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A Geometric Invariant Characterising Initial Data for the Kerr–Newman Spacetime

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Abstract. We describe the construction of a geometric invariant characterising initial data for the Kerr–Newman spacetime. This geometric invariant vanishes if and only if the initial data set corresponds to exact Kerr–Newman initial data, and so characterises this type of data. We first illustrate the characterisation of the Kerr–Newman spacetime in terms of Killing spinors. The space-spinor formalism is then used to obtain a set of four independent conditions on an initial Cauchy hypersurface that guarantee the existence of a Killing spinor on the development of the initial data. Following a similar analysis in the vacuum case, we study the properties of solutions to the approximate Killing spinor equation and use them to construct the geometric invariant.

1. Introduction

The Kerr–Newman solution to the Einstein–Maxwell equations, describing a stationary charged rotating black hole, is one of the most interesting and well-studied exact solutions in general relativity, and yet there still remain several unresolved questions. For example, the current family of uniqueness results regarding the Kerr–Newman solution contain assumptions on the spacetime that are often considered too restrictive, such as analyticity—see, e.g. [13], for a review on the subject. Also, although there has been significant progress on the linear stability of the Kerr–Newman solution, the question of nonlinear stability has been far more stubborn—see, e.g. [15], for a discussion on this topic.

Making progress on these unresolved questions concerning electrovacuum black holes provides the motivation for finding characterisations of the Kerr–Newman spacetime. Different methods for characterising the exact solution can be tailored to emphasise specific properties, and so address each of these unresolved properties directly. One such characterisation is expressed in terms of Killing spinors, closely related to Killing–Yano tensors, which represent

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hidden symmetries of the spacetime. These symmetries cannot be expressed in terms of isometries of the spacetime. It has been shown in [14] that an asymptotically flat electrovacuum spacetime admitting a Killing spinor which satisfies a certain *alignment condition* with the Maxwell field must be isometric to the Kerr–Newman spacetime—see Theorem 1.

Once the motivation for a characterisation of the Kerr–Newman spacetime in terms of Killing spinors has been established, it is useful to investigate how the existence of such a spinor can be expressed in terms of initial data. The initial value problem in general relativity has played a crucial role in the systematic analysis of the properties of generic solutions to the Einstein field equations—see, e.g. [16,21,22]. It also provides the framework necessary for numerical simulations of spacetimes to be performed—see, e.g. [1,8].

Representing symmetries of a spacetime in terms of conditions on an initial hypersurface is not a new idea; the *Killing initial data (KID) equations*—see, e.g. [10]—are conditions on a spacelike Cauchy surface $\mathcal{S}$ which guarantee the existence of a Killing vector in the resulting development of the initial data. Thus, isometries of the whole spacetime can be encoded at the level of initial data. The resulting conditions form a system of overdetermined equations, so do not necessarily admit a solution for an arbitrary initial data set. In fact, it has been shown that the KID equations are non-generic, in the sense that generic solutions of the vacuum constraint Einstein equations do not possess any global or local spacetime Killing vectors—see [11]. An analogous construction can, in principle, be performed for Killing spinors. This analysis has been performed for the vacuum case giving explicitly the conditions relating the Killing spinor candidate and the Weyl curvature of the spacetime —see [17] and also [4]. These conditions are, like the KID equations, an overdetermined system and so do not necessarily admit a solution for an arbitrary initial surface. However, in [3,4] it has been shown that given an asymptotically Euclidean hypersurface it is always possible to construct a *Killing spinor candidate* which, whenever there exists a Killing spinor in the development, coincides with the restriction of the Killing spinor to the initial hypersurface. This approximate Killing spinor is obtained by solving a linear second-order elliptic equation which is the Euler–Lagrange equation of a certain functional over $\mathcal{S}$. The approximate Killing spinor can be used to construct a geometric invariant, which in some way parametrises the deviation of the initial data set from Kerr initial data. Variants of the basic construction in [4] have been given in [5,6].

The purpose of this article is to extend the analysis of [4] to the electrovacuum case. In doing so, we rely on the characterisation of the Kerr–Newman spacetime given in [14] which, in turn, builds upon the characterisation provided in [18] for the vacuum case and [24] for the electrovacuum case. As a result of our analysis, we find that the Killing spinor initial data equations remain largely unchanged, with extra conditions, ensuring that the electromagnetic content of the spacetime inherits the symmetry of the Killing spinor. These electrovacuum Killing spinor equations, together with an appropriate approximate Killing spinor, are used to construct an invariant expressed in

terms of suitable integrals over the hypersurface $\mathcal{S}$ whose vanishing characterises in a necessary and sufficient manner initial data for the Kerr–Newman spacetime. Our main result, in this respect, is given in Theorem 6.

Overview of the Article

Section 2 provides a brief overview of the theory of Killing spinors in electrovacuum spacetimes. Section 3 discusses the evolution equations governing the propagation of the Killing spinor equation in an electrovacuum spacetime. The main conclusion from this analysis is that the resulting system is homogeneous in a certain set of zero-quantities. The trivial data for these equations give rise to the conditions implying the existence of a Killing spinor in the development of some initial hypersurface. In Sect. 4, a space-spinor formalism is used to reexpress these conditions in terms of quantities intrinsic to the initial hypersurface. In addition, in this section the interdependence between the various conditions is analysed and a *minimal* set of *Killing spinor data equations* is obtained. Section 5 introduces the notion of approximate Killing spinors for electrovacuum initial data sets and discusses some basic ellipticity properties of the associated approximate Killing spinor equation. Section 6 discusses the solvability of the approximate Killing spinor equation in a class of asymptotically Euclidean manifolds. Finally, Sect. 7 brings together the analyses in the various sections to construct a geometric invariant characterising initial data for the Kerr–Newman spacetime. The main result of this article is given in Theorem 6.

Notation and Conventions

Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ denote an electrovacuum spacetime—i.e. a solution to the Einstein–Maxwell field equations. The signature of the metric in this article will be $(+, -, -, -)$, to be consistent with most of the existing literature using spinors. We use the spinorial conventions of [19]. The lowercase Latin letters a, b, c, ... are used as abstract spacetime tensor indices, while the uppercase letters A, B, C, ... will serve as abstract spinor indices. The Greek letters μ, ν, λ, ... will be used as spacetime coordinate indices, while α, β, γ, ... will serve as spatial coordinate indices. Finally, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, ... will be used as spinorial frame indices.

The conventions for the spinorial curvature are set via the expressions

$$\square_{AB}\mu_C = \Psi_{ABCD}\mu^D - 2\Lambda\mu_{(A}\epsilon_{B)C}, \qquad \square_{A'B'}\mu_C = \Phi_{ACA'B'}\mu^A. \quad (1)$$

We systematically use of the following expression for the (once contracted) second derivative of a spinor:

$$\nabla_{AQ'}\nabla_B{}^{Q'} = \frac{1}{2}\epsilon_{AB}\square + \square_{AB}. \quad (2)$$

2. Killing Spinors in Electrovacuum Spacetimes

In this section, we provide a systematic exposition of the properties of Killing spinors in an electrovacuum spacetime.

2.1. The Einstein–Maxwell Equations

Using standard spinorial notation, the Einstein–Maxwell equations are given by

$$\Phi_{ABA'B'} = 2\phi_{AB}\bar{\phi}_{A'B'}, \qquad \Lambda = 0, \tag{3a}$$

$$\nabla^A{}_{A'}\phi_{AB} = 0. \tag{3b}$$

In particular, from the Maxwell equation (3b) it follows that

$$\nabla_{A'B}\phi_{CD} = \nabla_{A'(B}\phi_{CD)}.$$

The Bianchi identity is given by

$$\nabla^A{}_{B'}\Psi_{ABCD} = \nabla_{(B}{}^{A'}\Phi_{CD)A'B'},$$

or, more explicitly

$$\nabla^A{}_{A'}\Psi_{ABCD} = 2\bar{\phi}_{A'B'}\nabla_B{}^{B'}\phi_{CD}. \tag{4}$$

Given an electrovacuum spacetime, applying the derivative $\nabla^{A'}{}_C$ to the Maxwell equation in the form $\nabla^A{}_{A'}\phi_{AB} = 0$, one obtains, after some standard manipulations, the following wave equation for the Maxwell spinor:

$$\Box\phi_{AB} = 2\Psi_{ABCD}\phi^{CD}. \tag{5}$$

2.2. Killing Spinors

A *Killing spinor* $\kappa_{AB} = \kappa_{(AB)}$ in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is a solution to the Killing spinor equation

$$\nabla_{A'(A}\kappa_{BC)} = 0. \tag{6}$$

In the sequel, a prominent role will be played by the integrability conditions implied by the Killing spinor equation. More precisely, one has the following:

Lemma 1. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ denote an electrovacuum spacetime endowed with a Killing spinor κ_{AB}. Then κ_{AB} satisfies the integrability conditions:*

$$\kappa_{(A}{}^Q\Psi_{BCD)Q} = 0, \tag{7a}$$

$$\Box\kappa_{AB} + \Psi_{ABCD}\kappa^{CD} = 0. \tag{7b}$$

Proof. The integrability conditions follow from applying the derivative $\nabla_D{}^{A'}$ to the Killing spinor equation (6), then using identity (2) together with the box commutators (1) and finally decomposing the resulting expression into its irreducible terms—the only non-trivial trace yields Eq. (7b), while the completely symmetric part gives Eq. (7a). $\qquad\square$

Remark 1. Observe that although every solution to the Killing spinor equation (6) satisfies the wave Eq. (7b), the converse is not true. In what follows, a symmetric spinor satisfying Eq. (7b), but not necessarily Eq. (6), will be called a *Killing spinor candidate*. This notion will play a central role in our subsequent analysis—in particular, we will be concerned with the question of the further conditions that need to be imposed on a Killing spinor candidate to be an actual Killing spinor.

A well-known property of Killing spinors in a vacuum spacetime is that the spinor

$$\xi_{AA'} \equiv \nabla^Q{}_{A'}\kappa_{QA} \tag{8}$$

is the counterpart of a (possibly complex) Killing vector ξ^a. A similar property holds for electrovacuum spacetimes—however, a further condition is required on the Killing spinor.

Lemma 2. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ denote an electrovacuum spacetime endowed with a Killing spinor κ_{AB}. Then $\xi_{AA'}$ as defined by Eq. (8) is the spinorial counterpart of a Killing vector ξ^a if and only if*

$$\kappa_{(A}{}^Q \phi_{B)Q} = 0. \tag{9}$$

Proof. The proof follows by direct substitution of definition (8) into the derivative $\nabla_{AA'}\xi_{BB'}$. Again, using the box commutators (1) one obtains, after some manipulations that

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = 12\bar{\phi}_{A'B'}\,\kappa_{(A}{}^Q\phi_{B)Q},$$

from which the result follows. □

Remark 2. Condition (9) implies that the Killing spinor κ_{AB} and the Maxwell spinor ϕ_{AB} are proportional to each other—thus, in what follows we refer to (9) as the *matter alignment condition.*

Remark 3. In the sequel, we will refer to a spinor $\xi_{AA'}$ obtained from a symmetric spinor κ_{AB} using expression (8) (not necessarily a Killing spinor) as the *Killing vector candidate* associated with κ_{AB}.

2.3. Zero-Quantities

In order to investigate the consequences of the Killing spinor Eq. (6) in a more systematic manner, it is convenient to introduce the following *zero-quantities*:

$$H_{A'ABC} \equiv 3\nabla_{A'(A}\kappa_{BC)}, \tag{10a}$$

$$S_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'}, \tag{10b}$$

$$\Theta_{AB} \equiv 2\kappa_{(A}{}^Q\phi_{B)Q}. \tag{10c}$$

Observe that if $H_{A'ABC} = 0$, then κ_{AB} is a Killing spinor. Similarly, if $S_{AA'BB'} = 0$, then $\xi_{AA'}$ is the spinor counterpart of a Killing vector, while if $\Theta_{AB} = 0$, then the matter alignment condition (9) holds.

The decomposition in irreducible components of $\nabla_{AA'}\kappa_{BC}$ can be expressed in terms of $H_{A'ABC}$ and $\xi_{AA'}$ as

$$\nabla_{AA'}\kappa_{BC} = \frac{1}{3}H_{A'ABC} - \frac{2}{3}\epsilon_{A(B}\xi_{C)A'}. \tag{11}$$

Similarly, a further computation shows that for $\xi_{AA'}$ as given by Eq. (8) one has the decomposition

$$\nabla_{AA'}\xi_{BB'} = \bar{\eta}_{A'B'}\epsilon_{AB} + \eta_{AB}\epsilon_{A'B'} + \frac{1}{2}S_{(AB)(A'B')}, \tag{12}$$

where
$$\eta_{AB} \equiv \frac{1}{2}\nabla_{AQ'}\xi_B{}^{Q'}.$$

If $\xi_{AA'}$ is a real Killing vector, then the spinor η_{AB} encodes the information of the so-called *Killing form*.

Remark 4. From Eq. (12), it readily follows by contraction that
$$\nabla^{AA'}\xi_{AA'} = 0,$$

independently of whether the alignment condition (9) holds or not—i.e. the Killing vector candidate $\xi_{AA'}$ defined by Eq. (8) is always divergence-free. This observation, in turn, implies that
$$S_{AA'}{}^{AA'} = 0,$$

so that one has the symmetry
$$S_{AA'BB'} = S_{(AB)(A'B')}. \tag{13}$$

Remark 5. The zero-quantities introduced in Eqs. (10a)–(10c) are a helpful bookkeeping device. In particular, a calculation analogous to that of the proof of Lemma 1 shows that
$$\nabla_{(A}{}^{A'}H_{|A'|BCD)} = -6\Psi_{Q(ABC}\kappa_{D)}{}^Q,$$
$$\nabla^{AA'}H_{A'ABC} = 2\big(\Box\kappa_{BC} + \Psi_{BCPQ}\kappa^{PQ}\big).$$

Thus, the integrability conditions of Lemma 1 can be written, alternatively, as
$$\nabla_{(A}{}^{A'}H_{|A'|BCD)} = 0, \qquad \nabla^{AA'}H_{A'ABC} = 0.$$

In particular, observe that if κ_{AB} is a Killing spinor candidate, then the zero-quantity $H_{A'ABC}$ is divergence-free.

2.4. A Characterisation of Kerr–Newman in Terms of Spinors

The following definition will play an important role in our subsequent analysis:

Definition 1. A stationary asymptotically flat 4-end in an electrovacuum space-time $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is an open submanifold $\mathcal{M}_\infty \subset \mathcal{M}$ diffeomorphic to $I \times (\mathbb{R}^3 \setminus \mathcal{B}_R)$ where $I \subset \mathbb{R}$ is an open interval and $\mathcal{B}_R$ is a closed ball of radius R. In the local coordinates (t, x^α) defined by the diffeomorphism, the components $g_{\mu\nu}$ and $F_{\mu\nu}$ of the metric $\boldsymbol{g}$ and the Faraday tensor $\boldsymbol{F}$ satisfy
$$|g_{\mu\nu} - \eta_{\mu\nu}| + |r\partial_\alpha g_{\mu\nu}| \le Cr^{-1}, \tag{14a}$$
$$|F_{\mu\nu}| + |r\partial_\alpha F_{\mu\nu}| \le C'r^{-2}, \tag{14b}$$
$$\partial_t g_{\mu\nu} = 0, \tag{14c}$$
$$\partial_t F_{\mu\nu} = 0, \tag{14d}$$

where C and C' are positive constants, $r \equiv (x^1)^2 + (x^2)^2 + (x^3)^2$, and $\eta_{\mu\nu}$ denote the components of the Minkowski metric in Cartesian coordinates.

Remark 6. It follows from condition (14c) in Definition 1 that the stationary asymptotically flat end $\mathcal{M}_\infty$ is endowed with a Killing vector ξ^a which takes the form ∂_t—a so-called *time translation*. From condition (14d), one has that the electromagnetic fields inherit the symmetry of the spacetime—that is $\mathcal{L}_\xi \boldsymbol{F} = 0$, with $\mathcal{L}_\xi$ the Lie derivative along ξ^a.

Of particular interest will be those stationary asymptotically flat ends *generated by a Killing spinor*:

Definition 2. A stationary asymptotically flat end $\mathcal{M}_\infty \subset \mathcal{M}$ in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ endowed with a Killing spinor κ_{AB} is said to be generated by a Killing spinor if the spinor $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ is the spinorial counterpart of the Killing vector ξ^a.

Remark 7. Stationary spacetimes have a natural definition of mass in terms of the Killing vector ξ^a that generates the isometry—the so-called *Komar mass* m defined as

$$m \equiv -\frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \epsilon_{abcd}\nabla^c\xi^d \mathrm{d}S^{ab},$$

where S_r is the sphere of radius r centred at $r = 0$ and $\mathrm{d}S^{ab}$ is the binormal vector to S_r. Similarly, one can define the *total electromagnetic charge* of the spacetime via the integral

$$q = -\frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r} F_{ab}\mathrm{d}S^{ab}.$$

Remark 8. In the asymptotic region, the components of the metric can be written in the form

$$g_{00} = 1 - \frac{2m}{r} + O(r^{-2}),$$
$$g_{0\alpha} = \frac{4\epsilon_{\alpha\beta\gamma}S^\beta x^\gamma}{r^3} + O(r^{-3}),$$
$$g_{\alpha\beta} = -\delta_{\alpha\beta} + O(r^{-1}),$$

where m is the Komar mass of ξ^a in the end $\mathcal{M}_\infty$, $\epsilon_{\alpha\beta\gamma}$ is the flat rank 3 totally antisymmetric tensor and S^β denotes a 3-dimensional tensor with constant entries. For the components of the Faraday tensor, one has that

$$F_{0\alpha} = \frac{q}{r^2} + O(r^{-3}),$$
$$F_{\alpha\beta} = O(r^{-3})$$

—see, e.g. [23]. Thus, to leading order any stationary electrovacuum spacetime is asymptotically a Kerr–Newman spacetime.

In [14], the following result has been proved:

Theorem 1. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ be a smooth electrovacuum spacetime satisfying the matter alignment condition with a stationary asymptotically flat end $\mathcal{M}_\infty$ generated by a Killing spinor κ_{AB}. Let both the Komar mass associated with the Killing vector $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ and the total electromagnetic charge in*

$\mathcal{M}_\infty$ be nonzero. Then, $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is locally isometric to a member of the Kerr–Newman family spacetimes.

The above result is a consequence of the characterisation of the Kerr–Newman spacetime given in [24]. It replaces the assumption on the self-dual Weyl tensor of the spacetime with the (stronger) assumption of the existence of a Killing spinor. The assumption that the electric charge of the spacetime in $\mathcal{M}_\infty$ is nonzero is required in the proof of the above result given in [14]; however, a further theorem from [14] proves a similar result for vacuum spacetimes, identifying the spacetime as a member of the Kerr family of solutions.

3. The Killing Spinor Evolution System in Electrovacuum Spacetimes

In this section, we systematically investigate the interrelations between the zero-quantities $H_{A'ABC}$, $S_{AA'BB'}$ and Θ_{AB}. The ultimate objective of this analysis is to obtain a system of homogeneous wave equations for the zero-quantities.

3.1. A Wave Equation for $\xi_{AA'}$

Given a Killing spinor candidate κ_{AB}, the wave Eq. (7b) naturally implies a wave equation for the Killing vector candidate $\xi_{AA'}$. We first notice the following alternative expression for the field $S_{AA'BB'}$:

Lemma 3. *Let κ_{AB} denote a symmetric spinor field in an electrovacuum $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then, one has that*

$$S_{AA'BB'} = 6\bar{\phi}_{A'B'}\Theta_{AB} - \frac{1}{2}\nabla_{PA'}H_{B'AB}{}^P - \frac{1}{2}\nabla_{PB'}H_{A'AB}{}^P. \qquad (15)$$

Proof. To obtain the identity one starts by substituting the expression $\xi_{AA'} = \nabla^Q{}_{A'}\kappa_{QA}$ into the definition of $S_{AA'BB'}$, Eq. (10b). One then commutes covariant derivatives using commutators (1) and makes use of the decompositions of $\nabla_{AA'}\kappa_{BC}$, $\nabla_{AA'}\xi_{BB'}$ and $S_{AA'BB'}$ given by Eqs. (11), (12) and (13), respectively, to simplify. $\qquad \square$

Remark 9. Observe that in the above result the spinor κ_{AB} is not assumed to be a Killing spinor candidate.

The latter is used, in turn, to obtain the main result of this section:

Lemma 4. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the Killing vector candidate $\xi_{AA'} \equiv \nabla^Q{}_{A'}\kappa_{AQ}$ satisfies the wave equation*

$$\Box\xi_{AA'} = -2\xi^{PP'}\Phi_{APA'P'} + \Phi^{PQ}{}_{A'}{}^{P'}H_{P'APQ}$$
$$-\Psi_{APQD}H_{A'}{}^{PQD} + 6\bar{\phi}_{A'}{}^{P'}\nabla_{PP'}\Theta_A{}^P. \qquad (16)$$

Proof. One makes use of the definition of $S_{AA'BB'}$ and identity (3) to write

$$\nabla^{AA'}\nabla_{AA'}\xi_{BB'} + \nabla^{AA'}\nabla_{BB'}\xi_{AA'} = 6\Theta_{AB}\nabla^{AA'}\bar{\phi}_{A'B'} + 6\bar{\phi}_{A'B'}\nabla^{AA'}\Theta_{AB}$$

$$-\frac{1}{2}\nabla^{AA'}\nabla_{CA'}H_{B'AB}{}^C - \frac{1}{2}\nabla^{AA'}\nabla_{CB'}H_{A'AB}{}^C.$$

The above expression can be simplified using the Maxwell equations. Moreover, commuting covariant derivatives in the terms $\nabla^{AA'}\nabla_{CA'}H_{B'AB}{}^C$ and $\nabla^{AA'}\nabla_{CB'}H_{A'AB}{}^C$ one arrives to

$$\Box\xi_{AA'} = -2\xi^{PP'}\Phi_{APA'P'} + \Phi^{PQ}{}_{A'}{}^{P'}H_{P'APQ} - \Psi_{APQD}H_{A'}{}^{PQD}$$

$$+ 6\bar{\phi}_{A'}{}^{P'}\nabla_{PP'}\Theta_A{}^P - \nabla_{AA'}\nabla_{PP'}\xi^{PP'} - \frac{1}{2}\nabla_{QA'}\nabla_{PP'}H^{P'}{}_A{}^{PQ}.$$

Finally, using that $\xi_{AA'}$ is a Killing vector candidate (see Remark 4) and that $\nabla^{AA'}H_{A'ABC} = 0$ (see Remark 5) the result follows. □

Remark 10. Important for the subsequent discussion is that the wave Eq. (17) takes, in tensorial terms, the form

$$\Box\xi_a = -2\Phi_{ab}\xi^b + J_a, \tag{17}$$

where J_a is defined in spinorial terms by

$$J_{AA'} \equiv \Phi^{PQ}{}_{A'}{}^{P'}H_{P'APQ} - \Psi_{APQD}H_{A'}{}^{PQD} + 6\bar{\phi}_{A'}{}^{P'}\nabla_{PP'}\Theta_A{}^P.$$

In terms of the zero-quantity $\zeta_{AA'}$ to be introduced in Eq. (19), one has

$$J_{AA'} \equiv \Phi^{PQ}{}_{A'}{}^{P'}H_{P'APQ} - \Psi_{APQD}H_{A'}{}^{PQD} - 6\bar{\phi}_{A'}{}^{P'}\zeta_{AP'}.$$

Thus, $J_{AA'}$ is an homogeneous expression of zero-quantities and does not involve their derivatives.

3.2. A Wave Equation for $H_{A'ABC}$

Lemma 5. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the zero-quantity $H_{A'ABC}$ satisfies the wave equation*

$$\Box H_{A'BCD} = 2\Psi_{CDAF}H_{A'B}{}^{AF} + 2\Psi_{BDAF}H_{A'C}{}^{AF} + 4\phi_D{}^A\bar{\phi}_{A'}{}^{B'}H_{B'BCA}$$

$$- 12\bar{\phi}_{A'}{}^{B'}\nabla_{DB'}\Theta_{BC} - 2\nabla_D{}^{B'}S_{(BC)(A'B')}. \tag{18}$$

Proof. We consider, again, identity (3) in the form

$$\nabla_{AB'}H_{A'BC}{}^A = 6\bar{\phi}_{A'B'}\Theta_{BC} - S_{(BC)(A'B')}.$$

Applying the derivative $\nabla_D{}^{B'}$ to the above expression, one readily finds that

$$\nabla_D{}^{B'}\nabla_{AB'}H_{A'BC}{}^A$$
$$= 6(\Theta_{BC}\nabla_D{}^{B'}\bar{\phi}_{A'B'} + \bar{\phi}_{A'B'}\nabla_D{}^{B'}\Theta_{BC}) - \nabla_D{}^{B'}S_{(BC)(A'B')}).$$

Using identity (2) and the box commutators (1), one obtains, after using the Maxwell equations to simplify, the desired equation. □

Remark 11. Observe that the right-hand side of the wave equation (18) is an homogeneous expression in the zero-quantity $H_{A'ABC}$ and the first-order derivatives of Θ_{AB} and $S_{AA'BB'}$.

3.3. A Wave Equation for Θ_{AB}

In order to compute a wave equation for the zero-quantity associated with the matter alignment condition, it is convenient to introduce a further zero-quantity:

$$\zeta_{AA'} \equiv \nabla^Q{}_{A'}\Theta_{AQ}. \tag{19}$$

Clearly, if the matter alignment condition (9) is satisfied, then $\zeta_{AA'} = 0$. The reason for introducing this further field will become clear in the sequel. Using the above definition, one obtains the following:

Lemma 6. *Let κ_{AB} denote a symmetric spinor field in an electrovacuum space-time $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then, one has that*

$$\Box\Theta_{AB} = 2\Psi_{ABPQ}\Theta^{PQ} - 2\nabla_B{}^{A'}\zeta_{AA'}. \tag{20}$$

Proof. The wave equation follows from applying the derivative $\nabla_B{}^{A'}$ to the definition of $\zeta_{AA'}$ and using identity (2) together with the box commutators (1). $\qquad\square$

Remark 12. A direct computation using the definitions of Θ_{AB} and $\zeta_{AA'}$ together with the expression for the irreducible decomposition of $\nabla_{AA'}\kappa_{BC}$ given by Eq. (11) and the Maxwell equations gives that

$$\zeta_{AA'} = -\nabla_{A'(A}\phi_{BC)}\kappa^{BC} + \frac{4}{3}\xi^B{}_{A'}\phi_{AB} + \frac{1}{3}H_{A'ABC}\phi^{BC}. \tag{21}$$

Remark 13. It follows directly from Eq. (20) that

$$\nabla^{AA'}\zeta_{AA'} = 0.$$

Alternatively, this property can be verified through a direct computation using identity (21).

As the right-hand side of Eq. (20) is an homogeneous expression in Θ_{AB} and a first-order derivative of $\zeta_{AA'}$, one needs to construct a wave equation for $\zeta_{AA'}$. The required expression follows from an involved computation—as it can be seen from the proof of the following lemma:

Lemma 7. *Let κ_{AB} denote a symmetric spinor field in an electrovacuum $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then, one has that*

$$\Box\zeta_{AA'}$$
$$= 4\zeta^{DB'}\phi_{AD}\bar{\phi}_{A'B'} + \frac{2}{3}\phi^{DB}\Psi_{DBCF}H_{A'A}{}^{CF} - \frac{2}{3}\phi^{DB}\Psi_{ABCF}H_{A'D}{}^{CF}$$
$$- \frac{4}{3}\phi_A{}^D\phi^{BC}\bar{\phi}_{A'}{}^{B'}H_{B'DBC} - \frac{2}{3}H^{B'DBC}\nabla_{AB'}\phi_{DA'BC} - \frac{2}{3}H_{A'}{}^{DBC}\nabla_{AB'}\phi_D{}^{B'}{}_{BC}$$
$$+ \frac{2}{3}\phi^{DB'BC}\nabla_{AB'}H_{A'DBC} + \frac{2}{3}\phi^D{}_{A'}{}^{BC}\nabla_{AB'}H^{B'}{}_{DBC} - \frac{4}{3}\phi^{DB}\nabla_A{}^{B'}S_{(BD)(A'B')}$$
$$- 4\phi^{DB}\bar{\phi}_{A'}{}^{B'}\nabla_{BB'}\Theta_{AD} - \frac{2}{3}\phi^{DB}\nabla_B{}^{B'}S_{(AD)(A'B')} + \frac{2}{3}\nabla_A{}^{B'}\phi^{DB}\nabla_{CB'}H_{A'DB}{}^C$$
$$- \frac{4}{3}\nabla_A{}^{B'}\phi^{DB}S_{(BD)(A'B')}. \tag{22}$$

where $\phi_{AA'BC} \equiv \nabla_{AA'}\phi_{BC}$.

Proof. Consider identity (21) and apply the derivative $\nabla^A{}_{B'}$ to obtain

$$\nabla^A{}_{B'}\zeta_{AA'} = -\kappa^{BC}\nabla^A{}_{B'}\nabla_{AA'}\phi_{BC} - \nabla_{AA'}\phi_{BC}\nabla^A{}_{B'}\kappa^{BC}$$
$$+ \frac{4}{3}(\phi_{AB}\nabla^A{}_{B'}\xi^B{}_{A'} + \xi^B{}_{A'}\nabla^A{}_{B'}\phi_{AB})$$
$$+ \frac{1}{3}(H_{A'ABC}\nabla^A{}_{B'}\phi^{BC} + \phi^{BC}\nabla^A{}_{B'}H_{A'ABC}).$$

Some further simplifications yield

$$\nabla^A{}_{B'}\zeta_{AA'} = \frac{1}{3}\nabla^A{}_{B'}\phi^{BC}H_{A'ABC} + \frac{1}{3}\nabla^A{}_{A'}\phi^{BC}H_{B'ABC}$$
$$- \frac{1}{3}\phi^{AB}\nabla_{CB'}H_{A'AB}{}^C + \frac{2}{3}\phi^{AB}S_{(AB)(A'B')}.$$

To obtain the required wave equation, we apply $\nabla_D{}^{B'}$ to the above expression and make use of decomposition (2) on the terms

$$\frac{1}{3}\nabla_D{}^{B'}\nabla^A{}_{B'}\phi^{BC}H_{A'ABC}, \quad \nabla_D{}^{B'}\nabla^A{}_{B'}\zeta_{AA'}, \quad -\frac{1}{3}\phi^{AB}\nabla_D{}^{B'}\nabla_{CB'}H_{A'AB}{}^C.$$

Finally, substitution of the wave equations for ϕ_{AB} and $H_{A'ABCD}$, Eqs. (5) and (18) yields the required expression homogeneous in zero-quantities. $\square$

3.4. A Wave Equation for $S_{AA'BB'}$

The discussion of the wave equation for the spinorial field $S_{AA'BB'}$ is best carried out in tensorial notation. Accordingly, let S_{ab} denote the tensorial counterpart of the (not necessarily Hermitian) spinor $S_{AA'BB'}$. Key to this computation is the wave equation for the Killing vector candidate ξ^a, as given in Eq. (17).

Lemma 8. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the zero-quantity S_{ab} satisfies the wave equation*

$$\Box S_{ab} = -2\mathcal{L}_\xi T_{ab} + 2T_b{}^c S_{ac} + 2T_a{}^c S_{bc} - T^{cd}S_{cd}g_{ab}$$
$$- T_{ab}S^c{}_c - 2C_{acbd}S^{cd} + \nabla_a J_b + \nabla_b J_a \tag{23}$$

where $\mathcal{L}_\xi T_{ab}$ denotes the Lie derivative of the energy momentum of the Faraday tensor.

Proof. The required expression follows from applying $\Box = \nabla_a\nabla^a$ to

$$S_{ab} = \nabla_a\xi_b + \nabla_b\xi_a,$$

commuting covariant derivatives, using the wave equation (17), the Einstein equation

$$R_{ab} = T_{ab},$$

the contracted Bianchi identity

$$\nabla^a C_{abcd} = \nabla_{[c}T_{d]b}$$

and the relation

$$\nabla_a \xi_b = \frac{1}{2} S_{ab} + \nabla_{[a} \xi_{b]}.$$

$$\square$$

A straightforward computation shows that the Lie derivative of the electromagnetic energy-momentum tensor can be expressed in terms of the Lie derivative of the Faraday tensor and the zero-quantity S_{ab} as

$$\mathcal{L}_\xi T_{ab} = -\frac{1}{4} F_{cd} F^{cd} S_{ab} - F_a{}^c F_b{}^d S_{cd} + \frac{1}{2} F_c{}^f F^{cd} g_{ab} S_{df}$$
$$+ F_b{}^c \mathcal{L}_\xi F_{ac} + F_a{}^c \mathcal{L}_\xi F_{bc} - \frac{1}{2} F^{cd} g_{ab} \mathcal{L}_\xi F_{cd}.$$

Furthermore, the Lie derivative of the Faraday tensor can be expressed in terms of the Lie derivative of the Maxwell spinor as

$$\mathcal{L}_\xi F_{AA'BB'} = \left(\mathcal{L}_\xi \phi_{AB} - \frac{1}{2} S_{AC'BD'} \bar{\phi}^{C'D'} \right) \epsilon_{A'B'} + \text{complex conjugate},$$

where the *Lie derivative of the Maxwell spinor* is defined by

$$\mathcal{L}_\xi \phi_{AB} \equiv \xi^{CC'} \nabla_{CC'} \phi_{AB} + \phi_{C(A} \nabla_{B)C'} \xi^{CC'} \tag{24}$$

—see Section 6.6 in [20]. This expression can be written in terms of zero-quantities by using the wave equations for the Killing and Maxwell spinors, the Maxwell equations and the identity

$$\kappa^D{}_{(A} \Psi_{B)DEF} \phi^{EF} = \frac{1}{2} \Psi_{ABCD} \Theta^{CD} + \frac{1}{3} \phi^{EF} \nabla_{(A|}{}^{A'} H_{A'|BEF)},$$

along with the wave equations for the Killing and Maxwell spinors and the Maxwell equations, Eqs. (7b) and (17), so as to obtain

$$\mathcal{L}_\xi \phi_{AB} = -\frac{3}{2} \nabla_{(A}{}^{A'} \zeta_{B)A'} + H_{A'CD(A} \nabla_{B)}{}^{A'} \phi^{CD} - \phi^{CD} \nabla_{(A|}{}^{A'} H_{A'|BCD)}.$$

From the previous discussion, it follows that:

Lemma 9. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the Lie derivative $\mathcal{L}_\xi T_{ab}$ can be expressed as an homogeneous expression in the zero-quantities*

$$S_{AA'BB'}, \quad \zeta_{AA'}, \quad H_{A'ABC}$$

and their first-order derivatives.

Remark 14. In the context of the present discussion, the object $\mathcal{L}_\xi \phi_{AB}$, as defined in (24), must be regarded as a convenient shorthand for a complicated expression. It is only consistent with the usual notion of Lie derivative of tensor fields if $\xi^{AA'}$ is the spinorial counterpart of a conformal Killing vector ξ^a—see [20], Section 6.6, for further discussion on this point.

3.5. Summary

We summarise the discussion of the present section in the following:

Proposition 1. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then the zero-quantities*

$$H_{A'ABC}, \quad \Theta_{AB}, \quad \zeta_{AA'}, \quad S_{AA'BB'}$$

satisfy a system of wave equations, consisting of Eqs. (18), (20), (22) and (23), which is homogeneous on the above zero-quantities and their first-order derivatives.

A direct consequence of the above and the uniqueness of solutions to homogeneous wave equations is the following:

Theorem 2. *Let κ_{AB} denote a Killing spinor candidate in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$, and let $\mathcal{S}$ denote a Cauchy hypersurface of $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. The spinor κ_{AB} is an actual Killing spinor, and $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ is a Killing vector if and only if on $\mathcal{S}$ one has that*

$$H_{A'ABC}|_\mathcal{S} = 0, \qquad\qquad \nabla_{EE'}H_{A'ABC}|_\mathcal{S} = 0 \qquad (25a)$$

$$S_{AA'BB'}|_\mathcal{S} = 0, \qquad\qquad \nabla_{EE'}S_{AA'BB'}|_\mathcal{S} = 0 \qquad (25b)$$

$$\Theta_{AB}|_\mathcal{S} = 0, \qquad\qquad \nabla_{EE'}\Theta_{AB}|_\mathcal{S} = 0 \qquad (25c)$$

$$\zeta_{AA'}|_\mathcal{S} = 0, \qquad\qquad \nabla_{EE'}\zeta_{AA'}|_\mathcal{S} = 0. \qquad (25d)$$

Proof. The initial data for the homogeneous system of wave equations for the fields $H_{A'ABC}$, Θ_{AB}, $\zeta_{AA'}$ and $S_{AA'BB'}$ given by Eqs. (18), (20), (22) and (23) consist of the values of these fields and their normal derivatives at the Cauchy surface $\mathcal{S}$. Because of the homogeneity of the equations, the unique solution to these equations with vanishing initial data is given by

$$H_{A'ABC} = 0, \quad \Theta_{AB} = 0, \quad \zeta_{AA'} = 0, \quad S_{AA'BB'} = 0.$$

Thus, if this is the case, the spinor κ_{AB} satisfies the Killing equation on $\mathcal{M}$ and, accordingly, it is a Killing spinor. Conversely, given a Killing spinor κ_{AB} over $\mathcal{M}$ such that $\xi_{AA'} = \nabla_{BA'}\kappa_{AB}$ is a Killing vector, its restriction to $\mathcal{S}$ satisfies conditions (25a)–(25d).

$\square$

Remark 15. As the spinorial zero-fields $H_{A'ABC}$, Θ_{AB}, $\zeta_{AA'}$ and $S_{AA'BB'}$ can be expressed in terms of the spinor κ_{AB}, it follows that conditions (25a)–(25d) are, in fact, conditions on κ_{AB}, and its (spacetime) covariant derivatives up to third order. In the next section, it will be shown how these conditions can be expressed in terms of objects intrinsic to the hypersurface $\mathcal{S}$.

4. The Killing Spinor Data Equations

The purpose of this section is to show how conditions (25a)–(25d) of Theorem 2 can be reexpressed as conditions which are intrinsic to the hypersurface $\mathcal{S}$. To this end, we make use of the space-spinor formalism outlined in [4] with some minor notational changes.

4.1. The Space-Spinor Formalism

In what follows assume that the spacetime $(\mathcal{M}, \boldsymbol{g})$ obtained as the development of Cauchy initial data $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ can be covered by a congruence of timelike curves with tangent vector τ^a satisfying the normalisation condition $\tau_a \tau^a = 2$—the reason for normalisation will be clarified in the following—see Eq. (28). Associated with the vector τ^a, one has the projector

$$h_a{}^b \equiv \delta_a{}^b - \frac{1}{2}\tau_a \tau^b$$

projecting tensors into the distribution $\langle \boldsymbol{\tau} \rangle^\perp$ of hyperplanes orthogonal to τ^a.

Remark 16. The congruence of curves needs not to be hypersurface orthogonal—however, for convenience, it will be assumed that the vector field τ^a is orthogonal to the Cauchy hypersurface $\mathcal{S}$.

Now, let $\tau^{AA'}$ denote the spinorial counterpart of the vector τ^a—by definition, one has that

$$\tau_{AA'}\tau^{AA'} = 2. \tag{26}$$

Let $\{o^A, \iota^A\}$ denote a normalised spin dyad satisfying $o_A \iota^A = 1$. In the following, we restrict the attention to spin dyads such that

$$\tau^{AA'} = o^A \bar{o}^{A'} + \iota^A \bar{\iota}^{A'}. \tag{27}$$

It follows then that

$$\tau_{AA'}\tau^{BA'} = \delta_A{}^B, \tag{28}$$

consistent with the normalisation condition (26). As a consequence of this relation, the spinor $\tau^{AA'}$ can be used to introduce a formalism in which all primed indices in spinors and spinorial equations are replaced by unprimed indices by suitable contractions with $\tau_A{}^{A'}$.

Remark 17. The set of transformations on the dyad $\{o^A, \iota^A\}$ preserving expansion (27) is given by the group $SU(2, \mathbb{C})$.

4.1.1. The Sen Connection.

The *space-spinor* counterpart of the spinorial covariant derivative $\nabla_{AA'}$ is defined as

$$\nabla_{AB} \equiv \tau_B{}^{A'}\nabla_{AA'}. \tag{29}$$

The derivative operator ∇_{AB} can be decomposed in irreducible terms as

$$\nabla_{AB} = \frac{1}{2}\epsilon_{AB}\mathcal{P} + \mathcal{D}_{AB}, \tag{30}$$

where

$$\mathcal{P} \equiv \tau^{AA'}\nabla_{AA'} = \nabla_Q{}^Q, \qquad \mathcal{D}_{AB} \equiv \tau_{(A}{}^{A'}\nabla_{B)A'} = \nabla_{(AB)}.$$

The operator $\mathcal{P}$ is the directional derivative of $\nabla_{AA'}$ in the direction of $\tau^{AA'}$, while $\mathcal{D}_{AB}$ corresponds to the so-called *Sen connection of the covariant derivative* $\nabla_{AA'}$ *implied by* $\tau^{AA'}$.

4.1.2. The Acceleration and the Extrinsic Curvature. Of particular relevance in the subsequent discussion is the decomposition of the covariant derivative of the spinor $\tau_{BB'}$, namely $\nabla_{AA'}\tau_{BB'}$. A calculation readily shows that the content of this derivative is encoded in the spinors

$$K_{AB} \equiv \tau_B{}^{A'}\mathcal{P}\tau_{AA'}, \qquad K_{ABCD} \equiv \tau_D{}^{C'}\mathcal{D}_{AB}\tau_{CC'},$$

corresponding, respectively, to the spinorial counterparts of the acceleration and the Weingarten tensor, expressed in tensorial terms as

$$K_a \equiv -\frac{1}{2}\tau^b\nabla_b\tau_a, \qquad K_{ab} \equiv -h_a{}^c h_b{}^d\nabla_c\tau_d.$$

It can be readily verified that

$$K_{AB} = K_{(AB)}, \qquad K_{ABCD} = K_{(AB)(CD)}. \tag{31}$$

In the sequel, it will be convenient to express K_{ABCD} in terms of its irreducible components. To this end, define

$$\Omega_{ABCD} \equiv K_{(ABCD)}, \qquad \Omega_{AB} \equiv K_{(A}{}^Q{}_{B)Q}, \qquad K \equiv K_{AB}{}^{CD},$$

so that one can define

$$K_{ABCD} = \Omega_{ABCD} - \frac{1}{2}\epsilon_{A(C}\Omega_{D)B} - \frac{1}{2}\epsilon_{B(C}\Omega_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}K.$$

If the vector field τ^a is hypersurface orthogonal, then one has that $\Omega_{AB} = 0$, and thus, the Weingarten tensor satisfies the symmetry $K_{ab} = K_{(ab)}$ so that it can be regarded as the extrinsic curvature of the leaves of a foliation of the spacetime $(\mathcal{M}, \boldsymbol{g})$. If this is the case, in addition to the second symmetry in (31) one has that

$$K_{ABCD} = K_{CDAB}.$$

In particular, K_{ABCD} restricted to the hypersurface $\mathcal{S}$ satisfies the above symmetry and one has $\Omega_{AB} = 0$—cfr. Remark 16.

In what follows denote by $D_{AB} = D_{(AB)}$, the spinorial counterpart of the Levi–Civita connection of the metric $\boldsymbol{h}$ on $\mathcal{S}$. The Sen connection $\mathcal{D}_{AB}$ and the Levi–Civita connection D_{AB} are related to each other through the spinor K_{ABCD}. For example, for a valence 1 spinor π_A one has that

$$\mathcal{D}_{AB}\pi_C = D_{AB}\pi_C + \frac{1}{2}K_{ABC}{}^Q\pi_Q,$$

with the obvious generalisations for higher-order spinors.

4.1.3. Hermitian Conjugation. Given a spinor π_A, its *Hermitian conjugate* is defined as

$$\widehat{\pi}_A \equiv \tau_A{}^{Q'}\bar{\pi}_{Q'}.$$

This operation can be extended in the obvious way to higher valence pairwise symmetric spinors. The operation of Hermitian conjugation allows to introduce a notion of *reality*. Given spinors $\nu_{AB} = \nu_{(AB)}$ and $\xi_{ABCD} = \xi_{(AB)(CD)}$, we say that they are *real* if and only if

$$\widehat{\nu}_{AB} = -\nu_{AB}, \qquad \widehat{\xi}_{ABCD} = \xi_{ABCD}.$$

If the spinors are real, then it can be shown that there exist real spatial 3-dimensional tensors ν_i and ξ_{ij} such that ν_{AB} and ξ_{ABCD} are their spinorial counterparts. We also note that

$$\nu_{AB}\widehat{\nu}^{AB} \geq 0, \qquad \xi_{ABCD}\widehat{\xi}^{ABCD} \geq 0$$

independently of whether ν_{AB} and ξ_{ABCD} are real or not.

Finally, it is observed that while the Levi–Civita covariant derivative D_{AB} is real in the sense that

$$\widehat{D_{AB}\pi_C} = -D_{AB}\widehat{\pi}_C,$$

the Sen connection $\mathcal{D}_{AB}$ is not. More precisely, one has that

$$\widehat{\mathcal{D}_{AB}\pi_C} = -\mathcal{D}_{AB}\widehat{\pi}_C + \frac{1}{2}K_{ABC}{}^Q\widehat{\pi}_Q.$$

4.1.4. Commutators. The main analysis of this section will require a systematic use of the commutators of the covariant derivatives $\mathcal{P}$ and $\mathcal{D}_{AB}$. In order to discuss these in a convenient manner, it is convenient to define the Hermitian conjugate of the Penrose box operator $\Box_{AB} \equiv \nabla_{C'(A}\nabla_{B)}{}^{C'}$ in the natural manner as

$$\widehat{\Box}_{AB} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\Box_{A'B'}.$$

From the definition of $\Box_{A'B'}$, it follows that

$$\widehat{\Box}_{AB}\pi_C = \tau_A{}^{A'}\tau_B{}^{B'}\Phi_{FCA'B'}\pi^F.$$

In terms of $\Box_{AB}$ and $\widehat{\Box}_{AB}$, the commutators of $\mathcal{P}$ and $\mathcal{D}_{AB}$ read

$$[\mathcal{P},\mathcal{D}_{AB}] = \widehat{\Box}_{AB} - \Box_{AB} - \frac{1}{2}K_{AB}\mathcal{P} + K^D{}_{(A}\mathcal{D}_{B)D} - K_{ABCD}\mathcal{D}^{CD},$$
$$(32a)$$

$$[\mathcal{D}_{AB},\mathcal{D}_{CD}] = \frac{1}{2}\big(\epsilon_{A(C}\Box_{D)B} + \epsilon_{B(C}\Box_{D)A}\big) + \frac{1}{2}\big(\epsilon_{A(C}\widehat{\Box}_{D)B} + \epsilon_{B(C}\widehat{\Box}_{D)A}\big)$$
$$+ \frac{1}{2}\big(K_{CDAB}\mathcal{P} - K_{ABCD}\mathcal{P}\big) + K_{CDF(A}\mathcal{D}_{B)}{}^F - K_{ABF(C}\mathcal{D}_{D)}{}^F.$$
$$(32b)$$

Remark 18. Observe that on the hypersurface $\mathcal{S}$ commutator (32b) involves only objects intrinsic to $\mathcal{S}$. Notice, also, that the Sen connection $\mathcal{D}_{AB}$ has torsion. Namely, for a scalar ϕ one has that

$$[\mathcal{D}_{AB},\mathcal{D}_{CD}]\phi = K_{CDF(A}\mathcal{D}_{B)}{}^F\phi - K_{ABF(C}\mathcal{D}_{D)}{}^F\phi.$$

4.2. Basic Decompositions

The purpose of this section is to provide a systematic discussion of the irreducible decompositions of the various spinorial fields and equations that will be required in the subsequent analysis.

4.2.1. Space-Spinor Decomposition of the Killing Spinor and Maxwell Equations. For reference, we provide a brief discussion of the space-spinor decomposition of the Killing equation (6), and the Maxwell equation (3b).

Contracting the Killing spinor equation (6) in the form $\nabla_{(A|A'|}\kappa_{CD)} = 0$ with $\tau_B{}^{A'}$, one obtains

$$\nabla_{(A|B|}\kappa_{CD)} = 0,$$

where ∇_{AB} is the differential operator defined in Eq. (29). Using decomposition (29), one further obtains

$$\frac{1}{2}\epsilon_{(A|B|}\mathcal{P}\kappa_{CD)} + \mathcal{D}_{(A|B|}\kappa_{CD)} = 0.$$

Taking, respectively, the trace and the totally symmetric part of the above expression, one readily obtains the equations

$$\mathcal{P}\kappa_{AB} + \mathcal{D}_{(A}{}^Q\kappa_{B)Q} = 0, \tag{33a}$$

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0. \tag{33b}$$

Equation (33a) can be naturally interpreted as an evolution equation for the spinor κ_{AB}, while Eq. (33b) plays the role of a *constraint*.

A similar calculation applied to the Maxwell equation, Eq. (3b), in the form $\nabla^A{}_{A'}\phi_{AC} = 0$ yields the equations

$$\mathcal{P}\phi_{AB} - 2\mathcal{D}^Q{}_{(A}\phi_{B)Q} = 0, \tag{34a}$$

$$\mathcal{D}^{AB}\phi_{AB} = 0. \tag{34b}$$

Again, Eq. (34a) is an evolution equation for the Maxwell spinor ϕ_{AB}, while (34b) is the spinorial version of the electromagnetic Gauss constraint.

Remark 19. The operation of Hermitian conjugation can be used to define, respectively, the *electric* and *magnetic* parts of the Maxwell spinor:

$$E_{AB} \equiv \frac{1}{2}\big(\widehat{\phi}_{AB} - \phi_{AB}\big), \qquad B_{AB} \equiv \frac{i}{2}\big(\phi_{AB} + \widehat{\phi}_{AB}\big).$$

It can be readily verified that

$$\widehat{E}_{AB} = -E_{AB}, \qquad \widehat{B}_{AB} = -B_{AB}.$$

Thus, E_{AB} and B_{AB} are the spinorial counterparts of three-dimensional tensors; E_i and B_i—the electric and magnetic parts of the Faraday tensor with respect to the normal to the hypersurface $\mathcal{S}$.

4.2.2. The Decomposition of the Components of the Curvature. Crucial for our subsequent discussion will be the fact that the restriction of the Weyl spinor Ψ_{ABCD} to an hypersurface $\mathcal{S}$ can be expressed in terms of quantities intrinsic to the hypersurface.

In analogy to the case of the Maxwell spinor ϕ_{AB}, the Hermitian conjugation operation can be used to decompose the Weyl spinor Ψ_{ABCD} into its electric and magnetic parts with respect to the normal to $\mathcal{S}$ as

$$E_{ABCD} \equiv \frac{1}{2}\big(\Psi_{ABCD} + \hat{\Psi}_{ABCD}\big), \qquad B_{ABCD} \equiv \frac{i}{2}\big(\hat{\Psi}_{ABCD} - \Psi_{ABCD}\big)$$

so that

$$\Psi_{ABCD} = E_{ABCD} + iB_{ABCD}.$$

The electrovacuum Bianchi identity (4) implies on $\mathcal{S}$ the constraint

$$\mathcal{D}^{AB}\Psi_{ABCD} = -2\widehat{\phi}^{AB}\mathcal{D}_{AB}\phi_{CD}.$$

Finally, using the Gauss–Codazzi and Codazzi–Mainardi equations one finds that

$$E_{ABCD} = -r_{(ABCD)} + \frac{1}{2}\Omega_{(AB}{}^{PQ}\Omega_{CD)PQ} - \frac{1}{6}\Omega_{ABCD}K + E_{(AB}E_{CD)},$$

$$B_{ABCD} = -iD^Q{}_{(A}\Omega_{BCD)Q},$$

where r_{ABCD} is the spinorial counterpart of the Ricci tensor of the intrinsic metric of the hypersurface $\mathcal{S}$.

4.2.3. Decomposition of the Spatial Derivatives of the Killing Spinor Candidate. Given a spinor κ_{AB} defined on the Cauchy hypersurface $\mathcal{S}$, it will prove convenient to define:

$$\xi \equiv \mathcal{D}^{AB}\kappa_{AB}, \tag{35a}$$

$$\xi_{AB} \equiv \frac{3}{2}\mathcal{D}_{(A}{}^C\kappa_{B)C}, \tag{35b}$$

$$\xi_{ABCD} \equiv \mathcal{D}_{(AB}\kappa_{CD)}. \tag{35c}$$

These spinors correspond to the irreducible components of the Sen derivative of κ_{AB}, as follows:

$$\mathcal{D}_{AB}\kappa_{CD} = \xi_{ABCD} - \frac{1}{3}\epsilon_{A(C}\xi_{D)B} - \frac{1}{3}\epsilon_{B(C}\xi_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\xi.$$

Using the commutation relation for the Sen derivatives, Eq. (32b), we can also calculate the derivatives of ξ and ξ_{AB}. The irreducible components of $\mathcal{D}_{AB}\xi_{CD}$ are given on $\mathcal{S}$—where $\Omega_{AB} = 0$—by

$$\mathcal{D}_{AB}\xi^{AB} = -\frac{1}{2}K\xi + \frac{3}{4}\Omega^{ABCD}\xi_{ABCD} + \frac{3}{2}\Theta_{AB}\widehat{\phi}^{AB}, \tag{36a}$$

$$\mathcal{D}_{A(B}\xi_{C)}{}^A = -\mathcal{D}_{BC}\xi - \frac{3}{2}\Psi_{BCAD}\kappa^{AD} + \frac{2}{3}K\xi_{BC} + \frac{1}{2}\Omega_{BCAD}\xi^{AD}$$
$$- \frac{3}{2}\Omega_{(B}{}^{ADF}\xi_{C)ADF} + \frac{3}{2}\mathcal{D}_{AD}\xi_{BC}{}^{AD} - 3\Theta_{A(B}\widehat{\phi}_{C)}{}^A, \tag{36b}$$

$$\mathcal{D}_{(AB}\xi_{CD)} = 3\Psi_{F(ABC}\kappa_{D)}{}^F + K\xi_{ABCD} - \frac{1}{2}\xi\Omega_{ABCD} + \Omega_{(ABC}{}^F\xi_{D)F}$$
$$- \frac{3}{2}\Omega_{(AB}{}^{PQ}\xi_{CD)PQ} + 3\mathcal{D}^F{}_{(A}\xi_{BCD)F} - 3\Theta_{(AB}\widehat{\phi}_{CD)}, \tag{36c}$$

where we have also used the *Hermitian conjugate* of the Maxwell spinor, defined by

$$\widehat{\phi}_{AB} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\bar{\phi}_{A'B'}.$$

Note that in (36b), the term $\mathcal{D}_{AB}\xi$ appears—there is no independent equation for the Sen derivative of ξ.

We can also calculate the second-order derivatives of ξ. Again, on the hypersurface $\mathcal{S}$ these take the form:

$$
\begin{aligned}
\mathcal{D}_{AB}\mathcal{D}^{AB}\xi = {}& -\frac{1}{6}K^2\xi + \frac{1}{2}K\widehat{\phi}^{AB}\Theta_{AB} - 2\widehat{\phi}^{AB}\phi_{AB}\xi \\
& + \frac{2}{3}\xi^{AB}\mathcal{D}_{AB}K + 3\Theta^{AB}\mathcal{D}_{BC}\widehat{\phi}_A{}^C \\
& - 4\widehat{\phi}^{AB}\phi_A{}^C\xi_{BC} - \frac{3}{2}\Psi^{ABCD}\xi_{ABCD} + 3\widehat{\phi}^{AB}\phi^{CD}\xi_{ABCD} \\
& - 3\widehat{\phi}^{AB}\Theta^{CD}\Omega_{ABCD} - \frac{1}{2}\Omega_{ABCD}\Omega^{ABCD}\xi + \frac{5}{4}K\Omega^{ABCD}\xi_{ABCD} \\
& + 3\kappa^{AB}\Psi_A{}^{CDF}\Omega_{BCDF} - \frac{3}{2}\Omega_{AB}{}^{FG}\Omega^{ABCD}\xi_{CDFG} - 3\kappa^{AB}\widehat{\phi}^{CD}\mathcal{D}_{BD}\phi_{AC} \\
& + 3\kappa^{AB}\widehat{\phi}_A{}^C\mathcal{D}_{CD}\phi_B{}^D - \frac{3}{2}\kappa^{AB}\mathcal{D}_{CD}\Psi_{AB}{}^{CD} \\
& + \frac{1}{2}\xi^{AB}\mathcal{D}_{CD}\Omega_{AB}{}^{CD} + \frac{3}{2}\mathcal{D}_{CD}\mathcal{D}_{AB}\xi^{ABCD} \\
& + \frac{3}{2}\xi^{ABCD}\mathcal{D}_{DF}\Omega_{ABC}{}^F - \frac{9}{2}\Omega^{ABCD}\mathcal{D}_{DF}\xi_{ABC}{}^F, & (37a)
\end{aligned}
$$

$$
\mathcal{D}^C{}_{(A}\mathcal{D}_{B)C}\xi = \frac{1}{2}\Omega_{ABCD}\mathcal{D}^{CD}\xi - \frac{1}{3}K\mathcal{D}_{AB}\xi, \qquad\qquad (37b)
$$

$$
\begin{aligned}
\mathcal{D}_{(AB}\mathcal{D}_{CD)}\xi = {}& \frac{1}{3}\widehat{\phi}^{EF}\Theta_{EF}\Omega_{ABCD} - \Psi_{ABCD}\xi - \frac{5}{9}K\Omega_{ABCD}\xi \\
& + \frac{1}{6}\Omega^{EFPQ}\Omega_{ABCD}\xi_{EFPQ} + \frac{8}{9}K^2\xi_{ABCD} + \frac{1}{3}\xi\mathcal{D}_{E(A}\Omega_{BCD)}{}^E \\
& - \frac{10}{3}K\mathcal{D}_{E(A}\xi_{BCD)}{}^E + \frac{3}{2}\mathcal{D}_{(AB}\mathcal{D}_{|EF|}\xi_{CD)}{}^{EF} + \frac{3}{2}\mathcal{D}_{F(A}\mathcal{D}_{B|E|}\xi_{CD)}{}^{EF} \\
& + \frac{1}{2}\mathcal{D}_{(A|F}\mathcal{D}_{E|}{}^F\xi_{BCD)}{}^E + \frac{8}{3}K\kappa_{(A}{}^E\Psi_{BCD)E} - \frac{3}{2}\kappa_{(A}{}^E\mathcal{D}_{B|F|}\Psi_{CD)E}{}^F \\
& - \frac{3}{2}\kappa^{EF}\mathcal{D}_{(AB}\Psi_{CD)EF} - \frac{1}{2}\kappa^{EF}\mathcal{D}_{F(A}\Psi_{BCD)E} + 2\xi\widehat{\phi}_{(AB}\phi_{CD)} \\
& - \frac{8}{3}K\widehat{\phi}_{(AB}\Theta_{CD)} + \Theta_{(AB}\mathcal{D}_{C|E|}\widehat{\phi}_{D)}{}^E + 3\Theta_{(A}{}^E\mathcal{D}_{BC}\widehat{\phi}_{D)E} \\
& + \Theta_{(A}{}^E\mathcal{D}_{B|E|}\widehat{\phi}_{CD)} + 2\Psi_{E(ABC}\xi_{D)}{}^E + \frac{1}{6}\xi\Omega_{(AB}{}^{EF}\Omega_{CD)EF} \\
& - \frac{14}{9}K\Omega_{E(ABC}\xi_{D)}{}^E - \frac{5}{3}K\Omega_{(AB}{}^{EF}\xi_{CD)EF} + \frac{2}{3}\Omega_{E(ABC}\mathcal{D}_{D)}{}^E\xi \\
& + \frac{3}{2}\Omega_{(ABC}{}^E\mathcal{D}^{FP}\xi_{D)EFP} - \Omega_{(AB}{}^{EF}\mathcal{D}_C{}^P\xi_{D)EFP} \\
& + \frac{1}{2}\Omega_{(AB}{}^{EF}\mathcal{D}_{|FP|}\xi_{CD)E}{}^P - \frac{3}{2}\Omega_{(A}{}^{EFP}\mathcal{D}_{BC}\xi_{D)EFP} \\
& + \frac{1}{2}\Omega_{(A}{}^{EFP}\mathcal{D}_{B|P|}\xi_{CD)EF} + \frac{2}{3}\xi_{(AB}\mathcal{D}_{CD)}K \\
& + \frac{1}{2}\xi_{(A}{}^E\mathcal{D}_{B|F|}\Omega_{CD)E}{}^F + \frac{1}{2}\xi^{EF}\mathcal{D}_{(AB}\Omega_{CD)EF} \\
& + \frac{1}{6}\xi^{EF}\mathcal{D}_{F(A}\Omega_{BCD)E} + \frac{2}{3}\xi_{E(ABC}\mathcal{D}_{D)}{}^EK + \frac{1}{2}\xi_{(AB}{}^{EF}\mathcal{D}_{C|P|}\Omega_{D)EF}{}^P \\
& + \frac{3}{2}\xi_{(A}{}^{EFP}\mathcal{D}_{BC}\Omega_{D)EFP} + \frac{1}{2}\xi_{(A}{}^{EFP}\mathcal{D}_{B|P|}\Omega_{CD)EF} \\
& + \kappa_{(A}{}^E\widehat{\phi}_{BC}\mathcal{D}_{D)F}\phi_E{}^F - 3\kappa_{(A}{}^E\widehat{\phi}_B{}^F\mathcal{D}_{CD)}\phi_{EF} + 2\kappa_{(A}{}^E\widehat{\phi}_B{}^F\mathcal{D}_{C|F|}\phi_{D)E} \\
& + \kappa^{EF}\widehat{\phi}_{(AB}\mathcal{D}_{C|F|}\phi_{D)E} + 3\kappa^{EF}\widehat{\phi}_{E(A}\mathcal{D}_{BC}\phi_{D)F} \\
& + \frac{1}{2}\kappa_{E(A}\Psi_B{}^{EFP}\Omega_{CD)FP} - \frac{1}{2}\kappa^{EF}\Psi^P{}_{E(AB}\Omega_{CD)FP}
\end{aligned}
$$

$$+ \frac{3}{2}\kappa^{EF}\Psi^P{}_{EF(A}\Omega_{BCD)P} + \frac{10}{3}\widehat{\phi}_{(AB}\phi_C{}^E\xi_{D)E}$$

$$+ \frac{2}{3}\widehat{\phi}_{(A}{}^E\phi_{BC}\xi_{D)E} + \frac{2}{3}\phi_{E(A}\widehat{\phi}_B{}^E\xi_{CD)}$$

$$- \widehat{\phi}_{(AB}\phi^{EF}\xi_{CD)EF} + \widehat{\phi}_{(A}{}^E\phi_B{}^F\xi_{CD)EF} + 3\phi_E{}^F\widehat{\phi}_{(A}{}^E\xi_{BCD)F}$$

$$+ \frac{1}{6}\widehat{\phi}_{(AB}\Theta^{EF}\Omega_{CD)EF}$$

$$+ \frac{2}{3}\widehat{\phi}_{(A}{}^E\Theta_B{}^F\Omega_{CD)EF} + \frac{3}{2}\Theta_E{}^F\widehat{\phi}_{(A}{}^E\Omega_{BCD)F}$$

$$+ \frac{1}{6}\widehat{\phi}^{EF}\Theta_{(AB}\Omega_{CD)EF} + \frac{3}{2}\widehat{\phi}^{EF}\Theta_{E(A}\Omega_{BCD)F} + \frac{1}{2}\Omega_{(ABC}{}^P\Omega_{D)EFP}\xi^{EF}$$

$$+ \frac{1}{6}\Omega_{EFP(A}\Omega_{BC}{}^{FP}\xi_{D)}{}^E - \frac{3}{4}\Omega_{(ABC}{}^E\Omega_{D)}{}^{FPQ}\xi_{EFPQ}$$

$$- \frac{3}{4}\Omega_E{}^{FPQ}\Omega_{(ABC}{}^E\xi_{D)FPQ} + \frac{1}{12}\Omega_{(AB}{}^{EF}\Omega_{CD)}{}^{PQ}\xi_{EFPQ}$$

$$+ \frac{1}{3}\Omega_{E(A}{}^{PQ}\Omega_{BC}{}^{EF}\xi_{D)FPQ} + \frac{1}{12}\Omega_{EF}{}^{PQ}\Omega_{(AB}{}^{EF}\xi_{CD)PQ}. \tag{37c}$$

Remark 20. It is of interest to remark that Eq. (37b) is just the statement that the Sen connection has torsion—cf. Remark 18.

An important and direct consequence of the above expressions is the following:

Lemma 10. *Assume that* $\Omega_{AB} = 0$ *and* $\mathcal{D}_{(AB}\kappa_{CD)} = 0$ *on* $\mathcal{S}$. *Then*

$$\mathcal{D}_{AB}\mathcal{D}_{CD}\mathcal{D}_{EF}\kappa_{GH} = H_{ABCDEFGH}$$

on $\mathcal{S}$, *where* $H_{ABCDEFGH}$ *is a linear combination of* κ_{AB}, $\mathcal{D}_{AB}\kappa_{CD}$ *and* $\mathcal{D}_{AB}\mathcal{D}_{CD}\kappa_{EF}$ *with coefficients depending on* Ψ_{ABCD}, K_{ABCD}, ϕ_{AB}, $\widehat{\phi}_{AB}$ *and* $\mathcal{D}_{AB}\phi_{CD}$.

Proof. The proof of the above result follows from direct inspection of Eqs. (36a)–(36c) and (37a)–(37c). □

Remark 21. We observe that the above result is strictly not true if $\xi_{ABCD} = \mathcal{D}_{(AB}\kappa_{CD)} \neq 0$.

4.3. The Decomposition of the Killing Spinor Data Equations

In this section, we provide a systematic discussion of the decomposition of the *Killing initial data conditions* in Theorem 2. The main purpose of this decomposition is to untangle the interrelations between the various conditions and to obtain a *minimal* set of equations which is intrinsic to the Cauchy hypersurface $\mathcal{S}$.

For the ease of the discussion we make explicit the assumptions we assume to hold throughout this section:

Assumption 1. *Given a Cauchy hypersurface* $\mathcal{S}$ *of an electrovacuum spacetime* $(\mathcal{M}, \boldsymbol{g})$, *we assume that the hypothesis and conclusions of Theorem 2 hold.*

Also, to ease the calculations, without loss of generality we assume:

Assumption 2. *The spinor* $\tau^{AA'}$ *which on* $\mathcal{S}$ *is normal to* $\mathcal{S}$ *is extended off the initial hypersurface in such a way that it is the spinorial counterpart of*

the tangent vector to a congruence of $\boldsymbol{g}$-geodesics. Accordingly, one has that $K_{AB} = 0$ —that is, the acceleration vanishes.

4.3.1. Decomposing $\boldsymbol{H_{A'ABC} = 0}$. Splitting the expression $\tau_D{}^{A'} H_{A'ABC}$ into irreducible parts, and using definitions (35a)–(35c) gives that the condition $H_{A'ABC} = 0$ is equivalent to

$$\xi_{ABCD} = 0, \tag{38a}$$

$$\mathcal{P}\kappa_{AB} = -\frac{2}{3}\xi_{AB}. \tag{38b}$$

Equation (38a) is a condition intrinsic to the hypersurface, while (38b) is extrinsic—i.e. it involves derivatives in the direction normal to $\mathcal{S}$.

Remark 22. Observe that conditions (38a) and (38b) are essentially Eqs. (33a) and (33b).

4.3.2. Decomposing $\boldsymbol{\nabla_{EE'} H_{A'ABC} = 0}$. If $H_{A'ABC} = 0$ on $\mathcal{S}$, it readily follows that $\mathcal{D}_{EF} H_{A'ABC} = 0$ on $\mathcal{S}$. Thus, in order investigate the consequences of the second condition in (25a) it is only necessary to consider the transverse derivative $\mathcal{P}H_{A'ABC}$. It follows that

$$\tau_D{}^{A'} \mathcal{P} H_{A'ABC} = \mathcal{P}\big(\tau_D{}^{A'} H_{A'ABC}\big) - H_{A'ABC} \mathcal{P}\tau_D{}^{A'}$$

and so as $H_{A'ABC}|_{\mathcal{S}} = 0$, the irreducible parts of $\tau_D{}^{A'} \mathcal{P} H_{A'ABC} = 0$ are given by

$$\mathcal{P}\xi_{ABCD} = 0, \tag{39a}$$

$$\mathcal{P}^2 \kappa_{AB} = -\frac{2}{3}\mathcal{P}\xi_{AB}. \tag{39b}$$

Taking Eq. (39a) and commuting the $\mathcal{D}_{AB}$ and $\mathcal{P}$ derivatives, and using Eqs. (38a) and (38b), gives

$$\mathcal{P}\xi_{ABCD} = \mathcal{P}\mathcal{D}_{(AB}\kappa_{CD)}$$

$$= 2\Psi^F{}_{(ABC}\kappa_{D)F} - \frac{1}{3}\xi\Omega_{ABCD} + \frac{2}{3}\Omega^F{}_{(ABC}\xi_{D)F}$$

$$- \frac{2}{3}\mathcal{D}_{(AB}\xi_{CD)} - 2\Theta_{(AB}\widehat{\phi}_{CD)}.$$

Substituting for the derivative of ξ_{AB} using (36c), and using Eqs. (38a) and (38b) again, gives

$$\mathcal{P}\xi_{ABCD} = 4\Psi^F{}_{(ABC}\kappa_{D)F} = 0. \tag{40}$$

To reexpress condition (39b), we use the following result which is obtained by commuting the $\mathcal{D}_{AB}$ and $\mathcal{P}$ derivatives:

$$\mathcal{P}\xi_{AB} = \frac{3}{2}\kappa^{CD}\Psi_{ABCD} - 3\Theta_{C(A}\widehat{\phi}_{B)}{}^C - \frac{1}{3}K\xi_{AB}$$

$$+ \frac{1}{2}\Omega_{ABCD}\xi^{CD} - \frac{3}{2}\mathcal{D}_{C(A}\mathcal{P}\kappa_{B)}{}^C. \tag{41}$$

Recall that the Killing spinor candidate κ_{AB} satisfies the homogeneous wave equation (7b). We can use the space-spinor decomposition to split the wave operator into Sen and normal derivative operators. The result is:

$$
\begin{aligned}
\mathcal{P}^2 \kappa_{AB} = &- 2\kappa^{CD}\Psi_{ABCD} + \frac{1}{3}K_{AB}\xi + \frac{2}{3}\Omega_{AB}\xi - \tfrac{2}{3}K_{(A}{}^C\xi_{B)C} \\
&- \frac{4}{3}\Omega_{(A}{}^C\xi_{B)C} + K^{CD}\xi_{ABCD} + 2\Omega^{CD}\xi_{ABCD} \\
&- K\mathcal{P}\kappa_{AB} - \frac{2}{3}\mathcal{D}_{AB}\xi + \frac{4}{3}\mathcal{D}_{(A}{}^C\xi_{B)C} - 2\mathcal{D}_{CD}\xi_{AB}{}^{CD}
\end{aligned}
$$

Applying conditions (38a) and (38b) to the right-hand side of the latter, evaluating at $\mathcal{S}$ (where $\Omega_{AB} = 0$) and setting $K_{AB} = 0$ gives

$$
\mathcal{P}^2 \kappa_{AB} = -2\kappa^{CD}\Psi_{ABCD} + \frac{2}{3}K\xi_{AB} - \frac{2}{3}\mathcal{D}_{AB}\xi + \frac{4}{3}\mathcal{D}_{(A}{}^C\xi_{B)C}.
$$

Then, using Eqs. (36b) and (41), as well as (38a) and (38b) as needed, it can be shown that

$$
\mathcal{P}^2 \kappa_{AB} = -\frac{2}{3}\mathcal{P}\xi_{AB}, \tag{42}
$$

which is exactly the condition we needed. Thus, we have shown that condition (39b) is purely a consequence of the evolution equation for the Killing spinor candidate, along with the conditions arising from $H_{A'ABC}|_{\mathcal{S}} = 0$.

In summary, if κ_{AB} satisfies $\Box\kappa_{AB} + \Psi_{ABCD}\kappa^{CD} = 0$, then:

$$
H_{A'ABC}|_{\mathcal{S}} = \mathcal{P}H_{A'ABC}|_{\mathcal{S}} = 0 \iff \xi_{ABCD} = 0,
$$

$$
\mathcal{P}\kappa_{AB} + \frac{2}{3}\xi_{AB} = 0, \quad \Psi^F{}_{(ABC}\kappa_{D)F} = 0.
$$

4.3.3. Decomposing $\Theta_{AB} = 0$.
As Θ_{AB} has no unprimed indices, it is already in a space-spinor compatible form—we have the condition:

$$
\Theta_{AB} = \kappa_{(A}{}^C\phi_{B)C} = 0. \tag{43}
$$

4.3.4. Decomposing $\nabla_{EE'}\Theta_{AB} = 0$.
If $\Theta_{AB}|_{\mathcal{S}} = 0$, one only needs to consider the normal derivative $\mathcal{P}\Theta_{AB}$. Using the evolution equation for the spinor ϕ_{AB} implied by Maxwell equations, Eq. (34a), along with (38b) in the condition $\mathcal{P}\Theta_{AB} = 0$, gives the spatially intrinsic condition

$$
\kappa_{(A|}{}^C\mathcal{D}_{CD}\phi_{|B)}{}^D = \frac{1}{3}\phi_{(A}{}^C\xi_{B)C}. \tag{44}
$$

In summary, assuming (38b) holds, then:

$$
\Theta_{AB}|_{\mathcal{S}} = \mathcal{P}\Theta_{AB}|_{\mathcal{S}} = 0 \iff \kappa_{(A}{}^C\phi_{B)C} = 0,
$$

$$
\kappa_{(A|}{}^C\mathcal{D}_{CD}\phi_{|B)}{}^D = \frac{1}{3}\phi_{(A}{}^C\xi_{B)C}.
$$

4.3.5. Decomposing $S_{AA'BB'} = 0$. Our point of departure to decompose the condition $S_{AA'BB'}|_{\mathcal{S}} = 0$ is the relation linking $S_{AA'BB'}$ to Θ_{AB} and the derivative of $H_{A'ABC}$ given by Eq. (3). Splitting the derivative of $H_{A'ABC}$ into normal and tangential parts gives

$$S_{AA'BB'} = -6\bar{\phi}_{A'B'}\Theta_{AB} + \frac{1}{2}\tau^C{}_{(A'}\mathcal{P}H_{B')ABC} + \tau_{D(A'}\mathcal{D}^{DC}H_{B')ABC}. \quad (45)$$

We already have conditions ensuring that $\Theta_{AB}|_{\mathcal{S}} = H_{A'ABC}|_{\mathcal{S}} = \mathcal{P}H_{A'ABC}|_{\mathcal{S}} = 0$, and so as a consequence we automatically have that $S_{AA'BB'}|_{\mathcal{S}} = 0$.

4.3.6. Decomposing $\nabla_{EE'}S_{AA'BB'} = 0$. Again as $S_{AA'BB'}|_{\mathcal{S}} = 0$, one only needs to consider the normal derivative $\mathcal{P}S_{AA'BB'}$. Taking the normal derivative of Eq. (45) and using that one has a Gaussian gauge gives on $\mathcal{S}$ that

$$\mathcal{P}S_{AA'BB'} = -6\mathcal{P}\bar{\phi}_{A'B'}\Theta_{AB} - 6\bar{\phi}_{A'B'}\mathcal{P}\Theta_{AB} + \tau^C{}_{(A'}\mathcal{P}^2 H_{B')ABC}$$
$$+ \tau_{D(A'}\mathcal{P}\mathcal{D}^{DC}H_{B')ABC}.$$

The first and second terms on the right-hand side are zero as a consequence of conditions (43) and (44). The last term can also be shown to be zero by commuting the derivatives and using (38a), (38b) and (40). This leaves

$$0 = \mathcal{P}S_{AA'BB'} = \tau^C{}_{(A'}\mathcal{P}^2 H_{B')ABC}. \quad (46)$$

Eliminating the primed indices by multiplying by factors of $\tau_{AA'}$ gives

$$\tau_{(C|}{}^{A'}\mathcal{P}^2 H_{A'AB|D)} = 0.$$

Thus, if this condition is satisfied on $\mathcal{S}$, then we have that $\mathcal{P}S_{AA'BB'}|_{\mathcal{S}} = 0$. In the following, we investigate further the consequences of this condition. As in a Gaussian gauge $\mathcal{P}\tau_{AA'} = 0$ it readily follows that, in fact, one has

$$\mathcal{P}^2\left(\tau_{(C|}{}^{A'}H_{A'AB|D)}\right) = 0.$$

Splitting into irreducible parts, one obtains two necessary conditions:

$$\mathcal{P}^2\xi_{ABCD} = 0, \quad (47a)$$

$$\mathcal{P}^2\left(\mathcal{P}\kappa_{AB} + \frac{2}{3}\xi_{AB}\right) = 0. \quad (47b)$$

Let us first consider condition (47a). We can commute the Sen derivative with one of the normal derivatives to obtain

$$\mathcal{P}(\mathcal{P}\xi_{ABCD}) = \mathcal{P}(\mathcal{P}\mathcal{D}_{(AB}\kappa_{CD)})$$
$$= \mathcal{P}\left(2\Psi_{(ABC}{}^F\kappa_{D)F} - 2\Theta_{(AB}\hat{\bar{\phi}}_{CD)} - \frac{1}{3}\Omega_{ABCD}\xi - \frac{2}{3}\Omega_{F(ABC}\xi^F{}_{D)}\right.$$
$$\left. - \frac{1}{3}\Omega_{(AB}\xi_{CD)} - \frac{1}{3}K\xi_{ABCD} + \Omega^F{}_{(A}\xi_{BCD)F} - \Omega_{(AB}{}^{EF}\xi_{CD)EF} + \mathcal{D}_{(AB}\mathcal{P}\kappa_{CD)}\right).$$

Now, we can use our previous conditions on $\mathcal{S}$ to eliminate terms. For example, the second term in the bracket is zero from conditions (43) and (44). The fifth, sixth and seventh terms vanish from (38a) and (40). We can also use (38b) and (42) to replace the last term—alternatively, one can commute the derivatives, use the substitution and then commute back; the result is the same. From this substitution, one obtains a factor $\mathcal{D}_{(AB}\xi_{CD)}$ inside the normal

derivative, which can be replaced using (36c)—this equation is valid on the whole spacetime rather than just the hypersurface, so one is allowed to take normal derivatives of it.

Proceeding as above, condition (47a) can be reduced to

$$\mathcal{P}^2\xi_{ABCD} = \mathcal{P}\left(4\Psi_{(ABC}{}^F\kappa_{D)F}\right) = 0. \tag{48}$$

Now, splitting the covariant derivatives in the Bianchi identity (4) into normal and tangential components gives the following space-spinor version:

$$\mathcal{P}\Psi_{ABCD} = -4\widehat{\phi}_{F(A}\mathcal{D}^F{}_B\phi_{CD)} - 4\widehat{\phi}_{(AB}\mathcal{D}^F{}_C\phi_{D)F} - 2\mathcal{D}_{F(A}\Psi_{BCD)}{}^F.$$

One can use the latter expression to further reduce condition (48) to

$$\Psi_{F(ABC}\xi_{D)}{}^F + 6\widehat{\phi}_{F(A}\kappa^E{}_B\mathcal{D}^F{}_C\phi_{D)E} + 6\widehat{\phi}_{(AB}\kappa^E{}_C\mathcal{D}^F{}_{D)}\phi_{EF}$$
$$+3\kappa_{(A}{}^F\mathcal{D}_{B|E}\Psi_{F|CD)}{}^E = 0. \tag{49}$$

This is an intrinsic condition on $\mathcal{S}$.

In order to obtain insight into condition (47b), we make use, again, of the wave Eq. (7b) for the spinor κ_{AB}. Taking a normal derivative of this equation, one obtains

$$\mathcal{P}\left(\Box\kappa_{AB} + \Psi_{ABCD}\kappa^{CD}\right) = 0.$$

Splitting the spacetime derivatives into normal and tangential parts and rearranging gives

$$\mathcal{P}\left(\mathcal{P}^2\kappa_{AB}\right) = \mathcal{P}\Big(-2\kappa^{CD}\Psi_{ABCD} + \frac{2}{3}\Omega_{AB}\xi - \frac{4}{3}\Omega_{(A}{}^C\xi_{B)C} + 2\Omega^{CD}\xi_{ABCD}$$
$$- K\mathcal{P}\kappa_{AB} - \frac{2}{3}\mathcal{D}_{AB}\xi - \frac{4}{3}\mathcal{D}_{C(A}\xi_{B)}{}^C - 2\mathcal{D}_{CD}\xi_{AB}{}^{CD}\Big).$$

As before, we can use our previous conditions to eliminate terms. The fourth and eight terms on the right-hand side vanish due to (38a) and (40). Also, we can use Eq. (36b) to replace the seventh term—this is because relation (36b) holds on the whole spacetime, and so one can take normal derivatives of it freely. These steps give

$$\mathcal{P}\left(\mathcal{P}^2\kappa_{AB}\right) = \mathcal{P}\left(\frac{2}{3}\Omega_{(A}{}^C\xi_{B)C} - \frac{2}{9}K\xi_{AB} - \frac{2}{3}\Omega_{ABCD}\xi^{CD} + \frac{2}{3}\mathcal{D}_{AB}\xi\right).$$

Alternatively, consider the second derivative of ξ_{AB}, given by applying a normal derivative to Eq. (41)—note that Eq. (41) applies on the whole spacetime), so one can take the normal derivative. This yields

$$\mathcal{P}^2\xi_{AB} = \mathcal{P}\Big(\frac{3}{2}\kappa^{CD}\Psi_{ABCD} - 3\Theta_{C(A}\widehat{\phi}_{B)}{}^C - \frac{1}{2}\Omega_{AB}\xi - \frac{1}{3}K\xi_{AB}$$
$$+ \frac{1}{2}\Omega_{(A}{}^C\xi_{B)C} + \frac{1}{2}\Omega_{ABCD}\xi^{CD}$$
$$+ \frac{3}{4}\Omega^{CD}\xi_{ABCD} - \frac{3}{2}\Omega_{(A}{}^{CDF}\xi_{B)CDF} - \frac{3}{2}\mathcal{D}_{C(A}\mathcal{P}\kappa_{B)}{}^C\Big).$$

As before, we can use conditions (38a), (38b), (40) and (42), and identity (36b) to reduce this to

$$\mathcal{P}^2 \xi_{AB} = \mathcal{P}\left(\frac{1}{3} K \xi_{AB} - \Omega_{(A}{}^C \xi_{B)C} + \Omega_{ABCD} \xi^{CD} - \mathcal{D}_{AB}\xi\right).$$

By comparing terms, we find that

$$\mathcal{P}^3 \kappa_{AB} = -\frac{2}{3} \mathcal{P}^2 \xi_{AB}$$

which is exactly the second condition (47b). So, no further conditions are needed to be prescribed on the hypersurface—this condition arises naturally from the evolution equation for the Killing spinor.

4.3.7. Decomposing $\zeta_{AA'} = 0$. Recalling the definition of $\zeta_{AA'}$, Eq. (19), and splitting the spacetime spinorial derivative into normal and tangential parts, one obtains

$$\zeta_{AA'} = \nabla^B{}_{A'} \Theta_{AB}$$
$$= \frac{1}{2} \tau^B{}_{A'} \mathcal{P} \Theta_{AB} - \tau^C{}_{A'} \mathcal{D}_C{}^B \Theta_{AB}.$$

From conditions (43) and (44), it then follows that $\zeta_{AA'}|_{\mathcal{S}} = 0$.

4.3.8. Decomposing $\nabla_{EE'} \zeta_{AA'} = 0$. Again, if $\zeta_{AA'}|_{\mathcal{S}} = 0$ then one only needs to consider the transverse derivative $\mathcal{P}\zeta_{AA'}$. By definition, one has that

$$\mathcal{P}\zeta_{AA'} = \mathcal{P}\nabla^B{}_{A'} \Theta_{AB}$$
$$= \mathcal{P}\left(-\tau^C{}_{A'} \mathcal{D}^B{}_C + \frac{1}{2} \tau^B{}_{A'} \mathcal{P}\right) \Theta_{AB}$$
$$= \frac{1}{2} \tau^B{}_{A'} \mathcal{P}^2 \Theta_{AB}$$

where the last equation has been obtained by commuting the Sen and normal derivatives, and using (44). Therefore, one only needs to show that

$$\mathcal{P}^2 \Theta_{AB} = 0.$$

Now, recalling the wave equation for Θ_{AB}, Eq. (20), one readily notices that the right-hand side vanishes on $\mathcal{S}$ as a consequence of (38a), (38b) and (40), so that one is left with

$$\Box \Theta_{AB}|_{\mathcal{S}} = 0.$$

Finally, expanding the left-hand side one finds that on $\mathcal{S}$

$$\Box \Theta_{AB} = \nabla^{CC'} \nabla_{CC'} \Theta_{AB}$$
$$= \left(-\tau^{BC'} \mathcal{D}^C{}_B + \frac{1}{2} \tau^{CC'} \mathcal{P}\right)\left(-\tau^B{}_{C'} \mathcal{D}_{BC} + \frac{1}{2} \tau_{CC'} \mathcal{P}\right) \Theta_{AB}$$
$$= \frac{1}{4} \tau^{CC'} \tau_{CC'} \mathcal{P}^2 \Theta_{AB}$$

where the last line follows by commuting the derivatives where appropriate and using conditions (43) and (44). Finally, as $\tau^{CC'} \tau_{CC'} = 2$ by definition, we get that $\mathcal{P}^2 \Theta_{AB} = 0$ as a consequence of the evolution equation for Θ_{AB}.

4.4. Eliminating Redundant Conditions

The discussion of the previous subsections can be summarised in the following:

Theorem 3. *Let κ_{AB} denote a Killing spinor candidate on an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Then κ_{AB} satisfies on a Cauchy hypersurface $\mathcal{S}$ the intrinsic conditions*

$$\xi_{ABCD} = 0, \tag{50a}$$

$$\Psi_{F(ABC}\kappa_{D)}{}^{F} = 0, \tag{50b}$$

$$\kappa_{(A}{}^{C}\phi_{B)C} = 0, \tag{50c}$$

$$\kappa_{(A|}{}^{C}\mathcal{D}_{CD}\phi_{|B)}{}^{D} = \frac{1}{3}\phi_{(A}{}^{C}\xi_{B)C}, \tag{50d}$$

$$3\kappa_{(A}{}^{F}\mathcal{D}_{B}{}^{E}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F} = 6\widehat{\phi}_{F(A}\kappa^{E}{}_{B}\mathcal{D}^{F}{}_{C}\phi_{D)E}$$
$$+ 6\widehat{\phi}_{(AB}\kappa^{E}{}_{C}\mathcal{D}^{F}{}_{D)}\phi_{EF}, \tag{50e}$$

with normal derivative on $\mathcal{S}$ given by

$$\mathcal{P}\kappa_{AB} = -\frac{2}{3}\xi_{AB},$$

if and only if κ_{AB} is, in fact, a Killing spinor, and the vector $\xi_{AA'} = \nabla^{B}{}_{A'}\kappa_{AB}$ is a Killing vector.

Remark 23. We note that

$$\Theta_{AB} = \kappa_{(A}{}^{C}\phi_{B)C} = 0 \quad \text{implies} \quad \phi_{AB} \propto \kappa_{AB}$$

Using this fact, one can show that (50d) and (50e) can be more simply expressed as a condition on the proportionality between the Killing spinor κ_{AB} and the Maxwell spinor ϕ_{AB}.

In order to simplify the conditions in Theorem 3 and to analyse their various interrelations, we proceed by looking at the different algebraic types that the Killing spinor can have. First, we consider the algebraically general case:

Lemma 11. *Assume that a symmetric spinor κ_{AB} satisfies the conditions*

$$\kappa_{AB}\kappa^{AB} \neq 0, \qquad \xi_{ABCD} = \Psi_{F(ABC}\kappa_{D)}{}^{F} = \kappa_{(A}{}^{C}\phi_{B)C} = 0$$

on an open subset $\mathcal{U} \subset \mathcal{S}$. Then, there exists a spin basis $\{o^{A}, \iota^{A}\}$ with $o_{A}\iota^{A} = 1$ such that the spinors κ_{AB} and ϕ_{AB} can be expanded as

$$\kappa_{AB} = e^{\varkappa}o_{(A}\iota_{B)}, \qquad \phi_{AB} = \varphi o_{(A}\iota_{B)}.$$

Furthermore, if $\mathfrak{Q} \equiv \varphi e^{2\varkappa}$ is a constant on $\mathcal{U}$, then conditions (50d) and (50e) are satisfied on $\mathcal{U}$.

Proof. The first part of the lemma follows directly from $\kappa_{AB}\kappa^{AB} \neq 0$, and the fact that $\kappa_{(A}{}^{C}\phi_{B)C} = 0$ implies that $\phi_{AB} \propto \kappa_{AB}$. The condition $\Psi_{F(ABC}\kappa_{D)}{}^{F} = 0$ also allows us to expand the Weyl spinor in the same basis:

$$\Psi_{ABCD} = \psi o_{(A}o_{B}\iota_{C}\iota_{D)}.$$

To show the redundancy of (50d) and (50e), we first decompose the equation $\mathcal{D}_{(AB}\kappa_{CD)} = 0$ into irreducible components. To simplify the notation, we borrow the D, Δ, δ symbols from the Newman-Penrose formalism to represent directional derivatives:

$$D \equiv o^A o^B \mathcal{D}_{AB}, \qquad \Delta \equiv \iota^A \iota^B \mathcal{D}_{AB}, \qquad \delta \equiv o^A \iota^B \mathcal{D}_{AB}. \qquad (51)$$

The components of $\mathcal{D}_{(AB}\kappa_{CD)} = 0$ then become:

$$o^C D o_C = 0, \qquad (52a)$$

$$o^C \delta o_C = -\frac{1}{2} D \varkappa, \qquad (52b)$$

$$\iota^C D \iota_C - o^C \Delta o_C = 2\delta \varkappa, \qquad (52c)$$

$$\iota^C \delta \iota_C = \frac{1}{2} \Delta \varkappa, \qquad (52d)$$

$$\iota^C \Delta \iota_C = 0. \qquad (52e)$$

Using these, one can show that

$$e^{-\varkappa} \xi_{AB} = -3 o_A o_B \iota^F \delta \iota_F - 3 \iota_A \iota_B o^F \delta o_F + \frac{3}{2} o_{(A} \iota_{B)} \left(\iota^F D \iota_F + o^F \Delta o_F \right).$$

Now, using the electromagnetic Gauss constraint, Eq. (34b), together with the expansion for ϕ_{AB} one obtains that using the basis expansion for ϕ_{AB} one obtains

$$\delta \varphi + 2\varphi \delta \varkappa = 0 \qquad (53)$$

on $\mathcal{S}$. Now, the spacetime Bianchi identity (4) implies the constraint

$$\mathcal{D}^{CD} \Psi_{ABCD} = -2\hat{\phi}^{CD} \mathcal{D}_{CD} \phi_{AB} \qquad (54)$$

on $\mathcal{S}$. To find the basis expansion of the Hermitian conjugate $\widehat{\phi}_{AB}$, note that:

$$o_A \widehat{o}^A \equiv o_A \tau^{AA'} \bar{o}_{A'} = \tau_{AA'} o^A \bar{o}^{A'} = \tau_a k^a,$$

where $k_a \equiv o_A \bar{o}_{A'}$. As τ_a is timelike and k_a is null, this scalar product is nonzero, and the pair $\{o_A, \widehat{o}_A\}$ forms a basis. We expand the spinor ι^A in this basis as

$$\iota^A = \alpha \widehat{o}^A + \beta o^A.$$

Contracting this with o_A, we find $1/\alpha = o_A \widehat{o}^A \geq 0$, and so $\alpha \geq 0$. Performing a Lorentz transformation on the basis $\{o_A, \iota_A\}$ parametrised by the complex function one has that

$$o^A \mapsto \tilde{o} = \frac{1}{\lambda} o^A,$$

$$\iota^A \mapsto \tilde{\iota} = \lambda \iota^A.$$

This transformation preserves the value of $o_A \iota^A$ and the symmetrised product $o_{(A} \iota_{B)}$, and thus, it preserves the form of the basis expansions of κ_{AB} and ϕ_{AB}. Moreover, one has that

$$\tilde{\iota}^A = \alpha |\lambda|^2 \widehat{\tilde{o}}^A + \beta \lambda^2 \tilde{o}^A.$$

So, by choosing $|\lambda|^2 = 1/\alpha$ and $\tilde{\beta} = \beta\lambda^2$, and dropping the tildes, we get

$$\iota^A = \widehat{o}^A + \beta o^A,$$
$$\widehat{\iota}^A = -o^A + \bar{\beta}\widehat{o}^A.$$

Using the above expressions, we can find the basis expansion of $\widehat{\phi}_{AB}$. Namely, one has that:

$$\widehat{\phi}_{AB} = \frac{1}{2}\bar{\varphi}(\widehat{o}_A\widehat{\iota}_B + \widehat{\iota}_A\widehat{o}_B)$$
$$= \frac{1}{2}\bar{\varphi}(-o_A\widehat{o}_B - \widehat{o}_A o_B + 2\bar{\beta}\widehat{o}_A\widehat{o}_B)$$
$$= \bar{\varphi}\bar{\beta}\iota_A\iota_B + \bar{\varphi}\beta(1 + |\beta|^2)o_A o_B - \bar{\varphi}(1 + 2|\beta|^2)o_{(A}\iota_{B)}.$$

Now, using the basis expansion for the Weyl spinor, contracting with combinations of o^A and ι^A and using the relations given in (52a) and (53), the components of (54) become

$$D\psi + 3\psi D\varkappa = 6|\varphi|^2(1 + 2|\beta|^2)D\varkappa + 12\bar{\beta}|\varphi|^2 o^A\Delta o_A,$$
$$\Delta\psi + 3\psi\Delta\varkappa = 6|\varphi|^2(1 + 2|\beta|^2)\Delta\varkappa - 12\beta|\varphi|^2(1 + |\beta|^2)\iota^A D\iota_A,$$
$$\delta\psi + 3\psi\delta\varkappa = -6|\varphi|^2(1 + 2|\beta|^2)\delta\varkappa - 3\beta\bar{\varphi}(1 + |\beta|^2)D\varphi - 3\bar{\beta}\bar{\varphi}\Delta\varphi.$$

Exploiting conditions (52a), the expansions of the Maxwell and the Bianchi constraints, it can be shown that condition (50e) can be decomposed into the following non-trivial irreducible parts:

$$\bar{\beta}\bar{\varphi}\left(D\varphi + 2\varphi D\varkappa\right) = 0,$$
$$\bar{\varphi}(1 + 2|\beta|^2)\left(D\varphi + 2\varphi D\varkappa\right) = 0,$$
$$\bar{\varphi}(1 + 2|\beta|^2)\left(\Delta\varphi + 2\varphi\Delta\varkappa\right) = 0,$$
$$\beta\bar{\varphi}(1 + |\beta|^2)\left(\Delta\varphi + 2\varphi\Delta\varkappa\right) = 0.$$

Assuming $\varphi \neq 0$, these conditions along with the Maxwell constraint (53) are equivalent to the following basis-independent expression, also independent of the value of β:

$$\mathcal{D}_{AB}\varphi + 2\varphi\mathcal{D}_{AB}\varkappa = 0.$$

The latter can be written as

$$\mathcal{D}_{AB}\left(\varphi e^{2\varkappa}\right) = \mathcal{D}_{AB}\mathfrak{Q} = 0.$$

Therefore, under the hypotheses of the present lemma, Eq. (50e) is equivalent to the requirement of $\mathfrak{Q}$ being constant in a domain $\mathcal{U} \subset \mathcal{S}$. In a similar way, substituting the above relations in Eq. (50d) and splitting into irreducible parts gives the following set of equivalent conditions:

$$e^{\varkappa}\left(D\varphi + 2\phi D\varkappa\right) = 0,$$
$$e^{\varkappa}\left(\Delta\varphi + 2\varphi\Delta\varkappa\right) = 0,$$
$$e^{\varkappa}\left(\delta\varphi + 2\varphi\delta\varkappa\right) = 0.$$

As $e^{\varkappa}$ is nonzero, this set of conditions is again equivalent to the constancy of $\mathfrak{Q}$ in $\mathcal{U} \subset \mathcal{S}$. □

Next, we consider the case when the Killing spinor is algebraically special:

Lemma 12. *Assume the symmetric spinor κ_{AB} satisfies the conditions*

$$\kappa_{AB}\kappa^{AB} = 0, \quad \kappa_{AB}\hat{\kappa}^{AB} \neq 0, \quad \xi_{ABCD} = \Psi_{F(ABC}\kappa_{D)}{}^{F} = \kappa_{(A}{}^{C}\phi_{B)C} = 0$$

on an open subset $\mathcal{U} \subset \mathcal{S}$. Then, there exists a normalised spin basis $\{o^A, \iota^A\}$ such that the spinors κ_{AB} and ϕ_{AB} can be expanded as

$$\kappa_{AB} = e^{\varkappa}o_A o_B, \qquad \phi_{AB} = \varphi o_A o_B.$$

Furthermore, Eqs. (50d) and (50e) are satisfied on $\mathcal{U} \subset \mathcal{S}$.

Proof. The first part of the lemma follows directly from the hypothesis $\kappa_{AB}\kappa^{AB} = 0$, $\kappa_{AB}\hat{\kappa}^{AB} \neq 0$, and the fact that $\kappa_{(A}{}^{C}\phi_{B)C} = 0$ implies $\phi_{AB} \propto \kappa_{AB}$. The condition $\Psi_{F(ABC}\kappa_{D)}{}^{F} = 0$ also allows us to expand the Weyl spinor in the same basis as

$$\Psi_{ABCD} = \psi o_A o_B o_C o_D.$$

In this basis, the components of the equation $\mathcal{D}_{(AB}\kappa_{CD)} = 0$ become

$$o^A D o_A = 0,$$
$$D\varkappa + 4o^A \delta o_A + 2\iota^A D o_A = 0,$$
$$\delta\varkappa + o^A \Delta o_A + 2\iota^A \delta o_A = 0,$$
$$\Delta\varkappa + 2\iota^A \Delta o_A = 0.$$

Using these relations, one can show that

$$e^{-\varkappa}\xi_{AB} = 3o_A o_B o^C \Delta o_C - 6o_{(A}\iota_{B)}o^C \delta o_C.$$

The Maxwell constraint, Eq. (34b), on $\mathcal{S}$ is equivalent to

$$D\phi - \phi D\varkappa - 6\phi o^A \delta o_A = 0,$$

and the $o_{(A}\iota_{B)}$ component of the Bianchi constraint

$$\mathcal{D}^{CD}\Psi_{ABCD} = -2\hat{\phi}^{CD}\mathcal{D}_{CD}\phi_{AB}$$

on $\mathcal{S}$, as a consequence of the previous relations, is equivalent to the following condition:

$$|\varphi|^2 o^A \Delta o_A - 2\beta|\varphi|^2 o^A \delta o_A = 0.$$

Then, by substituting all the relevant basis expansions into (50d) and (50e), and splitting the equations into irreducible parts, one finds that both conditions are automatically satisfied as a result of the above relations.

$\square$

We round up the discussion of this section with the following electrovacuum analogue of Theorem in [6]:

Lemma 13. *Assume that one has a symmetric spinor κ_{AB} satisfying the conditions*

$$\mathcal{D}_{(AB}\kappa_{CD)} = \Psi_{F(ABC}\kappa_{D)}{}^{F} = \kappa_{(A}{}^{C}\phi_{B)C} = 0$$

on the Cauchy hypersurface $\mathcal{S}$ and that the complex function

$$\mathfrak{Q}^2 \equiv \left(\kappa_{AB}\kappa^{AB}\right)^2 \phi_{AB}\phi^{AB}$$

is constant on $\mathcal{S}$. Then the domain of dependence, $D^+(\mathcal{S})$, of the initial data set $(\mathcal{S}, \boldsymbol{g}, \boldsymbol{K}, \boldsymbol{F})$ will admit a Killing spinor.

Proof. Let $\mathcal{U}_1$ be the set of all points in $\mathcal{S}$ where $\kappa_{AB}\kappa^{AB} \neq 0$ and $\mathcal{U}_2$ be the set of all points in $\mathcal{S}$ where $\kappa_{AB}\widehat{\kappa}^{AB} \neq 0$. The scalar functions $\kappa_{AB}\kappa^{AB} : \mathcal{S} \to \mathbb{C}$ and $\kappa_{AB}\widehat{\kappa}^{AB} : \mathcal{S} \to \mathbb{R}$ are continuous. Therefore, $\mathcal{U}_1$ and $\mathcal{U}_2$ are open sets. Now, let $\mathcal{V}_1$ and $\mathcal{V}_2$ denote, respectively, the interiors of $\mathcal{S}\backslash\mathcal{U}_1$ and $\mathcal{V}_1\backslash\mathcal{U}_2$. On the open set $\mathcal{V}_1 \cap \mathcal{U}_2$, we have that $\kappa_{AB}\kappa^{AB} = 0$ and $\kappa_{AB}\widehat{\kappa}^{AB} \neq 0$. Hence, by Lemma 12, conditions (50d) and (50e) are satisfied on $\mathcal{V}_1 \cap \mathcal{U}_2$. Similarly, by Lemma 11, conditions (50d) and (50e) are satisfied on $\mathcal{U}_1$. On the open set $\mathcal{V}_2$, we have that $\kappa_{AB} = 0$ and therefore (50d) and (50e) are trivially satisfied on $\mathcal{V}_2$. Using the above sets, the 3-manifold $\mathcal{S}$ can be split as

$$\text{int}\mathcal{S} = \mathcal{U}_1 \cup (\mathcal{V}_1 \cap \mathcal{U}_2) \cup \mathcal{V}_2 \cup \partial\mathcal{U}_1 \cup \partial\mathcal{V}_2.$$

By hypothesis, all terms in conditions (50d) and (50e) are continuous, and the conditions themselves are satisfied on the open sets $\mathcal{U}_1$, $\mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{U}_2$. By continuity, the conditions are also satisfied on the boundaries $\partial\mathcal{U}_1$ and $\partial\mathcal{V}_2$. Therefore, (50d) and (50e) are satisfied on $\text{int}\,\mathcal{S}$, and by continuity this extends to the whole of $\mathcal{S}$. $\qquad\square$

4.5. Summary

We can summarise the discussion of the present section calculations in the following theorem:

Theorem 4. *Let* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{F})$ *be an initial data set for the Einstein–Maxwell field equations where* $\mathcal{S}$ *is a Cauchy hypersurface. Then the conditions*

$$\xi_{ABCD} = 0, \tag{55a}$$

$$\Psi_{F(ABC}\kappa_{D)}{}^F = 0, \tag{55b}$$

$$\kappa_{(A}{}^C\phi_{B)C} = 0, \tag{55c}$$

$$\mathfrak{Q}^2 \equiv \left(\kappa_{AB}\kappa^{AB}\right)^2 \phi_{AB}\phi^{AB} = constant, \tag{55d}$$

are satisfied on $\mathcal{S}$ *if and only if the development of the initial data set admits a Killing spinor* κ_{AB} *in the domain of dependence of* $\mathcal{S}$, *such that* $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ *is a Killing vector. The Killing spinor is obtained by evolving* (7b) *with initial data on* $\mathcal{S}$ *satisfying the above conditions and*

$$\mathcal{P}\kappa_{AB} = -\frac{2}{3}\xi_{AB}.$$

5. The Approximate Killing Spinor Equation

In the previous section, we have identified the conditions that need to be satisfied by an initial data set for the Einstein–Maxwell equations so that its development is endowed with a Killing spinor —see Theorem 4. Together with

the characterisation of the Kerr–Newman spacetime given by Theorem 1, the latter provide a way of characterising initial data for the Kerr–Newman spacetime. The key equation in this characterisation is the *spatial Killing spinor equation*

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0. \tag{56}$$

As it will be seen in the following, this equation is overdetermined and thus admits no solution for a generic initial data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{F})$. Following the discussion of Section 5 in [4], in this section we show how to construct a elliptic equation for a spinor κ_{AB} over $\mathcal{S}$ which can always be solved and which provides, in some sense, a best fit to a spatial Killing spinor. This *approximate Killing spinor* will be used, in turn, to measure the deviation of the electrovacuum initial data set under consideration from initial data for the Kerr–Newman spacetime.

5.1. Basic Identities

In the present section, we provide a brief discussion of the basic ellipticity properties of the spatial Killing equation. In what follows, let $\mathfrak{S}_{(AB)}(\mathcal{S})$ and $\mathfrak{S}_{(ABCD)}(\mathcal{S})$ denote, respectively, the space of totally symmetric valence 2 and 4 spinor fields over the 3-manifold $\mathcal{S}$. Given $\mu_{AB}, \nu_{AB} \in \mathfrak{S}_{(AB)}(\mathcal{S})$, $\zeta_{ABCD}, \chi_{ABCD} \in \mathfrak{S}_{(ABCD)}(\mathcal{S})$ one can use the Hermitian structure induced on $\mathcal{S}$ by $\tau^{AA'}$ to define an inner product in $\mathfrak{S}_{(AB)}(\mathcal{S})$ and $\mathfrak{S}_{(ABCD)}(\mathcal{S})$, respectively, via

$$\langle \boldsymbol{\mu}, \boldsymbol{\nu} \rangle \equiv \int_{\mathcal{S}} \mu_{AB}\widehat{\nu}^{AB}\mathrm{d}\mu, \qquad \langle \boldsymbol{\zeta}, \boldsymbol{\chi} \rangle \equiv \int_{\mathcal{S}} \zeta_{ABCD}\widehat{\chi}^{ABCD}\mathrm{d}\mu, \tag{57}$$

where $\mathrm{d}\mu$ denotes volume form of the 3-metric $\boldsymbol{h}$.

Let now Φ denote the *spatial Killing spinor operator*

$$\Phi : \mathfrak{S}_{(AB)}(\mathcal{S}) \longrightarrow \mathfrak{S}_{(ABCD)}(\mathcal{S}), \qquad \Phi(\boldsymbol{\kappa}) \equiv \mathcal{D}_{(AB}\kappa_{CD)}.$$

The inner product (57) allows to define $\Phi^* : \mathfrak{S}_{(ABCD)}(\mathcal{S}) \longrightarrow \mathfrak{S}_{(AB)}(\mathcal{S})$, the *formal adjoint of* Φ, through the condition

$$\langle \Phi(\boldsymbol{\kappa}), \boldsymbol{\zeta} \rangle = \langle \boldsymbol{\kappa}, \Phi^*(\boldsymbol{\zeta}) \rangle.$$

In order to evaluate the above condition, one makes use of the identity (*integration by parts*)

$$\int_{\mathcal{U}} \mathcal{D}^{AB}\kappa^{CD}\widehat{\zeta}_{ABCD}\mathrm{d}\mu - \int_{\mathcal{U}} \kappa^{AB}\mathcal{D}^{\widehat{CD}\zeta}_{ABCD}\mathrm{d}\mu + \int_{\mathcal{U}} 2\kappa^{AB}\Omega^{CDF}{}_A\widehat{\zeta}_{BCDF}\mathrm{d}\mu$$

$$= \int_{\partial\mathcal{U}} n^{AB}\kappa^{CD}\widehat{\zeta}_{ABCD}\mathrm{d}S \tag{58}$$

with $\mathcal{U} \subset \mathcal{S}$ and where $\mathrm{d}S$ denotes the area element of $\partial\mathcal{U}$, n_{AB} is the spinorial counterpart of its outward pointing normal and ζ_{ABCD} is a totally symmetric

spinorial field. Now, observing that

$$\langle\Phi(\boldsymbol{\kappa}),\boldsymbol{\zeta}\rangle = \int_{\mathcal{S}} \mathcal{D}_{(AB}\kappa_{CD)}\widehat{\zeta}^{ABCD}\mathrm{d}\mu$$

$$= \int_{\mathcal{S}} \mathcal{D}_{AB}\kappa_{CD}\widehat{\zeta}^{ABCD}\mathrm{d}\mu,$$

it follows then from identity (58) that

$$\Phi^*(\boldsymbol{\zeta}) = \mathcal{D}^{AB}\zeta_{ABCD} - 2\Omega^{ABF}{}_{(C}\zeta_{D)ABF}.$$

Definition 3. The composition operator $L \equiv \Phi^* \circ \Phi : \mathfrak{S}_{(AB)}(\mathcal{S}) \longrightarrow \mathfrak{S}_{(AB)}(\mathcal{S})$ given by

$$L(\boldsymbol{\kappa}) \equiv \mathcal{D}^{AB}\mathcal{D}_{(AB}\kappa_{CD)} - \Omega^{ABF}{}_{(A}\mathcal{D}_{|DF|}\kappa_{B)C} - \Omega^{ABF}{}_{(A}\mathcal{D}_{B)F}\kappa_{CD} \qquad (59)$$

will be called the *approximate Killing spinor operator* and the equation

$$L(\boldsymbol{\kappa}) = 0$$

the *approximate Killing spinor equation.*

Remark 24. A direct computation shows that the approximate Killing spinor Eq. (59) is, in fact, the Euler–Lagrange equation of the functional

$$J \equiv \int_{\mathcal{S}} \mathcal{D}_{(AB}\kappa_{CD)}\widehat{\mathcal{D}^{AB}\kappa^{CD}}\mathrm{d}\mu.$$

5.2. Ellipticity of the Approximate Killing Spinor Equation

The key observation concerning the approximate Killing spinor operator is given in the following:

Lemma 14. *The operator L is a formally self-adjoint elliptic operator.*

Proof. It is sufficient to look at the principal part of the operator L given by

$$P(L)(\boldsymbol{\kappa}) = \mathcal{D}^{AB}\mathcal{D}_{(AB}\kappa_{CD)}.$$

The symbol for this operator is given by

$$\sigma_L(\boldsymbol{\xi}) \equiv \xi^{AB}\xi_{(AB}\kappa_{CD)}$$

where the argument ξ_{AB} satisfies $\xi_{AB} = \xi_{(AB)}$ and $\widehat{\xi}_{AB} = -\xi_{AB}$—i.e. ξ is a real symmetric spinor. Also, define the inner product $\langle\,,\rangle$ on the space of symmetric valence-2 spinors by

$$\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle \equiv \widehat{\xi}^{AB}\eta_{AB}.$$

The operator L is elliptic if the map

$$\sigma_L(\boldsymbol{\xi}) : \kappa_{AB} \mapsto \xi^{CD}\xi_{(CD}\kappa_{AB)}$$

is an isomorphism when $|\boldsymbol{\xi}|^2 \equiv \langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle \neq 0$. As the above mapping is linear and between vector spaces of the same dimension, one only needs to verify injectivity—in other words, that if $\xi^{AB}\xi_{(AB}\kappa_{CD)} = 0$, then $\kappa_{AB} = 0$. To show this, first expand the symmetrisation in the symbol to obtain

$$-\kappa_{CD}|\boldsymbol{\xi}|^2 - \langle\boldsymbol{\xi},\boldsymbol{\kappa}\rangle\xi_{CD} + 2\xi^{AB}\xi_{CB}\kappa_{AD} + 2\xi^{AB}\xi_{DB}\kappa_{AC} = 0,$$

where we have used the reality condition $\widehat{\xi}_{AB} = -\xi_{AB}$. Note also that the Jacobi identity implies that

$$\xi^{AB}\xi_{CB} = -\frac{1}{2}\delta_C{}^A|\boldsymbol{\xi}|^2,$$

which reduces the above equation to

$$3\kappa_{CD}|\boldsymbol{\xi}|^2 + \xi_{CD}\langle\boldsymbol{\xi},\boldsymbol{\kappa}\rangle = 0.$$

Contracting this with $\widehat{\kappa}^{CD}$, and using the conjugate symmetry of the inner product, we obtain

$$3|\boldsymbol{\kappa}|^2|\boldsymbol{\xi}|^2 + |\langle\boldsymbol{\xi},\boldsymbol{\kappa}\rangle|^2 = 0.$$

Both of these terms are positive, and so the equality can only hold if each term vanishes individually. Taking the first of these, one sees that when $|\boldsymbol{\xi}|^2 \neq 0$, we must have $|\boldsymbol{\kappa}|^2 = 0$. This is equivalent to $\kappa_{AB} = 0$, completing the proof of injectivity and establishing the ellipticity of L. $\qquad\square$

6. The Approximate Killing Spinor Equation in Asymptotically Euclidean Manifolds

The purpose of this section is to discuss the solvability of the approximate Killing spinor equation, Eq. (59), in asymptotically Euclidean manifolds. As a result of this analysis, one concludes that for this type of initial data sets for the Einstein–Maxwell equations it is always possible to construct an approximate Killing spinor.

6.1. Weighted Sobolev Norms

The discussion of asymptotic boundary conditions for the approximate Killing equation makes use of *weighted Sobolev norms* and *spaces*. In this section, we introduce the necessary terminology and conventions to follow the discussion. The required properties of these objects for the present analysis are discussed in detail in Section 6.2 of [4] to which the reader is directed for further reference.

Given u a scalar function over $\mathcal{S}$ and $\delta \in \mathbb{R}$, let $\| u \|_\delta$ denote the weighted L^2 Sobolev norm of u. All throughout we make use of Bartnik's conventions for the weights—see [7]—so that, in particular $\| u \|_{-3/2}$ is the *standard L^2* norm of u. Similarly, let H_δ^s with s a non-negative index denote the weighted Sobolev space of functions for which the norm

$$\| u \|_{s,\delta} \equiv \sum_{0 \leq |\alpha| \leq s} \| D^\alpha u \|_{\delta-|\alpha|},$$

is finite where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex and $|\alpha| \equiv \alpha_1 + \alpha_2 + \alpha_3$. We say that $u \in H_\delta^\infty$ if $u \in H_\delta^s$ for all s. We will say that a spinor or a tensor belongs to a function space if its norm does—so, for instance $\zeta_{AB} \in H_\delta^s$ is a shorthand for $(\zeta_{AB}\hat{\zeta}^{AB} + \zeta_A{}^A\hat{\zeta}_B{}^B)^{1/2} \in H_\delta^s$. A property of the weighted Sobolev spaces that will be used repeatedly is the following: if $u \in H_\delta^\infty$, then u is smooth (i.e. C^∞ over $\mathcal{S}$) and has a fall off at infinity such that $D^\alpha u = o(r^{\delta-|\alpha|})$.[1] In

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[1] Recall that the *small o* indicates that if $f(x) = o(x^n)$, then $f(x)/x^n \to 0$ as $x \to 0$.

a slight abuse of notation, if $u \in H_\delta^\infty$, then we will often say that $u = o_\infty(r^\delta)$ at a given asymptotic end.

6.2. Asymptotically Euclidean Manifolds

We begin by spelling out our assumptions on the class of Einstein–Maxwell initial data sets to be considered in the remainder of this article. The Einstein–Maxwell constraint equations are given by

$$r - K^2 + K_{ij}K^{ij} = 2\rho,$$
$$D^j K_{ij} - D_i K = j_i,$$
$$D^i E_i = 0,$$
$$D^i B_i = 0,$$

where D_i denotes the Levi–Civita connection of the 3-metric $\boldsymbol{h}$, r is the associated Ricci scalar, K_{ij} is the extrinsic curvature, $K \equiv K_i{}^i$, ρ is the energy-density of the electromagnetic field, j_i is the associated Poynting vector and E_i and B_i denote the electric and magnetic parts of the Faraday tensor with respect to the unit normal of $\mathcal{S}$.

Assumption 3. *In the remainder of this article, it will be assumed that one has initial data* $(\boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ *for the Einstein–Maxwell equations which is asymptotically Reissner–Nordström in the sense that in each asymptotic end of $\mathcal{S}$ there exist* asymptotically Cartesian coordinates (x^α) *and two constants* $m, q \neq 0$, *for which*

$$h_{\alpha\beta} = -\left(1 + \frac{2m}{r}\right)\delta_{\alpha\beta} + o_\infty(r^{-3/2}), \tag{60a}$$

$$K_{\alpha\beta} = o_\infty(r^{-5/2}), \tag{60b}$$

$$E_\alpha = \frac{qx_\alpha}{r^2} + o_\infty(r^{-5/2}), \tag{60c}$$

$$B_\alpha = o_\infty(r^{-5/2}). \tag{60d}$$

Remark 25. The asymptotic conditions spelled in Assumption 3 ensure that the total mass and electric charge of the initial data are non-vanishing. In particular, it contains standard initial data for the Kerr–Newman spacetime in, say, Boyer–Lindquist coordinates as an example. More generally, the assumptions are consistent with the notion of *stationary asymptotically flat end* provided in Definition 1. In the case $m = 0$, the spacetime is guaranteed to be isomorphic to the Minkowski spacetime as a result of the positive energy theorem; in the case $q = 0$, the Einstein–Maxwell equations reduce to the vacuum Einstein equations, and the proceeding analysis is the same as in [4].

Remark 26. The above class of initial data is not the most general one could consider. In particular, conditions (60a)–(60d) exclude boosted initial data. In order to do so, one would require that

$$K_{\alpha\beta} = o_\infty(r^{-3/2}).$$

The Einstein–Maxwell constraint equations would then require one to modify the leading behaviour of the 3-metric $h_{\alpha\beta}$. The required modifications for this extension of the present analysis are discussed in [4].

6.3. Asymptotic Behaviour of the Approximate Killing Spinor

In this section, we discuss the asymptotic behaviour of solutions to the spatial Killing spinor equation (Eq. (56) start of Sect. 5) on asymptotically Euclidean manifolds of the type described in Assumption 3. To this end, we first consider the behaviour of the Killing spinor in the Kerr–Newman spacetime. In a second stage, we impose the same asymptotic solution to the approximate Killing spinor equation on slices of a more general spacetime. In what follows, we concentrate our discussion to an asymptotic end.

6.3.1. Asymptotic Behaviour in the Exact Kerr–Newman Spacetime.

For the exact Kerr–Newman spacetime with mass m, angular momentum a and charge q, it is possible to introduce a NP frame $\{l^a, n^a, m^a, \bar{m}^a\}$ with associated spin dyad $\{o^A, \iota^A\}$ such that the spinors κ_{AB}, ϕ_{AB} and Ψ_{ABCD} admit the expansion

$$\kappa_{AB} = \varkappa o_{(A} \iota_{B)}, \qquad \phi_{AB} = \varphi o_{(A} \iota_{B)}, \qquad \Psi_{ABCD} = \psi o_{(A} o_B \iota_C \iota_{D)},$$

with

$$\varkappa = \frac{2}{3}(r - ia \cos\theta),$$

$$\varphi = \frac{q}{(r - ia \cos\theta)^2},$$

$$\psi = \frac{6}{(r - ia \cos\theta)^3} \left(\frac{q^2}{r + ia \cos\theta} - m \right),$$

where r denotes the standard *Boyer–Lindquist* radial coordinate—see [2] for more details. A further computation shows that the spinorial counterpart, $\xi^{AA'}$, of the Killing vector ξ^a takes the form

$$\xi_{AA'} = -\frac{3}{2}\varkappa(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} + \tau \iota_A \bar{o}_{A'} - \rho \iota_A \bar{\iota}_{A'}) \tag{61}$$

where the NP spin connection coefficients μ, π, τ and ρ satisfy the conditions

$$\bar{\mu}\bar{\varkappa} = \mu\varkappa, \qquad \bar{\tau}\bar{\varkappa} = \varkappa\pi, \qquad \bar{\rho}\bar{\varkappa} = \varkappa\rho$$

which ensure that $\xi_{AA'}$ is a Hermitian spinor —i.e. $\xi_{AA'} = \bar{\xi}_{AA'}$. Despite the conciseness of the above expressions, the basis of *principal spinors* given by $\{o^A, \iota^A\}$ is not well adapted to the discussion of asymptotics on a stationary end of the Kerr–Newman spacetime—in particular, the asymptotic behaviour of the NP frame $\{l^a, n^a, m^a, \bar{m}^a\}$ is not related to the asymptotic behaviour of the stationary Killing vector of the spacetime.

From the point of view of asymptotics, a better representation of the Kerr–Newman spacetime is obtained using a NP frame $\{l'^a, n'^a, m'^a, \bar{m}'^a\}$ with associated spin dyad $\{o'^A, \iota'^A\}$ such that

$$\tau^a = l'^a + n'^a = \sqrt{2}(\partial_t)^a,$$

where the vector τ^a is the tensorial counterpart of the spinor $\tau^{AA'}$. It follows from the above that

$$\tau^{AA'} = o'^A \bar{o}'^{A'} + \iota'^A \bar{\iota}'^{A'}. \tag{62}$$

Notice, in particular, that from the above expression it follows that $\iota'_A = \hat{o}'_A$. As $\tau_{AA'} = \sqrt{2}\xi_{AA'}$, one can use expressions (61) and (62) to compute the leading terms of the Lorentz transformation relating the NP frames $\{l^a, n^a, m^a, \bar{m}^a\}$ and $\{l'^a, n'^a, m'^a, \bar{m}'^a\}$. The details of this tedious computation will not be presented here—just the main result.

In what follows it will be convenient to denote the spinors of the basis $\{o'^A, \iota'^A\}$ in the form $\{\epsilon_{\boldsymbol{A}}{}^A\}$ where

$$\epsilon_{\boldsymbol{0}}{}^A = o'^A, \qquad \epsilon_{\boldsymbol{1}}{}^A = \iota'^A.$$

Moreover, let $\kappa_{\boldsymbol{AB}} \equiv \epsilon_{\boldsymbol{A}}{}^A \epsilon_{\boldsymbol{B}}{}^B \kappa_{AB}$ denote the components of κ_{AB} with respect to the basis $\{\epsilon_{\boldsymbol{A}}{}^A\}$. It can then be shown that for Kerr–Newman initial data satisfying the asymptotic conditions (60a)–(60d) one can choose asymptotically Cartesian coordinates $(x^\alpha) = (x^1, x^2, x^3)$ and orthonormal frames on the asymptotic ends such that

$$\kappa_{\boldsymbol{AB}} = \mp \frac{\sqrt{2}}{3} x_{\boldsymbol{AB}} \mp \frac{2\sqrt{2}m}{3r} x_{\boldsymbol{AB}} + o_\infty(r^{-1/2}), \tag{63}$$

with

$$x_{\boldsymbol{AB}} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -x^1 + \mathrm{i}x^2 & x^3 \\ x^3 & x^1 + \mathrm{i}x^2 \end{pmatrix}.$$

From the above expressions, one finds that on the asymptotic ends

$$\xi = \pm\sqrt{2} + o_\infty(r^{-1/2}),$$

$$\xi_{\boldsymbol{AB}} = o_\infty(r^{-1/2}),$$

where $\xi_{\boldsymbol{AB}} \equiv \epsilon_{\boldsymbol{A}}{}^A \epsilon_{\boldsymbol{B}}{}^B \xi_{AB}$. Moreover, for any electrovacuum initial data set satisfying conditions (60a)–(60d) a spinor of form (63) satisfies

$$\mathcal{D}_{(\boldsymbol{AB}}\kappa_{\boldsymbol{CD})} = o_\infty(r^{-3/2}).$$

6.3.2. Asymptotic Behaviour for Non-Kerr Data. Not unsurprisingly, given electrovacuum initial data satisfying conditions (60a)–(60d), it is always possible to find a spinor κ_{AB} satisfying expansion (63) in the asymptotic region. More precisely, one has:

Lemma 15. *For any asymptotic end of an electrovacuum initial data set satisfying* (60a)–(60d), *there exists a spinor* κ_{AB} *such that*

$$\kappa_{AB} = \mp \frac{\sqrt{2}}{3} x_{\boldsymbol{AB}} \mp \frac{2\sqrt{2}m}{3r} x_{\boldsymbol{AB}} + o_\infty(r^{-1/2})$$

with

$$\xi = \pm\sqrt{2} + o_\infty(r^{-1/2}), \tag{64a}$$

$$\xi_{AB} = o_\infty(r^{-1/2}), \tag{64b}$$

$$\xi_{ABCD} = o_\infty(r^{-3/2}). \tag{64c}$$

Proof. The proof follows the same structure of Theorem 17 in [4] where the vacuum case is considered. □

Remark 27. The spinors obtained from the previous lemma can be cut off so that they are zero outside the asymptotic end. One can then add them to yield a real spinor $\mathring{\kappa}_{AB}$ on the whole of $\mathcal{S}$ such that

$$\mathcal{D}_{(AB}\mathring{\kappa}_{CD)} \in H^{\infty}_{-3/2}$$

and asymptotic behaviour given by (63) at each end.

In the analysis of the solvability of the approximate Killing spinor equation, it is crucial that there exist no non-trivial spatial Killing spinor that goes to zero at infinity. More precisely, one has the following:

Lemma 16. *Let $\nu_{AB} \in H^{\infty}_{-1/2}$ be a solution to $\mathcal{D}_{(AB}\nu_{CD)} = 0$ on an electrovacuum initial data set satisfying the asymptotic conditions (60a)–(60d). Then $\nu_{AB} = 0$ on $\mathcal{S}$.*

Proof. From Lemma 10, one can write $\mathcal{D}_{AB}\mathcal{D}_{CD}\mathcal{D}_{EF}\kappa_{GH}$ as a linear combination of lower order derivatives, with smooth coefficients. Direct inspection shows that the coefficients in this linear combination have the decay conditions to make use of Theorem 20 from [4] with $m = 2$. It then follows that ν_{AB} must vanish on $\mathcal{S}$. □

6.4. Solving the Approximate Killing Spinor Equation

In the reminder of this section, we will consider solutions to the approximate Killing spinor equation of the form:

$$\kappa_{AB} = \mathring{\kappa}_{AB} + \theta_{AB}, \qquad \theta_{AB} \in H^{\infty}_{-1/2} \tag{65}$$

with $\mathring{\kappa}_{AB}$ the spinor discussed in Remark 27. For this Ansatz, one has the following:

Theorem 5. *Given an electrovacuum asymptotically Euclidean initial data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ satisfying the asymptotic conditions (60a)–(60d) there exists a smooth unique solution to the approximate Killing spinor equation (59) of the form (65).*

Proof. The proof is analogous to that of Theorem 25 in [4] and is presented for completeness as this is the main result of this article.

Substitution of Ansatz (65) into equation (59) yields the equation

$$L(\theta_{AB}) = -L(\mathring{\kappa}_{AB}) \tag{66}$$

for the spinor θ_{AB}. Due to elliptic regularity, any solution to the above equation of class $H^{2}_{-1/2}$ is, in fact, a solution of class $H^{\infty}_{-1/2}$. Thus, if a solution θ_{AB} exists, then it must be smooth. By construction—see Remark 27—it follows that $\mathcal{D}_{(AB}\kappa_{CD)} \in H^{\infty}_{-3/2}$ so that

$$F_{AB} \equiv -L(\mathring{\kappa}_{AB}) \in H^{\infty}_{-5/2}.$$

In order to discuss the existence of solutions, we make use of the *Fredholm alternative* for weighted Sobolev spaces. In the particular case of equation (66), there exists a unique solution of class $H^{-1/2}$ if

$$\int_{\mathcal{S}} F_{AB} \widehat{\nu}^{AB} \mathrm{d}\mu = 0,$$

for all $\nu_{AB} \in H^2_{-1/2}$ satisfying

$$L^*(\nu_{CD}) = L(\nu_{CD}) = 0.$$

It will now be shown that a spinor ν_{AB} satisfying the above must be trivial. Using identity (58) with $\zeta_{ABCD} = \mathcal{D}_{(AB} \nu_{CD)}$ and assuming that $L(\nu_{CD}) = 0$, one obtains

$$\int_{\mathcal{S}} \mathcal{D}^{AB} \nu^{CD} \widehat{\mathcal{D}_{(AB} \nu_{CD)}} \mathrm{d}\mu = \int_{\partial \mathcal{S}_\infty} n^{AB} \nu^{CD} \widehat{\mathcal{D}_{(AB} \nu_{CD)}} \mathrm{d}S,$$

where $\partial \mathcal{S}_\infty$ denotes the sphere at infinity. Now, using that by assumption $\nu_{AB} \in H^2_{-1/2}$, it follows that $\mathcal{D}_{(AB} \nu_{CD)} \in H^\infty_{-3/2}$ and that

$$n^{AB} \nu^{CD} \widehat{\mathcal{D}_{(AB} \nu_{CD)}} = o(r^{-2}).$$

The integration of the latter over a finite sphere is of type $o(1)$. Accordingly, the integral over the sphere at infinity $\partial \mathcal{S}_\infty$ vanishes and, moreover,

$$\int_{\mathcal{S}} \mathcal{D}^{AB} \nu^{CD} \widehat{\mathcal{D}_{(AB} \nu_{CD)}} \mathrm{d}\mu = 0.$$

Thus, one concludes that

$$\mathcal{D}_{(AB} \nu_{CD)} = 0 \qquad \text{over} \qquad \mathcal{S}$$

so that ν_{AB} is a Killing spinor candidate. Now, Lemma 16 shows that there are no non-trivial Killing spinor candidates that go to zero at infinity.

It follows from the discussion in the previous paragraph that the kernel of the approximate Killing spinor operator is trivial and that the Fredholm alternative imposes no obstruction to the existence of solutions to (66). Thus, one obtains a unique solution to the approximate Killing spinor equation with the prescribed asymptotic behaviour at infinity. □

7. The Geometric Invariant

In this section, we make use of the approximate Killing spinor constructed in the previous section to construct an invariant measuring the deviation of a given electrovacuum initial data set satisfying the asymptotic conditions (60a)–(60d) from initial data for the Kerr–Newman spacetime.

In the following, let κ_{AB} denote the approximate Killing spinor obtained from Theorem 5, and let

$$J \equiv \int_{\mathcal{S}} \mathcal{D}_{(AB}\kappa_{CD)}\widehat{\mathcal{D}^{AB}\kappa^{CD}}\mathrm{d}\mu, \tag{67a}$$

$$I_1 \equiv \int_{\mathcal{S}} \Psi_{(ABC}{}^F\kappa_{D)F}\widehat{\Psi^{ABCG}\kappa^D}{}_G\mathrm{d}\mu, \tag{67b}$$

$$I_2 \equiv \int_{\mathcal{S}} \Theta_{AB}\widehat{\Theta^{AB}}\mathrm{d}\mu, \tag{67c}$$

$$I_3 \equiv \int_{\mathcal{S}} \mathcal{D}_{AB}\mathfrak{Q}^2\widehat{\mathcal{D}^{AB}\mathfrak{Q}^2}\mathrm{d}\mu, \tag{67d}$$

where following the notation of Sect. 4 one has

$$\Theta_{AB} \equiv 2\kappa_{(A}{}^Q\phi_{B)Q}, \qquad \mathfrak{Q}^2 \equiv \left(\kappa_{AB}\kappa^{AB}\right)^2\phi_{AB}\phi^{AB}.$$

The above integrals are well defined. More precisely, one has that:

Lemma 17. *Given the approximate Killing spinor κ_{AB} obtained from Theorem 5, one has that*

$$J, I_1, I_2, I_3 < \infty.$$

Proof. By construction, one has that the spinor κ_{AB} obtained from Theorem 5 satisfies $\mathcal{D}_{(AB}\kappa_{CD)} \in H^0_{-3/2}$. It follows then from the definition of the weighted Sobolev norm that

$$\parallel \nabla_{(AB}\kappa_{CD)}\parallel_{H^0_{-3/2}} = \parallel \nabla_{(AB}\kappa_{CD)}\parallel_{L^2} = J < \infty.$$

To verify the boundedness of I_1 one notices that by assumption $\Psi_{ABCD} \in H^\infty_{-3+\varepsilon}$, $\kappa_{AB} \in H^\infty_{1+\varepsilon}$ it follows by the multiplication properties of weighted Sobolev spaces (see, e.g. Lemma 14 in [4]) that

$$\Psi_{(ABC}{}^F\kappa_{D)F} \in H^\infty_{-3/2},$$

so that, in fact, $I_1 < \infty$.

We now look at the boundedness of I_2. By construction and due to the asymptotic conditions (60a)–(60d), one can choose asymptotically Cartesian coordinates and orthonormal frames on the asymptotic ends such that the approximate Killing spinor and Maxwell spinor satisfy

$$\kappa_{\boldsymbol{AB}} = \mp\frac{\sqrt{2}}{3}x_{\boldsymbol{AB}} + o_\infty\left(r^{1/2}\right)$$

$$\phi_{\boldsymbol{AB}} = \frac{q}{\sqrt{2}r^3}x_{\boldsymbol{AB}} + o_\infty\left(r^{-5/2}\right).$$

Therefore,

$$\Theta_{\boldsymbol{AB}} = \kappa_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\phi_{\boldsymbol{B})\boldsymbol{Q}}$$

$$= \mp\frac{q}{3r^3}x_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}x_{\boldsymbol{B})\boldsymbol{Q}} + o_\infty\left(r^{-3/2}\right)$$

$$= o_\infty\left(r^{-3/2}\right).$$

and so $\Theta_{AB} \in H^\infty_{-3/2}$, and $I_2 < \infty$.

Finally, to show the boundedness of I_3, note that in the asymptotically Cartesian coordinates and orthonormal frames used above, we have

$$\left(\kappa_{AB}\kappa^{AB}\right)^2 = \frac{4}{81}r^4 + o_\infty\left(r^{-7/2}\right),$$

$$\phi_{AB}\phi^{AB} = \frac{q^2}{2r^4} + o_\infty\left(r^{-9/2}\right),$$

and so the quantity $\mathfrak{Q}$ satisfies

$$\mathfrak{Q}^2 = \frac{2}{81}q^2 + o_\infty\left(r^{-1/2}\right).$$

Taking a derivative, one obtains

$$\mathcal{D}_{AB}\mathfrak{Q}^2 = o_\infty\left(r^{-3/2}\right),$$

and therefore $\mathcal{D}_{AB}\mathfrak{Q}^2 \in H^\infty_{-3/2}$ and $I_3 < \infty$. □

The integrals J, I_1, I_2 and I_3 are then used to define the following geometric invariant:

$$I = J + I_1 + I_2 + I_3. \tag{68}$$

One has the following result combining the whole analysis of this article:

Theorem 6. *Let $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ denote a smooth asymptotically Euclidean initial data set for the Einstein–Maxwell equations satisfying on each of its two asymptotic ends the decay conditions* (60a)–(60d) *with non-vanishing mass and electromagnetic charge. Let I be the invariant defined by equation* (68) *where κ_{AB} is the unique solution to equation* (59) *with asymptotic behaviour at each end given by* (63). *The invariant I vanishes if and only if $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ is locally an initial data set for a member of the Kerr–Newman family of spacetimes.*

Remark 28. Theorem 6 is the electrovacuum generalisation of the characterisation of initial data for the Kerr spacetime given in Theorem 28 in [4].

Proof. The proof follows the same strategy of Theorem 28 in [4]. It follows from our assumptions that if $I = 0$ then the electrovacuum Killing spinor data Eqs. (55a)–(55d) are satisfied on the whole of the hypersurface $\mathcal{S}$. Thus, from Theorem 4 the development of the electrovacuum initial data $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ will have, at least on a slab a Killing spinor.

Now, the idea is to make use of Theorem 1 to conclude that the development will be the Kerr–Newman spacetime. For this, one has to conclude that the spinor $\xi_{AA'} \equiv \nabla^Q{}_A\kappa_{BQ}$ is Hermitian so that it corresponds to the spinorial counterpart of a real Killing vector. By assumption, it follows from expansions (64a)–(64c) that

$$\xi - \hat{\xi} = o_\infty(r^{-1/2}), \qquad \xi_{AB} + \hat{\xi}_{AB} = o_\infty(r^{-1/2}).$$

Together, the last two expressions correspond to the Killing initial data for the imaginary part of $\xi_{AA'}$—thus, the imaginary part of $\xi_{AA'}$ goes to zero at infinity. It is well known that for electrovacuum spacetimes there exist no

non-trivial Killing vectors of this type [9,12]. Thus, $\xi_{AA'}$ is the spinorial counterpart of a real Killing vector. By construction, $\xi_{AA'}$ tends, asymptotically, to a time translation at infinity. Accordingly, the development of the electrovacuum initial data $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \boldsymbol{E}, \boldsymbol{B})$ contains two asymptotically stationary flat ends $\mathcal{M}_\infty$ and $\mathcal{M}'_\infty$ generated by the Killing spinor κ_{AB}. As the Komar mass and the electromagnetic charge of each end is, by assumption, nonzero, one concludes from Theorem 1 that the development $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is locally isometric to the Kerr–Newman spacetime.

Conversely, if the initial data set corresponds to initial data for a member of the Kerr–Newman family of solutions, then we know that there exists a Killing spinor κ_{AB} in the development $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$, such that $\kappa_{AB}|_\mathcal{S}$ has the asymptotic behaviour given in (65), and the vector $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ is a Killing vector in $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$. Furthermore, this restriction must satisfy the spatial Killing spinor equation on $\mathcal{S}$, and so also the approximate Killing spinor equation. Therefore, $\kappa_{AB}|_\mathcal{S}$ is the unique solution described in Theorem 5, which is used to construct the invariant I. Finally, Theorem 4 tells us that conditions (55a)–(55d) must be satisfied on $\mathcal{S}$, and so the invariant I must vanish. $\qquad\square$

8. Conclusions

As a natural extension to the vacuum case described by Bäckdahl and Valiente Kroon [4], the formalism presented above for the electrovacuum case has similar applications and possible modifications. For example, the use of asymptotically hyperboloidal rather than asymptotically flat slices can now be analysed for the full electrovacuum case, applying to the more general Kerr–Newman solution. Another interesting alternative to asymptotically flat slices would be to obtain necessary and sufficient conditions for the existence of a Killing spinor in the future development of a pair of intersecting null hypersurfaces. For instance, one could take a pair of event horizons intersecting at a bifurcation surface and obtain a system of conditions intrinsic to the horizon that ensures the black hole interior is isometric to the Kerr–Newman solution.

A motivation for the above analysis was also to provide a way of tracking the deviation of initial data from exact Kerr–Newman data in numerical simulations. However, in order to be a useful tool, one would still have to show that the geometric invariant is suitably behaved under time evolution (such as monotonicity). As highlighted in [4], a major problem is that it is hard to find a evolution equation for κ_{AB} such that the elliptic equations (59) are satisfied on each leaf in the foliation. If these issues can be resolved, then this formalism may be of some use in the study of nonlinear perturbations of the Kerr–Newman solution and the black hole stability problem.

Finally, the ethos of this article is to show that the characterisation of black hole spacetimes using Killing spinors is still a fruitful avenue of investigation. In future, we hope to show that this method can be used to investigate other open questions, such as the Penrose inequality and black hole uniqueness.

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