(5\psi_1, 5\psi_2))}{m(F_t(\rho_i, \rho_j))} \ll \frac{\psi_1(u_s)^{\delta_1}\psi_2(u_s)^{\delta_2}}{\rho(u_t)^\delta} .$$

This estimate of N_4, combined with (57), (56), and then (53), implies that

$$\begin{aligned} m(A_s \cap A_t) &\ll \frac{m(B)}{\rho(u_s)^\delta} \; N_4 \; m(F_t(\psi_1, \psi_2)) \\ &\ll m(B) \; \frac{\psi_1(u_s)^{\delta_1}\psi_2(u_s)^{\delta_2}}{\rho(u_s)^\delta} \; \frac{\psi_1(u_t)^{\delta_1}\psi_2(u_t)^{\delta_2}}{\rho(u_t)^\delta} \\ &\ll \frac{1}{m(B)} \, m(A_s) \, m(A_t) . \end{aligned}$$

Thus for Case B4 we have pairwise quasi-independence.

Now recall the following sum from the statement of the lemma and note that it breaks down into the four cases as follows.

$$\sum_{s,t=1}^{Q} m(A_s \cap A_t) =$$

$$\sum_{s=1}^{Q} m(A_s) + 2\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B1}}}^{Q} m(A_s \cap A_t) + 2\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B2}}}^{Q} m(A_s \cap A_t)$$

$$+ \; 2\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B3}}}^{Q} m(A_s \cap A_t) + 2\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B4}}}^{Q} m(A_s \cap A_t) .$$

Note that, from the argument of Case B4,

$$\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B4}}}^{Q} m(A_s \cap A_t) \ll \frac{1}{m(B)} \sum_{s=1}^{Q-1} m(A_s) \sum_{t=s+1}^{Q} m(A_t)$$

$$\ll \frac{1}{m(B)} \left(\sum_{s=1}^{Q} m(A_s) \right)^2 .$$

Also note that by (55), for Q large enough,

$$\sum_{s=1}^{Q} m(A_s) \ll \frac{1}{m(B)} \left(\sum_{s=1}^{Q} m(A_s) \right)^2 . \tag{63}$$

Thus the lemma follows on showing that

$$\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case } B^*}}^{Q} m(A_s \cap A_t) \ll \sum_{s=1}^{Q} m(A_s) \tag{64}$$

where B^* represents $B1, B2$ or $B3$. We proceed to prove (64) first under the hypothesis that ψ_1 and ψ_2 are both u-regular, and then again under the alternate hypothesis that ρ_i and ρ_j are u-regular. Suppose ψ_1 and ψ_2 are both u-regular. This means that, for $t > s$, with s sufficiently large,

$$\psi_1(u_t) \leq \lambda_1^{t-s} \psi_1(u_s) \quad \text{and} \quad \psi_2(u_t) \leq \lambda_2^{t-s} \psi_2(u_s)$$

for constants $0 < \lambda_1, \lambda_2 < 1$. Thus, by this property,

$$\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B1}}}^{Q} m(A_s \cap A_t)$$

$$\ll m(B) \sum_{s=1}^{Q-1} \frac{1}{\rho(u_s)^{\delta}} \sum_{t=s+1}^{Q} \psi_1(u_t)^{\delta_1} \psi_2(u_t)^{\delta_2} \tag{65}$$

$$\ll \sum_{s=1}^{Q-1} \frac{\psi_1(u_s)^{\delta_1} \psi_2(u_s)^{\delta_2}}{\rho(u_s)^{\delta}} \sum_{t=s+1}^{Q} \left(\lambda_1^{\delta_1} \lambda_2^{\delta_2}\right)^{t-s}$$

$$\ll \sum_{s=1}^{Q} m(A_s) ,$$

since $\lambda_1^{\delta_1}\lambda_2^{\delta_2} < 1$. Hence (64) is satisfied for Case B1. For Case B2 we have that

$$\begin{aligned}
&\sum_{s=1}^{Q-1}\sum_{\substack{t=s+1\\ \text{case B2}}}^{Q} m\left(A_s \cap A_t\right) \\
&\quad \ll \; m(B)\sum_{s=1}^{Q-1}\frac{\psi_2(u_s)^{\delta_2}}{\rho(u_s)^{\delta}}\sum_{t=s+1}^{Q}\frac{\psi_1(u_t)^{\delta_1}\psi_2(u_t)^{\delta_2}}{\rho_j(u_t)^{\delta_2}} \qquad (66)\\
&\quad \ll \; \sum_{s=1}^{Q-1}\frac{\psi_1(u_s)^{\delta_1}\psi_2(u_s)^{\delta_2}}{\rho(u_s)^{\delta}}\sum_{t=s+1}^{Q}\lambda_1^{\delta_1(t-s)}\,\frac{\psi_2(u_t)^{\delta_2}}{\rho_j(u_t)^{\delta_2}}\\
&\quad \ll \; \sum_{s=1}^{Q} m(A_s)
\end{aligned}$$

since $\psi_2(u_t) < \rho_j(u_t)$. Thus (64) is satisfied for Case B2. Similarly Case B3 is resolved as follows:

$$\begin{aligned}
&\sum_{s=1}^{Q-1}\sum_{\substack{t=s+1\\ \text{case B3}}}^{Q} m\left(A_s \cap A_t\right) \\
&\quad \ll \; m(B)\sum_{s=1}^{Q-1}\frac{\psi_1(u_s)^{\delta_1}}{\rho(u_s)^{\delta}}\sum_{t=s+1}^{Q}\frac{\psi_1(u_t)^{\delta_1}\psi_2(u_t)^{\delta_2}}{\rho_i(u_t)^{\delta_1}} \qquad (67)\\
&\quad \ll \; \sum_{s=1}^{Q-1}\frac{\psi_1(u_s)^{\delta_1}\psi_2(u_s)^{\delta_2}}{\rho(u_s)^{\delta}}\sum_{t=s+1}^{Q}\lambda_2^{\delta_2(t-s)}\,\frac{\psi_1(u_t)^{\delta_1}}{\rho_i(u_t)^{\delta_1}}\\
&\quad \ll \; \sum_{s=1}^{Q} m(A_s)\,.
\end{aligned}$$

The lemma is therefore proved under the assumption of the u-regularity of ψ_1 and ψ_2.

Next suppose ρ_i and ρ_j are u-regular. For $t > s$ we have positive constants $\lambda_i, \lambda_j < 1$ such that, for s sufficiently large,

$$\rho_i(u_t) \leq \lambda_i^{t-s}\rho_i(u_s) \quad \text{and} \quad \rho_j(u_t) \leq \lambda_j^{t-s}\rho_j(u_s)\,.$$

For each case we require a simple rearrangement of the double sum; otherwise the arguments are virtually the same as those for the ψ_1, ψ_2 u-regularity case. However, for completeness we proceed to resolve the lemma under the alternate hypothesis. Beginning with Case B1 and following from (65) and (U1) we have

$$\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B1}}}^{Q} m(A_s \cap A_t)$$

$$\ll \sum_{n=2}^{Q} \frac{\psi_1(u_n)^{\delta_1} \psi_2(u_n)^{\delta_2}}{\rho_i(u_n)^{\delta_1} \rho_j(u_n)^{\delta_2}} \sum_{m=1}^{n-1} \left(\lambda_i^{\delta_1} \lambda_j^{\delta_2}\right)^{n-m}$$

$$\ll \sum_{n=1}^{Q} m(A_n) .$$

Next, following from (66)

$$\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B2}}}^{Q} m(A_s \cap A_t)$$

$$\ll \sum_{n=2}^{Q} \frac{\psi_1(u_n)^{\delta_1} \psi_2(u_n)^{\delta_2}}{\rho_i(u_n)^{\delta_1} \rho_j(u_n)^{\delta_2}} \sum_{m=1}^{n-1} \lambda_i^{\delta_1(n-m)} \frac{\psi_2(u_m)^{\delta_2}}{\rho_j(u_m)^{\delta_2}}$$

$$\ll \sum_{n=1}^{Q} m(A_n) .$$

Finally, from (67), we have

$$\sum_{s=1}^{Q-1} \sum_{\substack{t=s+1 \\ \text{case B3}}}^{Q} m(A_s \cap A_t)$$

$$\ll \sum_{n=2}^{Q} \frac{\psi_1(u_n)^{\delta_1} \psi_2(u_n)^{\delta_2}}{\rho_i(u_n)^{\delta_1} \rho_j(u_n)^{\delta_2}} \sum_{m=1}^{n-1} \lambda_j^{\delta_2(n-m)} \frac{\psi_1(u_m)^{\delta_1}}{\rho_i(u_m)^{\delta_1}}$$

$$\ll \sum_{n=1}^{Q} m(A_n) .$$

Therefore (64) holds for Cases B1, B2 and B3 as required to prove the lemma under the hypothesis of u-regularity of ρ_i and ρ_j. This concludes the proof of Lemma 16 and hence completes the proof of Theorem 10.

□

5.9 Proof of Corollary 14

For this proof we carry forward the notation from §5.4 and we restrict our attention to begin with to the intersection of $W(\psi_1, \psi_2)$ and the unit square.

Convergence part: We begin by noting that

$$m(\Delta_l^u(\psi_1, \psi_2; n)) \leq \sum_{k^{n-1}<q\leq k^n} \sum_{p_1,p_2\leq q} F\left(\left(\tfrac{p_1}{q}, \tfrac{p_2}{q}\right); \psi_1(q), \psi_2(q)\right)$$

$$\ll k^{3(n-1)} \psi_1(k^{n-1})\psi_2(k^{n-1}) .$$

Then, by Fact 1 of §2.1,

$$\sum_{n=1}^{\infty} m(\Delta_l^u(\psi_1, \psi_2; n)) < \infty \quad \text{if} \quad \sum_{q=1}^{\infty} q^2\psi_1(q)\psi_2(q) < \infty . \tag{68}$$

By the first Borel–Cantelli lemma, if the left-hand side of (68) is true then

$$m(W(\psi_1, \psi_2) \cap [0, 1]^2) = 0 .$$

This argument gives the same result when $[0, 1]^2$ is replaced by any square in $\mathbb{R}^2$ of unit size. Thus $W(\psi_1, \psi_2)$ is a countable union of zero measure sets. It therefore follows that $W(\psi_1, \psi_2)$ has measure zero, under the convergent sum hypothesis of the corollary.

Divergence part: Proposition 13 states that the system $(\mathcal{R}, \beta)$ is a local m-ubiquitous system relative to (Φ, l, u). Trivially, any ρ_i, ρ_j is k^n-regular, for any $k > 1$, since

$$\rho_t(k^{n+1}) \leq \tfrac{1}{k}\, \rho_t(k^n) \qquad \text{for } t = i, j;\ n \geq 1.$$

Hence, Theorem 10 implies

$$m\left(W(\psi_1, \psi_2) \cap [0,1]^2\right) = 1 ,$$

under the divergent sum hypothesis. We can replace $[0,1]^2$ by any square of unit size i.e. $\Omega := [a, a+1]^2$ for any integer a. From the set up its clear that $W(\psi_1, \psi_2) \cap [a, a+1]^2$ can be written as a lim sup set. The ubiquity condition can also be seen to hold, for this Ω, by a slight adjustment of the proof of Proposition 13. Therefore, since

$$W(\psi_1, \psi_2) = \bigcup_{a \in \mathbb{Z}} W(\psi_1, \psi_2) \cap [a, a+1]^2 ,$$

it follows that $m\left(\mathbb{R}^2 \setminus W(\psi_1, \psi_2)\right) = 0$. Hence $W(\psi_1, \psi_2)$ has full m-measure.

5.10 Proof of Corollary 15

The proof of Corollary 15 follows in a similar way to that of Corollary 14. Let $\Omega := [0,1]^3 = \Omega_1 \times \Omega_2 = [0,1]^2 \times [0,1]$. Define a rectangle $F(c; l_1, l_2)$ centred at $c = (c_1, c_2, c_3)$ in $[0,1]^3$ and of side lengths $l_1, l_2 \in \mathbb{R}^+$ to be the set

$$\{x = (x_1, x_2, x_3) \in [0,1]^3 \ :\ |x_1 - c_1| \leq l_1,\ |x_2 - c_2| \leq l_1,\ |x_3 - c_3| \leq l_2,\ \} .$$

For $n \geq 1$, let

$$\Delta_l^u(\psi_1, \psi_2; n) := \bigcup_{k^{n-1} < q \leq k^n} \ \bigcup_{p_1, p_2, p_3 \leq q} F\left(\left(\tfrac{p_1}{q}, \tfrac{p_2}{q}; \tfrac{p_3}{q}\right); \psi_1(q), \psi_2(q)\right) \cap [0,1]^3$$

where the sequences l, u are such that $l_{n+1} = u_n = k^n$, for some $k > 1$. Thus we can write

$$W(\psi_1, \psi_1, \psi_2) \cap [0,1]^3 = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Delta_l^u(\psi_1, \psi_2; n) .$$

To identify this set with the general framework, let

$$J := \{(p_1, p_2, p_3, q) \in \mathbb{N}^4 : p_1, p_2, p_3 \leq q\} , \quad \alpha := (p_1, p_2, p_3, q) \in J ,$$

$$\beta : (p_1, p_2, p_3, q) \mapsto q , \qquad R_\alpha := \left(\tfrac{p_1}{q}, \tfrac{p_2}{q}; \tfrac{p_3}{q}\right) .$$

Let the measure m be three-dimensional Lebesgue measure. Naturally, let m_1, m_2 be two-dimensional and one-dimensional Lebesgue measure, respectively. Each satisfies the measure condition (A). Thus $\delta_1 = 2$, $\delta_2 = 1$ and $\delta = 3$. Note that

$$m(\Delta_l^u(\psi_1, \psi_2; n)) \ll k^{4(n-1)} \, \psi_1(k^{n-1})^2 \psi_2(k^{n-1}) .$$

Thus, by the first Borel–Cantelli lemma,

$$m\left(W(\psi_1, \psi_1, \psi_2) \cap [0,1]^3\right) = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} k^{4n} \, \psi_1(k^n)^2 \psi_2(k^n) < \infty .$$

The above sum is convergent as a result of the convergent sum hypothesis and by Fact 1 of §2.1. This argument gives the same result when $[0,1]^3$ is replaced by any square in $\mathbb{R}^3$ of unit size. Thus $W(\psi_1, \psi_1, \psi_2)$ is a countable union of zero measure sets. It therefore follows that $W(\psi_1, \psi_1, \psi_2)$ has measure zero, under the convergent sum hypothesis of the corollary. For the divergence part of the proof, we proceed to show all the conditions of Theorem 10 are satisfied.

Proposition 17. *The pair $(\mathcal{R}, \beta)$ is a local m-ubiquitous system relative to (Φ, l, u) where Φ is the collection of pairs (ρ_i, ρ_j) such that, for a constant*

$k > 1$,

$$\rho_i(r) := k\,r^{-(1+i)}, \quad \rho_j(r) := k\,r^{-(1+j)}$$

where $0 \le i \le \frac{1}{2}$, $0 \le j \le 1$ *and* $2i + j = 1$.

From [BDV03, §12.1] it follows that the pair $(\mathcal{R}, \beta)$ is a local m-ubiquitous system relative to (ρ, l, u) where $\rho(r) :=$ constant $\times\, r^{-4/3}$. Thus we see that (U1) is satisfied since, for any pair (ρ_i, ρ_j),

$$\rho_i(r)^2\, \rho_j(r) = \rho(r)^3 \,.$$

We postpone the remainder of the proof of Proposition 17. For any i, j, the functions ρ_i, ρ_j are both k^n-regular with regularity constant $\lambda := 1/k < 1$. Thus the conditions of the Theorem 10 are satisfied and it follows that

$$m\left(W(\psi_1, \psi_1, \psi_2) \cap [0,1]^3\right) \;=\; 1 \quad \text{if} \quad \sum_{n=1}^{\infty} k^{4n}\, \psi_1(k^n)^2 \psi_2(k^n) \;=\; \infty \,. \qquad (69)$$

By Fact 1, the right-hand side of (69) diverges under the divergent sum hypothesis. The statement of (69) remains true when $[0,1]^3$ is replaced by any cube of unit size i.e. $\Omega := [a, a+1]^3$ for any integer a. From the set up its clear that $W(\psi_1, \psi_1, \psi_2) \cap [a, a+1]^3$ can be written as a lim sup set. The ubiquity condition can also be seen to hold, for this Ω, by a slight adjustment of the proceeding proof of Proposition 17. Therefore, since

$$W(\psi_1, \psi_1, \psi_2) \;=\; \bigcup_{a \in \mathbb{Z}} W(\psi_1, \psi_1, \psi_2) \cap [a, a+1]^3 \,,$$

it follows that $m\left(\mathbb{R}^3 \setminus W(\psi_1, \psi_1, \psi_2)\right) = 0$. Hence $W(\psi_1, \psi_1, \psi_2)$ has full m-measure. It remains to prove the condition (U2) is met.

Proof of Proposition 17: part (U2)

Let $B = [a,b]^2 \times [a,b] \subset [0,1]^3$. Fix i, j such that $0 \le i \le \frac{1}{2}$, $0 \le j \le 1$ and $2i + j = 1$. It is easily verified by the pigeonhole principle that for any $x = (x_1, x_2, x_3)$ in B, there exist integers p_1, p_2, p_3, q, with $(p_t, q) = 1$ $(t = 1, 2, 3)$ and $1 \le q \le k^n$, satisfying

$$\left|x_t - \tfrac{p_t}{q}\right| < \tfrac{1}{qk^{in}} \quad (t = 1, 2) \qquad \text{and} \qquad \left|x_3 - \tfrac{p_3}{q}\right| < \tfrac{1}{qk^{jn}} \ .$$

Clearly, $aq - 1 \le p_1, p_2, p_3 \le bq + 1$. Thus for a fixed integer q there exist at most $((b - a)q + 3)^3$ many triples (p_1, p_2, p_3). Thus, for n large enough,

$$m\left(B \cap \bigcup_{q \le k^{n-1}} \bigcup_{p_1, p_2, p_3 \le q} F\left(\left(\tfrac{p_1}{q}, \tfrac{p_2}{q}; \tfrac{p_3}{q}\right); \tfrac{1}{qk^{in}}, \tfrac{1}{qk^{jn}}\right)\right)$$

$$\le 2 \sum_{q \le k^{n-1}} ((b - a)q + 3)^3 \tfrac{1}{q^2 k^n} \quad \le \quad \tfrac{3}{k} m(B) \ .$$

It follows that for $k \ge 6$,

$$m(B \cap \Delta_l^u(\rho_i, \rho_j; n))$$

$$\ge m\left(B \cap \bigcup_{k^{n-1} < q \le k^n} \bigcup_{p_1, p_2, p_3 \le q} F\left(\left(\tfrac{p_1}{q}, \tfrac{p_2}{q}; \tfrac{p_3}{q}\right); \tfrac{1}{qk^{in}}, \tfrac{1}{qk^{jn}}\right)\right)$$

$$\ge m(B) - \tfrac{3}{k} m(B) \ \ge \ \tfrac{1}{2} m(B) \ .$$

□

Chapter 6

Conclusion

This thesis set out to explore uncharted territory in the field of metric Diophantine approximation.

We began by taking a look at a general approach to well-approximable numbers explored recently by Beresnevich, Dickinson and Velani [BDV03]. We employed their results to gain a new result for the Hausdorff measure of a set of well-approximable numbers with restricted sets which is analogous to the Lebesgue measure result for the same set established by Harman [Har88c].

A whole new approach to badly approximable sets in an abstract setting was established in the main section, Chapter 4, and along with it a dimension law was presented with several applications. This result was extended to encompass sets dependent on more than one approximation function giving the results a broader scope of application.

The extended theory in Chapter 4 inspired the study presented in the final chapter. In this chapter an extension of a measure result from [BDV03] was achieved for a general lim sup set dependent on two approximation functions. A dimension result was also conjectured for this set and therefore the basis for future work is set up. To make this study a complete extension further work needs to be done to achieve a Hausdorff measure law for the general lim sup set dependent on two approximation functions.

Ideally, future work on this lim sup set would also encompass resonant sets, as opposed to simply resonant points, in line with the general setting of [BDV03]. In addition, it would be nice to point towards the method for establishing measure results for any finite number of approximation functions, as achieved by the analogous classical result of Khintchine (1926a).

More interestingly, the author would like to stretch out this theory for lim sup sets with two approximation functions, particularly once the Hausdorff measure result is achieved, in order to investigate its implications for the tempting Littlewood's conjecture, as stated in §4.2.

Thus new territory in the field has been charted and the path is set for future investigation. Whatever is discovered next, we can say for certain that the enticing landscape of metric Diophantine approximation will keep on evolving.

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