Some Problems in the Representation Theory of Hyperoctahedral Groups and Related Algebras

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Abstract

We begin by using a version of Green correspondence due to Grabmeier to count the number of components of two permutation modules $V^{\otimes r}$ and $Y^{\otimes r}$ for the hyperoctahedral group. We quantize these actions to make $V^{\otimes r}$ and $Y^{\otimes r}$ into modules for the type B Hecke algebra $\mathcal{H}(r)$ and then show that, as $\mathcal{H}(r)$-modules, $Y^{\otimes r}$ is isomorphic to a direct sum of permutation modules M^{λ} as given by Du and Scott. This enables us to use our earlier results to show that in the group case, over odd characteristic, the q-Schur^2 algebra and the hyperoctahedral Schur algebra are Morita equivalent, as these algebras are respectively the centralizing algebras of the actions of the hyperoctahedral group on $Y^{\otimes r}$ and $V^{\otimes r}$.

We then attempt to construct a bialgebra, the dual of whose r^{th} homogeneous part is isomorphic to the q-Schur^2 algebra. We show that this is not possible by the usual methods unless $q = 1$, and give a full description in the group case.

Results of earlier chapters lead us to introduce the notion of a balanced Mackey system for a finite group G, and exhibit balanced Mackey systems for wreath products of H and the symmetric group, where H is any finite group, and a new balanced Mackey system for the symmetric group itself. We then use this as a basis for counting the number of simple modules for the partition algebra, and also derive a formula for the dimensions of these simple modules.

In the final chapter we conjecture how some of our results may extend to complex reflection groups and Ariki-Koike algebras.

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Contents

Introduction

Dipper and James first introduced the q-Schur algebra in [**4**] , and it has been extensively studied since then. They defined for each composition $\lambda \in \Lambda(n,r)$ a cyclic submodule M^λ of $\mathrm{Hec}(r)$, the Hecke algebra of type A. The q-Schur algebra, $\mathcal{S}_q(n,r)$, is the centralizing algebra of the action of $\mathrm{Hec}(r)$ on the direct sum $\bigoplus_{\lambda\in\Lambda(n,r)} M^\lambda$. If we put $q = 1$ then $\mathrm{Hec}(r)$ becomes $k\,\mathrm{Sym}(r)$, the group algebra of the symmetric group, and the q-Schur algebra is the classical Schur algebra, $\mathcal{S}(n,r)$, which was first considered in [**31**]. Now the term Schur algebra was coined by Green in [**18**], where he defined $\mathcal{S}(n,r)$ as the dual of a certain coalgebra $A(n,r)$. In [**3**] Dipper and Donkin quantized $A(n,r)$, giving us the bialgebra $A_q(n,r)$. They then showed that if E denotes the natural module for GL_n then the centralizing algebra of a certain action of $\mathrm{Hec}(r)$ on $E^{\otimes r}$ is isomorphic to the dual of $A_q(n,r)$, generalizing [**18**, 2.6]. Since they also show that $E^{\otimes r}$ is isomorphic to $\bigoplus_{\lambda\in\Lambda(n,r)} M^\lambda$, this gives us two ways of constructing the q-Schur algebra.

Central to this thesis is the idea of a p-Mackey system of a finite group G. A Mackey system is traditionally a set of subgroups, $\mathcal{M}$, of a given finite group, which is closed under both conjugation and intersection. (See [**13**, exercise 2.2] for example.) Let p be prime. If also the identity subgroup is a member of $\mathcal{M}$, and every Sylow p-subgroup of G is contained in a member of $\mathcal{M}$, we call $\mathcal{M}$ a p-Mackey system of G. In his doctoral thesis [**14**], Grabmeier proves a version of Green correspondence where all the vertices come from a given p-Mackey system. Using this he derives a method of counting the number of isomorphism types of indecomposable trivial source modules with vertices in a given

p-Mackey system, working over a field of characteristic p. He applies this count to the p-Mackey system of Young subgroups of the symmetric group, which enables him to count the number of components of the permutation module $E^{\otimes r}$ above and hence to derive results about the representation theory of the Schur algebras.

The main part of this thesis involves generalizations of all of the above to the type B situation, where we use $\mathcal{H}(r)$ to denote the type B Hecke algebra. Now many Schur algebras associated to $\mathcal{H}(r)$ have been introduced, and in this thesis we consider the following examples. In [**19**], Richard Green introduces the hyperoctahedral q-Schur algebra, $\mathcal{S}_q^{\mathrm{Hyp}}(n,r)$, as the centralizing algebra of the action of $\mathcal{H}(r)$ on a certain tensor power $V^{\otimes r}$. Independently of each other, Du and Scott [**11**] and Dipper, James and Mathas [**5**] introduced two Morita equivalent Schur algebras, called the q-Schur^2 algebra (pronounced q-Schur-two) and the (Q,q)-Schur algebra respectively. They are both centralizing algebras of the action of $\mathcal{H}(r)$ on direct sums of $\mathcal{H}(r)$-modules M^λ, which are defined for each pair of compositions $\lambda = (\mu;\nu)$, the direct sum in the former being over all such pairs of compositions, and in the later over a certain subset of these. In chapters 2, 3 and 4 we derive results to help us compare these algebras in certain cases.

We begin chapter 2 by introducing two systems of subgroups of the hyperoctahedral group, $\mathrm{Hyp}(r)$. These are $\mathcal{V}$, the set of Young subgroups of the hyperoctahedral group, and $\mathcal{Y}$, which we call the set of infant subgroups. Each Young subgroup is also an infant subgroup. We also introduce a $k\,\mathrm{Hyp}(r)$-module Y and its r^{th} tensor power $Y^{\otimes r}$. We show that the set of point stabilizers of the action of $\mathrm{Hyp}(r)$ on $Y^{\otimes r}$ is precisely the set of infant subgroups $\mathcal{Y}$, and analogously the set of point stabilizers of the action of $\mathrm{Hyp}(r)$ on $V^{\otimes r}$ is precisely the set of

Young subgroups $\mathcal{V}$. We prove then that for all primes p both $\mathcal{Y}$ and $\mathcal{V}$ are p-Mackey systems of $\mathrm{Hyp}(r)$, except in one particular case ($\mathcal{V}$ when $p = 2$) which we adapt to suit our methods. We apply Grabmeier's count to count the number of isomorphism types of indecomposable trivial source modules over both $\mathcal{Y}$ and $\mathcal{V}$, which we denote by $\mathrm{TSM}(\mathcal{Y})$ and $\mathrm{TSM}(\mathcal{V})$ respectively. We show when p is odd that $\mathrm{TSM}(\mathcal{Y}) = \mathrm{TSM}(\mathcal{V})$, but that this is not true when $p = 2$.

In chapter 3 we begin by quantizing our $\mathrm{Hyp}(r)$-action on $Y^{\otimes r}$ to make $Y^{\otimes r}$ into a module for the type B Hecke algebra $\mathcal{H}(r)$. Now $Y^{\otimes r}$ has a basis labelled by the set $I_B(n,r)$ and we define the content of an element of $I_B(n,r)$. This gives rise to the subspaces $Y_\lambda^{\otimes r}$, and we show that $Y^{\otimes r}$ decomposes as a certain direct sum of these subspaces. Then, by considering annihilators of certain elements, we show for each pair of compositions λ that both $Y_\lambda^{\otimes r}$ and M^λ are isomorphic to the same $\mathcal{H}(r)$-module, and hence are themselves isomorphic. This gives us that Du and Scott's direct sum $\bigoplus_\lambda M^\lambda$ and our $Y^{\otimes r}$ are isomorphic as $\mathcal{H}(r)$-modules, and so the q-Schur^2 algebra $\mathcal{S}_q^2(n,r)$ is isomorphic to $\mathrm{End}_{\mathcal{H}(r)}(Y^{\otimes r})$.

Chapter 4 mainly collects results implied by the previous two chapters. We first look at how Grabmeier's methods enable us to count the number of components of a permutation module. Then using this and some Morita theory we can show that in the group case (at $q = Q = 1$) the Schur^2 algebra and the hyperoctahedral Schur algebra are Morita equivalent over a field of odd characteristic, but that this is not true over characteristic 2. We then also apply Fitting's theorem to count the number of irreducible representations of each of the type B Schur algebras we consider, again when $q = Q = 1$.

In chapter 5 we mimic [**3**] to build ourselves a coalgebra from $Y^{\otimes r}$. We start with the free bialgebra, $F_B(3n)$, for which $Y^{\otimes r}$ is naturally a comodule and derive the relations needed in it to ensure that multiplication by any element of $\mathcal{H}(r)$ is a comodule map. We show that the ideal I generated by these relations is a biideal of $F_B(3n)$ so that $F(3n)_B/I$ is also a bialgebra, which we denote by $C_{q,Q}(3n)$. The fact that multiplication by any element of $\mathcal{H}(r)$ is a comodule map induces a homomorphism between $C_{q,Q}(3n,r)^*$, the dual of the r^{th} homogeneous part of $C_{q,Q}(3n)$, and the q-Schur^2 algebra $\mathcal{S}^2_q(n,r)$. When $q = Q = 1$ we show this is an isomorphism, so that we have a coalgebra construction of the Schur^2 algebra, analogously to [**18**, 2.6] and [**3**, 3.1.5]. Hence we can write down the dimension of $\mathcal{S}^2_q(n,r)$ for all values of q, Q, n and r, and can also exhibit a spanning set of $C_{q,Q}(3n,r)$ of this desired dimension. We show that when $q \neq 1$ this spanning set is linearly dependent, so that the dimension of $C_{q,Q}(3n,r)^*$ is strictly less that that of $\mathcal{S}^2_q(n,r)$ so that the two cannot be isomorphic in this case, and hence in particular we cannot follow the above methods to associate a quantum group to this set-up when $q \neq 1$.

In chapter 6 we return to Mackey systems and define the idea of a balanced p-Mackey system. This is an analogue of what happens in the characteristic zero case (or when $p \nmid |G|$). We then consider the so called complete monomial groups $G \wr \mathrm{Sym}(r)$, where G is any finite group, and look at how to view these as permutation groups. We show that the Young subgroups inside these groups form a p-Mackey system, providing that $p \nmid |G|$, and use Grabmeier's methods to show that this Mackey system is balanced.

In chapter 7 we continue to look at Mackey systems. We define a system of subgroups of $\mathrm{Sym}(n)$ which we call $\mathcal{P}$, and we show that

$\mathcal{P}$ is a p-Mackey system for all primes p. We use Grabmeier's count in the usual way, and show that $\mathcal{P}$ is balanced if and only if $n > p/2$. We then use this to count the number of components of a certain $k\,\mathrm{Sym}(n)$-module when char $k = p \nmid n$, and $n \leqslant r+1$. As the partition algebra $P_r(n)$ (see [**26**] for example) is isomorphic to the centralizing algebra of this module when $n \geqslant 2r$ this suggests a method by which we may derive results about the representation theory of $P_r(n)$. We end the chapter by using a lemma of Donkin [**7**, remark after 3.6] and Fitting's theorem to count the number of irreducible modules $U(\lambda)$ for the partition algebra when $n \geqslant 2r$ and $p \nmid n$, and also to calculate in these cases the dimension of each $U(\lambda)$ in terms of dimensions of weight spaces of simple GL_n-modules.

In the final chapter we define a k-space X_m, and show that its r^{th} tensor power $X_m^{\otimes r}$ is a module for the group algebra of the complex reflection group $\mathrm{G}(m,1,r)$. We conjecture that this is isomorphic to a certain direct sum of modules as in [**10**]. We also make a conjecture about a p-Mackey of the complex reflection groups $\mathrm{G}(m,a,r)$.

CHAPTER 1

Preliminaries

In this first chapter we introduce the structures we will be using throughout the thesis, and also review some of the results relating to them which will be useful later on.

1.1. Groups, Hecke algebras and representations

Let Ω be a set. Then the symmetric group on Ω, denoted $\mathrm{Sym}(\Omega)$, is the set of all permutations of Ω, and $\mathrm{Sym}(\Omega)$ is a group with the operation of composition. If $\Omega = \{1, 2, \ldots, r\}$, which we denote by $\underline{r}$, then we write $\mathrm{Sym}(r)$ for the symmetric group on Ω. For $1 \leqslant i \leqslant r-1$ let $s_i = (i \quad i+1)$, the element of $\mathrm{Sym}(r)$ which swaps i and $i+1$. Then $\mathrm{Sym}(r)$ is generated by the set $\{s_i \,|\, 1 \leqslant i \leqslant r-1\}$, with relations

$$s_i^2 = 1$$
$$s_i s_j = s_j s_i, \text{ for } |i-j| > 1$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

where $1 \leqslant i, j \leqslant r-1$, giving $\mathrm{Sym}(r)$ a Coxeter type presentation. ($\mathrm{Sym}(r)$ is the Coxeter group of type A.) Given any finite set Ω we could define a similar set which generates $\mathrm{Sym}(\Omega)$.

We now introduce the star of the show. Let X be a set with a regular involution $\bar{\ }$, called bar, so that X necessarily has $2r$ elements for some non negative integer r. Then the hyperoctahedral group, $\mathrm{Hyp}(X)$, is the subgroup of $\mathrm{Sym}(X)$ given by

$$\mathrm{Hyp}(X) = \{\sigma \in \mathrm{Sym}(X) \,|\, \sigma(\bar{i}) = \overline{\sigma(i)}\, \forall i \in X\}.$$

Of course, we have in mind that $\bar{\bar{i}} = i$.

If $X = \{1, 2, \ldots, r, \bar{1}, \bar{2}, \ldots, \bar{r}\}$ then we write $\mathrm{Hyp}(r)$ for the hyperoctahedral group on X. It is well known that $\mathrm{Hyp}(r)$ is isomorphic to the wreath product $\frac{\mathbb{Z}}{2\mathbb{Z}} \wr \mathrm{Sym}(r)$, and also to the automorphism group of an r-dimensional cube.

We can also present $\mathrm{Hyp}(r)$ as a Coxeter group. This time we define element $s_i \in \mathrm{Hyp}(r)$ via

$$s_i = \begin{cases} (1 \quad \bar{1}) & \text{if } i = 0 \\ (i \quad i+1)(\bar{i} \quad \overline{i+1}) & \text{if } 1 \leqslant i \leqslant r-1 \end{cases}$$

Then $\mathrm{Hyp}(r)$ has generators $s_0, s_1, \ldots, s_{r-1}$ and relations

$$s_i^2 = 1$$

$$s_i s_j = s_j s_i, \text{ for } |i - j| > 1$$, and if $j = 0$ and $1 < i < r$,

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$$

where $1 \leqslant i, j \leqslant r - 1$. $\mathrm{Hyp}(r)$ a Coxeter group of Type B. The concepts of a reduced expression for an element of a Coxeter group and for the Coxeter length of an element are defined in the usual way. For an element w of either $\mathrm{Sym}(r)$ or $\mathrm{Hyp}(r)$, we let $l(w)$ denote the Coxeter length of w. If $w \in \mathrm{Hyp}(r)$, then the number of times s_0 occurs in a reduced expression for w is constant, and we denote this by $n_0(w)$.

Note that if, for $i \in \underline{r}$, we put $t_i = s_{i-1}s_{i-2}\ldots s_0 s_1 \ldots s_{i-1} = (i \quad \bar{i})$, then $t_1, t_2, \ldots, t_r$ generate a subgroup of $\mathrm{Hyp}(r)$ isomorphic to $(\frac{\mathbb{Z}}{2\mathbb{Z}})^r$. Also the elements $s_1, s_2, \ldots, s_{r-1}$ generate a subgroup isomorphic to $\mathrm{Sym}(r)$. We now present another way of looking at that.

Again let X be a set with a regular involution $^-$. For any subset $X' = \{x_1, \ldots, x_m\}$ of X we write $\overline{X'}$ for the set $\{\bar{x}_1, \ldots, \bar{x}_m\}$. Now let

Z be a subset of X such that $Z \sqcup \overline{Z} = X$ i.e. with $Z \cup \overline{Z} = X$ and $Z \cap \overline{Z} = \emptyset$. Then we call Z a schism of X, and we can identify $\mathrm{Sym}(Z)$ with a subgroup of $\mathrm{Hyp}(X)$ via the following map. Let

$$\phi : \mathrm{Sym}(Z) \to \mathrm{Sym}(X) \text{ via } \phi(\sigma)(z) = \sigma z \text{ and } \phi(\sigma)(\overline{z}) = \overline{\sigma z},$$

for $\sigma \in \mathrm{Sym}(Z)$ and $z \in Z$. Now, for each $\sigma \in \mathrm{Sym}(Z)$ and $z \in Z$, we have that $\overline{\phi(\sigma)(z)} = \overline{\sigma z} = \phi(\sigma)(\overline{z})$ so that each $\phi(\sigma)$ is in fact a member of $\mathrm{Hyp}(X) \subset \mathrm{Sym}(X)$. We denote the image of $\mathrm{Sym}(Z)$ under ϕ by $S(Z)$, so that $S(Z)$ is a subgroup of $\mathrm{Hyp}(X)$ which is isomorphic to $\mathrm{Sym}(Z)$. Then $S(Z)$ and $\mathrm{Sym}(X)$ are reserved with this meaning, so that $\mathrm{Sym}(X)$ always denotes the full symmetric group on X, and $S(Z)$ is always the image of $\mathrm{Sym}(Z)$ inside $\mathrm{Hyp}(X)$. (Also, in chapter 5, we will use $S(Z)$ to denote the image of $\mathrm{Sym}(Z)$ inside a larger wreath product $H \wr \mathrm{Sym}(r)$, where H is any finite group.) Note that of course $S(Z)$ and $S(\overline{Z})$ are the same subgroup of $\mathrm{Hyp}(X)$.

Let $r, n \geqslant 1$, and unless otherwise stated we assume that $n \geqslant r$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a sequence of non negative integers λ_i, not all equal to zero. If $\lambda_1 + \cdots + \lambda_n = r$ then we call λ a composition of r, into at most n parts, and denote the set of these by $\Lambda(n, r)$. If in addition to this we have that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ then we call λ a partition of r, and denote the set of these by $\Lambda^+(n, r)$. If λ is a composition in $\Lambda(n, r)$ then we denote by $\overline{\lambda}$ the unique partition created from λ by reordering the parts in descending order. Note that a partition λ may end in a sequence of zeros, and we usually omit these. We also often use superscripts to denote a sequence of more than one λ_i. In this case the commas may be omitted, so that $(4, 4, 3, 3, 2, 0, \ldots, 0) = (4, 4, 3, 3, 2) = 4^2 3^2 2$.

We can represent a partition $\lambda \in \Lambda^+(n, r)$ by its Young diagram of crosses in the plane. For example, the Young diagram d^λ of $\lambda =$

$(6,4,4,2)$ is given by

$$d^\lambda \quad = \quad \begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & & \\ \times & \times & \times & \times & & \\ \times & \times & & & & \end{array} \quad .$$

We can flip a diagram over its main diagonal, and the corresponding partition is called the partition conjugate to λ, and is denoted by λ'. For example, flipping the Young diagram d^λ in the previous example gives

$$d^{\lambda'} \quad = \quad \begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \\ \times & \times & \times & \\ \times & & & \\ \times & & & \end{array}$$

so that $\lambda' = (4,4,3,3,1,1)$, is the partition conjugate to $(6,4,4,2)$.

Now let p be a prime. If $\lambda \in \Lambda^+(n,r)$ is such that $\lambda_i - \lambda_{i+1} < p$ for all $1 \leqslant i \leqslant n-1$ and $\lambda_n < p$ then we say that λ is a column p-regular partition of r, and we denote the set of these by $\Lambda^+(n,r)_{col}$. If for all $0 \leqslant i \leqslant n-p$ a partition $\lambda \in \Lambda^+(n,r)$ contains no sequence $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+p} > 0$ then we call λ a row p-regular partition, and denote the set of these by $\Lambda^+(n,r)_{row}$. Note that if $\lambda \in \Lambda^+(n,r)_{col}$ then $\lambda' \in \Lambda^+(n,r)_{row}$, and vice versa. (It may be that $\lambda = \lambda'$.) From this it is easy to see that $|\Lambda^+(n,r)_{col}| = |\Lambda^+(n,r)_{row}|$. It is also very useful for us to know that if p is prime then $\lambda \in \Lambda^+(n,r)$ can be uniquely written as

$$\lambda = \textstyle\sum_{i \geqslant 0} p^i \lambda(i),$$

with each $\lambda(i)$ a column p-regular partition. We call this the unique p-adic expansion of λ.

We associate to each partition a subgroup of the symmetric group. Let $\lambda \in \Lambda^+(n,r)$, with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then the standard Young subgroup of $\mathrm{Sym}(r)$ associated to λ is given by

$$\mathrm{Sym}(\lambda) = \mathrm{Sym}\{1,2,\ldots,\lambda_1\} \times \mathrm{Sym}\{\lambda_1+1, \lambda_1+2, \ldots, \lambda_1+\lambda_2\} \times \cdots$$
$$\cdots \times \mathrm{Sym}\{r-\lambda_n, \ldots, r\}.$$

We can similarly associate a group $\mathrm{Sym}(\lambda)$ to any composition of r. A Young subgroup of $\mathrm{Sym}(r)$ is any subgroup $\mathrm{Sym}(r)$-conjugate to a standard Young subgroup.

It is not only single partitions and compositions that will be useful in this thesis. The set of pairs of compositions (or bicompositions) of r, denoted by $\Lambda_2(n,r)$, is given by

$$\Lambda_2(n,r) = \{\lambda = (\mu;\nu) \mid \mu \in \Lambda(n,a),\ \nu \in \Lambda(n, r-a), 0 \leqslant a \leqslant r\}.$$

Note that each partition in the pair has at most n parts. If μ and ν are both partitions then we call $\lambda = (\mu;\nu)$ a pair of partitions (or a bipartition) of r, and denote the set of these by $\Lambda_2^+(n,r)$.

We can also extend these ideas to construct the set of m-compositions of r, denoted $\Lambda_m^+(n,r)$, which consists of m-tuples $\lambda = (\lambda(1);\ldots;\lambda(m))$ such that $\lambda(i) \in \Lambda(n, r_i)$, $0 \leqslant r_i \leqslant r$ and $\sum_{i \in \underline{m}} r_i = r$. Note again that each $\lambda(i)$ has at most n parts, and if each $\lambda(i)$ is a partition the we call $\lambda = (\lambda(1);\ldots;\lambda(m))$ an m-partition, the set of all such being denoted by $\Lambda_m^+(n,r)$.

We also consider Young subgroups inside the hyperoctahedral group, and other wreath products of the form $H \wr \mathrm{Sym}(r)$, for H a finite group. If $\mu \in \Lambda^+(n,r)$ then we write $S(\mu)$ for the image of the standard Young subgroup $\mathrm{Sym}(\mu)$ under the map ϕ as given earlier in this chapter. Now, if $\lambda = (\mu;\nu) \in \Lambda_2^+(n,r)$, and g is the transposition in $\mathrm{Hyp}(r)$ given by $g = (1, a+1)(\overline{1}, \overline{a+1})(2, a+2)(\overline{2}, \overline{a+2}) \cdots (r-a, r)(\overline{r-a}, \overline{r})$, where $a = |\mu|$, then we write $S(\lambda)$ for $S(\mu) \times S(\nu)^g \leqslant \mathrm{Hyp}(r)$. These are then the standard Young subgroups of $\mathrm{Hyp}(r)$ and in general a Young subgroup of $\mathrm{Hyp}(r)$ is one which is $\mathrm{Hyp}(r)$-conjugate to a standard Young subgroup. Of course in this way we have a Young subgroup of $\mathrm{Hyp}(r)$ associated to each composition of r. Note also that given $\lambda \in \Lambda_m^+(n,r)$ we can make a similar definition for Young subgroups of the wreath product $H \wr \mathrm{Sym}(r)$.

Let k be a field, which for the rest of the thesis we will assume is algebraically closed, and since all our k-algebras are finite dimensional k will in particular be a splitting field for these algebras. We now turn to irreducible kG-modules, or representations, in the cases $G = \mathrm{Sym}(r)$ and $G = \mathrm{Hyp}(r)$. In [**22**], James constructs for each $\lambda \in \Lambda^+(n,r)$ a $k\,\mathrm{Sym}(r)$-module S^λ, called a Specht module, and in turn a certain factor module denoted by D^λ. When $k = \mathbb{Q}$, the field of rational numbers, it is part of the folklore of representation theory that the set $\{S^\lambda \mid \lambda \in \Lambda^+(n,r)\}$ is a full set of irreducible $\mathbb{Q}\mathrm{Sym}(r)$-modules, see for example [**22**]. James also shows that over a field k of characteristic p, the set $\{D^\lambda \mid \lambda \in \Lambda^+(n,r)_{row}\}$ is a full set of $k\,\mathrm{Sym}(r)$-modules. This means that $\mathrm{Sym}(r)$ has $|\Lambda^+(n,r)|$ isomorphism types of ordinary irreducible representations, and as $|\Lambda^+(n,r)_{row}| = |\Lambda^+(n,r)_{col}|$, we also have that $\mathrm{Sym}(r)$ has $|\Lambda^+(n,r)_{col}|$ isomorphism types of p-modular representations.

As we have already seen, the hyperoctahedral group $\mathrm{Hyp}(r)$ is the wreath product $\frac{\mathbf{Z}}{2\mathbf{Z}} \wr \mathrm{Sym}(r)$, and therefore moreover it is a semi-direct product $(\frac{\mathbf{Z}}{2\mathbf{Z}})^r \rtimes \mathrm{Sym}(r)$. Then since $(\frac{\mathbf{Z}}{2\mathbf{Z}})^r$ is an abelian group, we could use the method of the "little groups of Wigner", as detailed in [**32**, 8.2] to construct and label the irreducibles for the hyperoctahedral group over the field $\mathbb{C}$. However, in [**1**], Morris et al. construct irreducible $k\,\mathrm{Hyp}(r)$-modules over fields of any characteristic. They define modules Y^λ and J^λ for each pair of partitions $\lambda = (\mu; \nu) \in \Lambda_2^+(n, r)$. Then over a field k of characteristic 0 the set $\{Y^\lambda \mid \lambda \in \Lambda_2^+(n, r)\}$ is a full set of $k\,\mathrm{Hyp}(r)$-modules, so that $\mathrm{Hyp}(r)$ has $|\Lambda_2^+(n, r)|$ isomorphism types of ordinary irreducible representations.

Before we look at the modules J^λ and the p-modular case, we need to define p-regular pairs of partitions. If $p \neq 2$ then $\lambda = (\mu; \nu)$ is row (resp. column) p-regular if both μ and ν are row (resp. column) p-regular. If $p = 2$ then λ is row (resp. column) p-regular if $\mu = \emptyset$ and ν is row (resp. column) p-regular. This gives us the sets $\Lambda_2^+(n, r)_{row}$ and $\Lambda_2^+(n, r)_{col}$, for any prime p. Then over a field k of prime characteristic the set $\{J^\lambda \mid \lambda \in \Lambda_2^+(n, r)_{row}\}$ is a full set of $k\,\mathrm{Hyp}(r)$-modules, so that $\mathrm{Hyp}(r)$ has $|\Lambda_2^+(n, r)_{row}| = |\Lambda_2^+(n, r)_{col}|$ isomorphism types of p-modular irreducible representations. Note that when $p = 2$ we have that $|\Lambda_2^+(n, r)_{col}| = |\Lambda^+(n, r)_{col}|$, so we have that $\mathrm{Hyp}(r)$ has $|\Lambda^+(n, r)_{col}|$ isomorphism types of 2-modular irreducible representations.

We will also consider Hecke algebras. These can be regarded as q-deformations of the group algebras $k\,\mathrm{Sym}(r)$ and $k\,\mathrm{Hyp}(r)$ of the symmetric and hyperoctahedral groups. We now make this more precise. If $0 \neq q, Q \in k$, then the Hecke algebra of type A, which we denote by $\mathrm{Hec}(r)$, is the free k-module with generators $T_{s_1}, T_{s_2}, \ldots, T_{s_{r-1}}$ and relations

$$(T_{s_i} - q)(T_{s_i} + 1) = 0$$
$$T_{s_i} T_{s_j} = T_{s_j} T_{s_i}, \text{ for } |i - j| > 1$$

$$T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}}$$

where $1 \leqslant i,j \leqslant r-1$. Note that by putting $q = 1$ we retrieve $k\,\mathrm{Sym}(r)$. In a similar way the Hecke algebra of type B, denoted $\mathcal{H}(r)$ or just $\mathcal{H}$, is the free k-module with generators $T_{s_0}, T_{s_1}, \ldots, T_{s_{r-1}}$ and relations

$$(T_{s_i} - q)(T_{s_i} + 1) = 0, \text{ for } 1 \leqslant i \leqslant r-1$$
$$(T_{s_0} - Q)(T_{s_0} + 1) = 0$$
$$T_{s_i}T_{s_j} = T_{s_j}T_{s_i}, \text{ for } |i-j| > 1$$, and if $j=0$ and $1<i<r$,
$$T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}}$$
$$T_{s_0}T_{s_1}T_{s_0}T_{s_1} = T_{s_1}T_{s_0}T_{s_1}T_{s_0}$$

where again $1 \leqslant i,j \leqslant r-1$, and this time if we put $q = Q = 1$ we retrieve the group algebra $k\,\mathrm{Hyp}(r)$. Note that the subalgebra of $\mathcal{H}(r)$ generated by $T_{s_1}, T_{s_2}, \ldots, T_{s_{r-1}}$ is isomorphic to $\mathrm{Hec}(r)$.

Now if $w = s_{a(1)} \cdots s_{a(m)}$ is a reduced expression for $w \in \mathrm{Hyp}(r)$ (resp. $w \in \mathrm{Sym}(r)$) then we write T_w for $T_{s_{a(1)}} \cdots T_{s_{a(m)}}$. Then as k-modules $\mathrm{Hec}(r)$ has k-basis $\{T_w \mid w \in \mathrm{Sym}(r)\}$ and $\mathcal{H}(r)$ has k-basis $\{T_w \mid w \in \mathrm{Hyp}(r)\}$.

The representations of $\mathrm{Hec}(r)$ and $\mathcal{H}(r)$ are labelled by partitions and pairs of partitions of r in an analogous way to the earlier group representations, but we do not consider them in this thesis so a description of them is not given here.

1.2. Endomorphism algebras

As stated in the introduction one of our aims is to compare the various Schur algebras in type B. We also intend to look at the partition algebras, and in the cases we consider these are isomorphic to certain endomorphism algebras. Therefore we now introduce all these structures and look at some useful theorems we will use to prove other theorems about them.

We begin with Schur algebras. The name Schur algebras was originally used by Green in his monograph [18]. He defined the Schur algebra, $\mathcal{S}(n,r)$, as the dual of a certain coalgebra $A(n,r)$, and we will consider this approach later on in both this chapter and the thesis itself. Green shows [**18**, 2.6] that $\mathcal{S}(n,r)$ is isomorphic to the centralizing algebra of a certain permutation module for the symmetric group. Here we look at that approach and see how it leads to the definition of other Schur algebras, these being related to the hyperoctahedral groups, and Hecke algebras of both types A and B. (Of course, there are others, but we only introduce the ones which play a role in this thesis.) Here we go.

Let E be an n-dimensional vector space over a field k with basis $\{e_1, e_2, \ldots, e_n\}$, and let $E^{\otimes r}$ denote its r^{th} tensor power $E \otimes E \otimes \cdots \otimes E$ (r times). If $I(n,r)$ denotes the set of r-tuples with entries from $\underline{n}$, and if $\mathbf{i} = (i_1, i_2, \ldots, i_r) \in I(n,r)$ then every basis element in $E^{\otimes r}$ can be written uniquely as

$$e_{\mathbf{i}} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}.$$

The symmetric group $\mathrm{Sym}(r)$ acts on the right of $I(n,r)$ via place permutation i.e. if $\mathbf{i} \in I(n,r)$ and $\pi \in \mathrm{Sym}(r)$ then

$$\mathbf{i}.\pi = (i_1, \ldots, i_r).\pi = (i_{(1)\pi}, \ldots, i_{(r)\pi}).$$

We can now transport this action to $E^{\otimes r}$ so that if $e_{\mathbf{i}}$ is a basis element in $E^{\otimes r}$, and $\pi \in \mathrm{Sym}(r)$ then

$$e_{\mathbf{i}}.\pi = (e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}).\pi = e_{i_{(1)\pi}} \otimes e_{i_{(2)\pi}} \otimes \cdots \otimes e_{i_{(r)\pi}} = e_{\mathbf{i}.\pi}$$

Extending this action linearly makes $E^{\otimes r}$ into a $k\,\mathrm{Sym}(r)$-module. Then the centralizing algebra of this action is the Schur algebra $\mathcal{S}(n,r)$ as above i.e. we have that $\mathcal{S}(n,r) = \mathrm{End}_{\mathrm{Sym}(r)}(E^{\otimes r})$. Note that the point stabilizers of basis elements of $E^{\otimes r}$ under this action are the Young subgroups, $\mathrm{Sym}(\lambda)$.

We can now quantize the above action, as in [**3**, 3.1.4] to give an action of the type A Hecke algebra, Hec(r), on $E^{\otimes r}$. Again we describe the action by specifying how each generator T_{s_j} of Hec(r) acts on $E^{\otimes r}$.

Let $q \in k$, and let $\mathbf{i} \in I(n,r)$, so that $e_{\mathbf{i}}$ is a basis element of $E^{\otimes r}$. Then T_{s_j} acts on $e_{\mathbf{i}}$ as follows:

$$e_{\mathbf{i}}T_{s_j} = \begin{cases} qe_{\mathbf{i}s_j} & \text{if } i_j \leqslant i_{j+1} \\ e_{\mathbf{i}s_j} + (q-1)e_{\mathbf{i}} & \text{if } i_j > i_{j+1} \end{cases}$$

Then $\mathrm{End}_{\mathrm{Hec}(r)}(E^{\otimes r})$ is called the q-Schur algebra, and is denoted by $\mathcal{S}_q(n,r)$. Note again that putting $q = 1$ we are back in the group case. The q-Schur algebras were introduced by Dipper and James in [**4**]. They defined for each composition $\lambda \in \Lambda(n,r)$ a certain Hec(r)-"permutation" module M^λ, and defined the q-Schur algebra to be $\mathrm{End}_{\mathrm{Hec}(r)}(\bigoplus_{\lambda\in\Lambda(n,r)} M^\lambda)$. Since $E^{\otimes r} \cong \bigoplus_{\lambda\in\Lambda(n,r)} M^\lambda$, by [**3**, 3.1.5] we can use either incarnation of this "permutation" module.

We now look at Schur algebras in type B. In [**19**], Richard Green takes the tensor power approach to constructing permutation modules. Let $L(n)$ denote the set $\{1,2,\ldots,n,\overline{n},\ldots,\overline{1}\}$ with elements in $L(n)$ being ordered via $1 < \cdots < n < \overline{n} < \cdots < \overline{1}$. Let V denote the $2n$-dimensional vector space over k with basis $\{v_i \mid i \in L(n)\}$, and let $V^{\otimes r}$ denote the r^{th} tensor power of V. Then let $I_V = I_V(2n,r)$ denote the set of r-tuples $\mathbf{i} = (i_1, i_2, .., i_r)$ with entries from $\{1,\ldots,n,\overline{n},\ldots,\overline{1}\}$. Similarly to above, $V^{\otimes r}$ has a k-basis consisting of elements $v_{\mathbf{i}}$, where $\mathbf{i} \in I_V$. Now Hyp(r) acts on the right of I_V via

$$\mathbf{i}.s_j = (i_1,..,i_r).s_j = \begin{cases} (i_1,\ldots,i_{j-1},i_{j+1},i_j,i_{j+2},\ldots,i_r) & \text{if } j \neq 0 \\ (\overline{i_1}, i_2, \ldots, i_r) & \text{if } j = 0 \end{cases}$$

and thus $\mathrm{Hyp}(r)$ acts on the right of $V^{\otimes r}$ via $v_{\mathbf{i}}.\pi = v_{\mathbf{i}.\pi}$ for every $\pi \in \mathrm{Hyp}(r)$ and $\mathbf{i} \in I_V$. Again we extend this action linearly to make $V^{\otimes r}$ into a right $k\,\mathrm{Hyp}(r)$-module. The hyperoctahedral Schur algebra, which we denote by $\mathcal{S}^{\mathrm{Hyp}}(n,r)$, is then the centralizing algebra of this action i.e. we have $\mathcal{S}^{\mathrm{Hyp}}(n,r) = \mathrm{End}_{k\,\mathrm{Hyp}(r)}(V^{\otimes r})$.

Again this action can be quantized. Let $q, Q \in k$, and let $\mathbf{i} \in I_V(2n,r)$ so that $v_{\mathbf{i}}$ is a basis element of $V^{\otimes r}$. Then the generators $T_{s_0}, \ldots, T_{s_{r-1}}$ act on $V^{\otimes r}$ as follows, by [**19**, 3.1.2].

$$v_i T_{s_j} = \begin{cases} q v_{\mathbf{i}s_j} & \text{if } i_j \leqslant i_{j+1} \\ v_{\mathbf{i}s_j} + (q-1)v_{\mathbf{i}} & \text{if } i_j > i_{j+1} \\ Q v_{\mathbf{i}s_j} & \text{if } j = 0 \text{ and } i_1 \in \{1, \ldots, n\} \\ v_{\mathbf{i}s_j} + (Q-1)v_{\mathbf{i}} & \text{if } j = 0 \text{ and } i_1 \in \{\overline{n}, \ldots, \overline{1}\} \end{cases}$$

Then the hyperoctahedral q-Schur algebra, denoted by $\mathcal{S}_q^{\mathrm{Hyp}}(n,r)$, is defined by $\mathcal{S}_q^{\mathrm{Hyp}}(n,r) = \mathrm{End}_{\mathcal{H}}(V^{\otimes r})$. It is no surprise to find that by putting $q = Q = 1$ we are back in the group case.

We now need something to compare this to. In [**11**] and [**5**] two type B Schur algebras were unveiled, Du and Scott's being the q-Schur^2 algebra, and Dipper et al. introducing the (Q,q)-Schur algebra. We will concentrate mainly on the former.

Let $\lambda = (\mu;\nu) \in \Lambda_2(n,r)$. We now define some elements of $\mathcal{H}$ related to λ. Let $|\mu| = a$. Then we let

$$\pi_\lambda = \textstyle\prod_{i=1}^{a}(q^{i-1} + T_{t_i}).$$

Also, let $x_\lambda = \sum_{w \in S(\lambda)} T_w$. Then we put $m_\lambda = x_\lambda \pi_\lambda$ and define M^λ to be the cyclic right $\mathcal{H}(r)$-module given by $M^\lambda = m_\lambda \mathcal{H}(r)$. Du and Scott then define the q-Schur^2-algebra, denoted $\mathcal{S}_q^2(n,r)$, to be

$$\mathcal{S}_q^2(n,r) = \mathrm{End}_{\mathcal{H}}(\textstyle\bigoplus_{\lambda \in \Lambda_2(n,r)} M^\lambda).$$

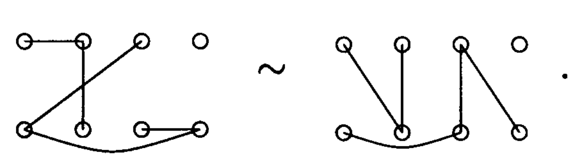
If $q = Q = 1$, so that we are back in the group case, we write $\mathcal{S}^2(n,r)$ for the Schur[2] algebra.

Dipper, James and Mathas's definition of the (Q,q)-Schur algebra is similar. Let $\Lambda_2^{\leqslant}(n,r)$ be the subset of $\Lambda_2(n,r)$ consisting of pairs of compositions $\lambda = (\mu;\nu)$ where between them μ and ν have at most n parts. Then the (Q,q)-Schur algebra, denoted $\mathcal{S}_{Q,q}(n,r)$, is defined to be

$$\mathcal{S}_{Q,q}(n,r) = \mathrm{End}_{\mathcal{H}}(\textstyle\bigoplus_{\lambda\in\Lambda_2^{\leqslant}(n,r)} M^{\lambda}).$$

$\mathcal{S}_{Q,q}(n,r)$ is a subalgebra of $\mathcal{S}_q^2(n,r)$, and again back in the group case $(q = Q = 1)$ we denote the corresponding Schur algebra by $\mathcal{S}_{1,1}(n,r)$. We will see a connection between $\mathcal{S}_{Q,q}(n,r)$ and $\mathcal{S}_q^2(n,r)$ later on in this chapter.

We now have one more endomorphism algebra to meet, called the partition algebra. We now consider graphs on two rows of r vertices, with each edge joining precisely 2 vertices, and with at most 1 edge between any two vertices. The connected components of each graph partition the vertices into m subsets, for some $1 \leqslant m \leqslant 2r$. We then define an equivalence relation $\sim$ on the set of all such graphs (with $2r$ vertices) by saying that two graphs are equivalent if they determine the same partition of the $2r$ vertices. So for example

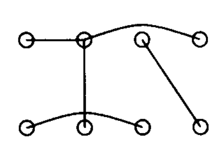


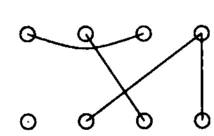
Then we use the term r-partition diagram, or just diagram, to mean the equivalence class under $\sim$ of the given graph.

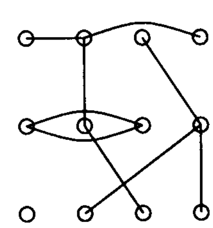
Let d_1 and d_2 be diagrams. If we let $n \in k$ we can multiply them, and the recipe for producing the product $d_2 d_1$ is,

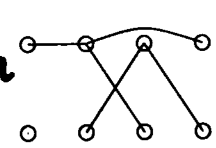
- Place d_1 above d_2 so that the bottom row of d_1 coincides with the top row of d_2. This gives us a diagram with a top, middle, and bottom row.
- Let α be the number of connected components of our 3 rowed diagram that lie entirely in the middle row.
- Make a new r-partition diagram by eliminating the middle row, but keeping the top and bottom rows and maintaining the connections between them.
- Then $d_2 d_1 = n^\alpha d_3$.

For example, let

$d_1 =$  ,

$d_2 =$  ,

then $d_2 d_1 =$  ,

$= n$  .

The partition algebra $P_r(n)$ is the algebra of k-linear combinations of r-partition diagrams, with multiplication as above.

We can now relate this to endomorphism algebras. We have already seen a right action of $\mathrm{Sym}(r)$ on $E^{\otimes r}$, but we can also make $\mathrm{Sym}(n)$ act diagonally on the left of $E^{\otimes r}$. More precisely, if $\pi \in \mathrm{Sym}(n)$, and

$e_{\mathfrak{i}}$ is a basis vector of $E^{\otimes r}$, then

$$\pi e_{\mathfrak{i}} = \pi(e_{i_1} \otimes \cdots e_{i_r}) = (e_{\pi(i_1)} \otimes \cdots e_{\pi(i_r)}).$$

Of course, we extend this action linearly to make $E^{\otimes r}$ into a left $k\,\mathrm{Sym}(n)$-module.

The partition algebra now becomes even more interesting due to the following theorem.

THEOREM 1.1. *When $n \geqslant 2r$, we have that*

$$P_r(n) \cong \mathrm{End}_{k\,\mathrm{Sym}(n)}(E^{\otimes r}).$$

PROOF. This was originally proved by Jones in [**24**], over the complex numbers $\mathbb{C}$. For a characteristic free version see [**26**, 2.7]. □

In a later chapter we will consider some aspects of the representation theory of $\mathrm{End}_{k\,\mathrm{Sym}(n)}(E^{\otimes r})$, giving us results about the partition algebra when $n \geqslant 2r$. We now look at some useful theorems.

1.3. Some useful theorems

We first look at two theorems concerning endomorphism algebras, and begin with some notation. Let A be a finite dimensional k-algebra, and let M be a right A-module. If we can find non zero A-modules M' and M'' such that $M = M' \bigoplus M''$ then we call M decomposable. Otherwise M is indecomposable. We call an indecomposable direct summand of M a component of M, and if M has d isomorphism types of component we record this by writing $\mathrm{Comp}(M) = d$. We have the following theorem, called Fitting's Theorem.

THEOREM 1.2. *Let M be a right A-module, and let*

$$M = M_1^{\oplus m_1} \bigoplus M_2^{\oplus m_2} \bigoplus \cdots \bigoplus M_d^{\oplus m_d}$$

be its decomposition into components, so that each $m_i \geqslant 1$. Then *there exist precisely d simple* $\mathrm{End}_A(M)$*-modules, and moreover they have dimensions* $m_1, m_2, \ldots, m_d$ *respectively.*

PROOF. This was originally proved by Fitting in [**12**], but a proof can also be found in [**16**, 5.2] and [35], for example. □

Writing mod(A) for the category of finite generated right A-modules, we say that two rings R and R' are Morita equivalent if there exists an equivalence of categories between mod(R) and mod(R'). This essentially means they have the same representation theory. The following is a standard tool for studying endomorphism algebras, and is useful in determining when two endomorphism algebras are Morita equivalent.

THEOREM 1.3. *Let M and N be A-modules such that*

$$M = M_1^{\oplus m_1} \bigoplus M_2^{\oplus m_2} \bigoplus \cdots \bigoplus M_d^{\oplus m_d}, \text{ and}$$
$$N = M_1 \bigoplus M_2 \bigoplus \cdots \bigoplus M_d$$

where $m_i \geqslant 1$*. Then* $\mathrm{End}_A(M)$ *and* $\mathrm{End}_A(N)$ *are Morita equivalent.*

Note that since Morita equivalence is transitive, the above theorem shows that if M and M' are two A-modules with the same components (but with possibly different multiplicities) then $\mathrm{End}_A(M)$ and $\mathrm{End}_A(M')$ are Morita equivalent. Then in particular the q-Schur^2 algebra $\mathcal{S}_q^2(n,r)$ and the (Q,q)-Schur algebra $\mathcal{S}_{Q,q}(n,r)$ are Morita equivalent.

We now review Grabmeier's work on Green correspondence and Mackey systems. We begin by defining the latter.

DEFINITION. Let $\mathcal{M}$ be a set of subgroups of a finite group G, and p a prime. Then $\mathcal{M}$ is a p-Mackey system of G if

(M1) $\{1\} \in \mathcal{M}$.

(M2) $\mathcal{M}$ is closed under both (i) conjugation, and (ii) intersection.

(M3) If S is a Sylow p-subgroup of G then $S \leqslant A$ for some $A \in \mathcal{M}$.

Some authors, for example Geck and Pfeiffer in [**13**, Exercise 2.2], call $\mathcal{M}$ a Mackey system if it satisfies the two closure axioms above. We will often abuse this and use the name Mackey system when really we should say p-Mackey system. Our Mackey systems always satisfy all axioms (M1), (M2) and (M3).

Grabmeier shows, in [**14**, 2.3(i)], that if H is a subgroup of G, and $\mathcal{M}$ is a p-Mackey system of G, then $\mathcal{M} \downarrow H = \{H \cap A \mid A \in \mathcal{M}\}$ is a p-Mackey system of H.

Now for some notation. Let p be a prime, let k have characteristic p, let G be a finite group, and let $\mathcal{M}$ be a p-Mackey system. Let X be a kG-module and H be a subgroup of G. Then X is H-projective (or projective relative to H) if X is a direct summand of $\mathrm{ind}_H^G W$ for some kH-module W. We now define the set $\mathcal{M}(X) = \{H \in \mathcal{M} \mid X \text{ is } H-\text{projective}\}$. Then $\mathcal{M}(X)$ is non-empty, and is also stable under G-conjugation. If X is indecomposable we call the minimal elements of $\mathcal{M}(X)$ the $\mathcal{M}$-vertices of X. These form a single conjugacy class. Now let $\mathcal{M}_0$ be the subset of $\mathcal{M}$ consisting of subgroups P of G such that p divides $|P : B|$ for all members B of $\mathcal{M}$ which are proper subgroups of P. In [**15**], Green shows that if $p \nmid |G : H|$ then every kG-module is H-projective. From this it follows that $P \in \mathcal{M}_0$ if and only if P occurs as an $\mathcal{M}$ vertex of some indecomposable kG-module. We call $\mathcal{M}_0$ the $\mathcal{M}$-vertices, and where this is unambiguous we call them simply the vertices.

Let P be an $\mathcal{M}$-vertex of an indecomposable kG-module X. We call a finite dimensional indecomposable kP-module W such that X is a component of $\mathrm{ind}_P^G W$ an $\mathcal{M}$-source (or simply source) of X. Then an indecomposable trivial source module is an indecomposable kG-module X which is a direct summand of $\mathrm{ind}_P^G k$ for some $P \in \mathcal{M}_0$, where k denotes the trivial kP-module. T is a trivial source module over $\mathcal{M}$ (or just trivial source module when the Mackey system involved is understood) if T is a direct sum of indecomposable trivial $\mathcal{M}$-source modules.

For $H \leqslant G$, we write $N_G(H)$ for the normalizer of H in G. Let P be an $\mathcal{M}$-vertex, and let $H \leqslant G$ with $N_G(P) \leqslant H$. Let

$$\mathfrak{X} = \{A \in \mathcal{M} \mid A \leqslant P \cap P^x \text{ for some } x \in G \setminus H\},$$
$$\mathfrak{Z} = \{A \in \mathcal{M} \downarrow H \mid A \leqslant H \cap P^x \text{ for some } x \in G \setminus H\},$$
$$\mathfrak{A} = \{A \in \mathcal{M} \mid A \leqslant P, \text{ but no } G\text{-conjugate of } A \text{ is in } \mathfrak{X}\}.$$

Note that we have $\mathfrak{X} \subseteq \mathfrak{Z}$ and $P \in \mathfrak{A}$.

Then for each kG-module V we define, up to isomorphism, the kH-module fV by $V \downarrow_H \cong fV \bigoplus V'$, where the kH-module V' is a direct summand of kH-modules with $\mathcal{M} \downarrow H$-vertex in $\mathfrak{Z}$, and also no component of fV has an $\mathcal{M} \downarrow H$-vertex in $\mathfrak{Z}$. If V is indecomposable and the $\mathcal{M}$-vertex of V is in $\mathfrak{A}$ then the kH-module fV is also indecomposable, and it has an $\mathcal{M} \downarrow H$-vertex in $\mathfrak{A}$.

Also if W is a kH-module we again define up to isomorphism the kG-module gW via $\mathrm{ind}_H^G W = gW \bigoplus W'$, where W' is a direct summand of kG-modules with $\mathcal{M}$-vertex in $\mathfrak{X}$, and no component of gW has $\mathcal{M}$-vertex in $\mathfrak{X}$. Similarly, if W is an indecomposable kH-module with $\mathcal{M} \downarrow H$-vertex in $\mathfrak{A}$ then gW is an indecomposable kG-module with vertex in $\mathfrak{A}$. Then, by [**14**, 3.7(i)] we have the following.

THEOREM 1.4. *Let V be an indecomposable kG-module. The assignment $V \mapsto fV$ (and similarly $W \mapsto gW$) gives us a one to one* *correspondence between isomorphism classes of finite dimensional indecomposable kG-modules with $\mathcal{M}$-vertex in $\mathfrak{A}$ and isomorphism classes of indecomposable kH-modules with $\mathcal{M} \downarrow H$-vertex in $\mathfrak{A}$. The modules V and fV have a common vertex, and fV is called the Green correspondent of V.*

Note that this is a generalization of the classical Green correspondence in which $\mathcal{M}$ is the Mackey system of all subgroups of G. Grabmeier now uses his version of Green correspondence to analyze indecomposable trivial source modules over $\mathcal{M}$. In [**14**, 4.5] he refines this correspondence to give the following.

THEOREM 1.5. *Let P be in $\mathcal{M}_0$, so that P is an $\mathcal{M}$-vertex. Then there exists a bijection between isomorphism classes of indecomposable kG-trivial source modules with vertex P, and isomorphism classes of projective indecomposable modules for $\frac{N_G(P)}{P}$.*

Note that in [**9**, 1.1(1)] Donkin proves an analogous result for modules with a linear source.

If G is a finite group we write $\#_p(G)$ for the number of p'-classes of G, so that $\#_p(G)$ is also the number of both irreducible and projective indecomposable kG-modules. We also write $\mathcal{M}_0'$ for the set of $\mathcal{M}$-vertices up to conjugacy. This gives us the following.

THEOREM 1.6. *The number of isomorphism types of indecomposable trivial source modules over a p-Mackey system M is given by*

$$\sum_{P\in\mathcal{M}_0'} \#_p\left(\frac{N_G(P)}{P}\right).$$

This result will be used time and again during this thesis, and will often be referred to as Grabmeier's count. Note that for ease we will denote the number of isomorphism types of indecomposable trivial source modules over a p-Mackey system $\mathcal{M}$ simply by $\text{TSM}(\mathcal{M})$.

Remark

If $\mathcal{M}$ is a p-Mackey system for G when p divides $|G|$ then it is also a p-Mackey system $\forall p \nmid |G|$, as if $p \nmid |G|$ then the identity subgroup is the unique Sylow p-subgroup. If $p \nmid |G|$ then the identity subgroup is also the unique $\mathcal{M}$-vertex, and $\frac{N_G(\{1\})}{\{1\}} = G$ so that when $p \nmid |G|$ we have that $\text{TSM}(\mathcal{M}) = \#_p(G) =$ the number of conjugacy classes of G. The story in characteristic zero is identical.

I.4. Coalgebras and Schur algebras

Green originally defined the Schur algebras as the dual of a certain coalgebra $A(n,r)$. In this section we recall some basic definitions and then look at the construction of the q-Schur algebra via the coalgebra approach. More details about coalgebras and their friends can be found in [**34**].

A k-coalgebra is a triple (A, Δ, ε) where A is a k-vector space and we have that $\Delta : A \to A \otimes A$ and $\varepsilon : A \to k$ are linear maps satisfying the following conditions:

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : A \to A \otimes A \otimes A \text{ (co-associativity)}$$

and

$$(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = 1 : A \to A \text{ (co-unit)}$$

where 1 denotes the identity map on A. A coalgebra (A, Δ, ε) is a bialgebra if in addition to the co-associativity and co-unit conditions being satisfied we also have that Δ and ε are algebra homomorphisms. We do not consider Hopf algebras in this thesis, but remark that a bialgebra (A, Δ, ε) is a Hopf algebra if there exists a linear map $\sigma : A \to A$ such that we have $m(\sigma \otimes 1)\delta = m(1 \otimes \sigma)\delta = \epsilon$ where $m : A \otimes A \to A$

is multiplication and we define the map $\epsilon : A \to A$ by $\epsilon(a) = \varepsilon(1)a$ for all $a \in A$. This is called the antipode condition, and if such a map σ exists, it is unique and is called the antipode of A. We will often abbreviate a coalgebra or bialgebra (A, Δ, ε) simply to A.

If (A, Δ, ε) is a bialgebra, then (I, Δ, ε), or just I, is called a biideal of A if I is an ideal of A and :

(i) $\Delta(I) \subseteq I \otimes A + A \otimes I$, and

(ii) $\varepsilon(I) = 0$.

If I is a biideal of A, then $(A/I, \Delta', \varepsilon')$ is a bialgebra, where Δ' and ε' are induced on A/I by their respective maps on A. This is a useful way of constructing bialgebras, as we will see shortly when we construct the q-Schur algebras.

Firstly some more definitions. Let (A, Δ, ε) be a k-coalgebra. Then a pair (V, τ), often abbreviated to just V, is a right A-comodule if V is a k-space and $\tau : V \to V \otimes A$ is a linear map such that

$$(\tau \otimes 1)\tau = (1 \otimes \Delta)\tau : V \to V \otimes A \otimes A$$

and

$$(1 \otimes \varepsilon)\tau = 1 : V \to V,$$

where of course 1 is the identity map on V.

Let (V, τ) and (V', τ') be right A-comodules. A linear map $\phi : V \to V'$ is a comodule homomorphism if $\tau'\phi = (\phi \otimes 1)\tau$.

Any right comodule (V, τ) also has associated to it another structure, called the coefficient space of V, denoted $\mathrm{cf}(V)$. Say V has k-basis $\{v_i \mid i \in I\}$. Then we have elements $c_{ij} \in A$, where $i, j \in I$, defined by

$$\tau(v_j) = \textstyle\sum_{i \in I} v_i \otimes c_{ij},$$

for each $j \in I$. Then the coefficient space $\mathrm{cf}(V)$ is defined by $\mathrm{cf}(V) := k - \mathrm{span}\{c_{ij} \mid i, j \in I\}$. Note that $\mathrm{cf}(V) \leqslant A$.

As in [**17**] if (A, Δ, ε) is a k-coalgebra, then its dual space $A^* = \mathrm{Hom}_k(A, k)$ is an associative k-algebra. Multiplication of two elements $\alpha, \beta \in A^*$ is given by

$$\alpha\beta = (\alpha \otimes \beta)\Delta.$$

Also, for an A-comodule (V, τ), we regard the k-vector space V as a left A^*-module via the product $\alpha v = (1 \otimes \alpha)\tau(v)$, for $\alpha \in A^*, v \in V$.

We are now in position to construct the q-Schur algebra $\mathcal{S}_q(n, r)$. This construction originally comes from [**3**].(A similar approach is taken in [**29**, 3.5], giving a different coalgebra but isomorphic Schur algebra.) Start with $F(n)$, the free k-algebra in non-commuting indeterminates $x_{i,j}$, where i and j run over $\underline{n}$. This is naturally a bialgebra, with $\Delta(x_{i,j}) = \sum_{m \in \underline{n}} x_{i,m} \otimes x_{m,j}$, and $\varepsilon(x_{i,j}) = \delta_{ij}$, the Kronecker delta. Let $q \in k$ and let J be the ideal of $F(n)$ generated by elements of the form

$$x_{i,l}x_{j,m} - qx_{j,m}x_{i,l} \qquad \text{for } i > j \text{ and } l \leqslant m,$$
$$x_{i,l}x_{j,m} - x_{j,m}x_{i,l} - (q-1)x_{j,l}x_{i,m} \qquad \text{for } i > j \text{ and } l > m,$$
$$x_{i,l}x_{i,m} - x_{i,m}x_{i,l} \qquad \text{for all } i, l, m,$$

where $i, j, l, m \in \underline{n}$. Then J is a biideal of $F(n)$, and we put $A_q(n) = F(n)/J$, so that writing $X_{i,j}$ for the canonical image $x_{i,j} + J$ of $x_{i,j}$ in $A_q(n)$ we get that in $A_q(n)$ we have the following relations

$$X_{i,l}X_{j,m} = qX_{j,m}X_{i,l} \qquad \text{for } i > j \text{ and } l \leqslant m,$$
$$X_{i,l}X_{j,m} = X_{j,m}X_{i,l} + (q-1)X_{j,l}X_{i,m} \qquad \text{for } i > j \text{ and } l > m,$$
$$X_{i,l}X_{i,m} = X_{i,m}X_{i,l} \qquad \text{for all } i, l, m,$$

where $i, j, l, m \in \underline{n}$. If we let $X_{i,j}$ have degree 1, then $A_q(n)$ is a graded algebra. We denote the r^{th} homogeneous part of $A_q(n)$ by $A_q(n, r)$. Then $E^{\otimes r}$ is an $A_q(n, r)$-comodule [**3**, 2.1.1] and the relations above ensure that multiplication by elements of $\mathrm{Hec}(r)$ is a comodule map. This induces an isomorphism

$$\theta : A_q(n, r)^* \to \mathrm{End}_{\mathrm{Hec}(r)}(E^{\otimes r}) = \mathcal{S}_q(n, r),$$

and hence we have an alternative way of constructing the q-Schur algebra. Putting $q = 1$ we are back in the group case, and we have the construction of the classical Schur algebra, as given in Green [**18**]. Note that Dipper and Donkin also localize at the determinant, d, to give us $A_q(n)_d$, which they show is a Hopf algebra.

In chapter 5 we will mimic this method in an attempt to construct a coalgebra whose dual is the q-Schur^2 algebra.

CHAPTER 2

Mackey systems for $\mathrm{Hyp}(r)$

Eventually, we aim to compare the above hyperoctahedral Schur algebra with Dipper and James' (Q, q)-Schur algebra, and with Du and Scott's q-Schur^2 algebra, both at $q = Q = 1$. To this end we introduce a new module for the hyperoctahedral group, called $Y^{\otimes r}$.

Let $Z(n)$ denote the set $\{\overline{n}, \overline{n-1}, \dots, \overline{1}, \widehat{1}, \widehat{2}, \dots, \widehat{n}, 1, 2, \dots, n\}$, with ordering $\overline{n} < \overline{n-1} < \cdots < \overline{1} < \widehat{1} < \widehat{2} < \cdots < \widehat{n} < 1 < 2 < \cdots < n$. Call elements $\overline{i}$ barred, and call the elements $\widehat{i}$ hatted. Then we let $I_B(n, r)$ denote the set of r-tuples with entries from $Z(n)$, so that the involution $^-$ acts on $Z(n)$, and we have that $\overline{\widehat{i}} = \widehat{i}$ and $\overline{\overline{i}} = i$ for all $i \in \underline{n}$.

LEMMA 2.1. $\mathrm{Hyp}(r)$ *acts on the right of* $I_B(n,r)$ *via*

$$\mathbf{i}.s_j = (i_1, \dots, i_{(j)s_j}, i_{(j+1)s_j}, \dots, i_r), \text{ if } 0 < j < r,$$
$$\text{and } \mathbf{i}.s_0 = (\overline{i}_1, i_2, .., i_r)$$

PROOF. It suffices to check the relation $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$, which splits into 4 cases. The other relations follow from **[18]**, **[19]**, and the fact that s_0 acts trivially on $\mathbf{i} \in I_B(n,r)$ if $i_1 \in \{\widehat{1}, \widehat{2}, \dots, \widehat{n}\}$.

Case 1 $i_1, i_2 \notin \{\widehat{1}, \widehat{2}, \dots, \widehat{n}\}$;

then $(i_1, i_2, \dots, i_r)s_0 s_1 s_0 s_1 = (\overline{i_1}, i_2, i_3, \dots, i_r)s_1 s_0 s_1 =$
$(i_2, \overline{i_1}, i_3, \dots, i_r)s_0 s_1 = (\overline{i_2}, \overline{i_1}, i_3, \dots, i_r)s_1 = (\overline{i_1}, \overline{i_2}, i_3, \dots, i_r)$
and

$(i_1, i_2, \ldots, i_r)s_1s_0s_1s_0 = (i_2, i_1, i_3, \ldots, i_r)s_0s_1s_0 =$
$(\overline{i_2}, i_1, i_3, \ldots, i_r)s_1s_0 = (i_1, \overline{i_2}, i_3, \ldots, i_r)s_0 = (\overline{i_1}, \overline{i_2}, i_3, \ldots, i_r)$

Case 2 $i_1, i_2 \in \{\widehat{1}, \widehat{2}, \ldots, \widehat{n}\}$;

as in case 1, but now $i_1 = \overline{i_1}$ and $i_2 = \overline{i_2}$, and so we get

$(i_1, i_2, \ldots, i_r)s_1s_0s_1s_0 = (i_1, i_2, \ldots, i_r) = (i_1, i_2, \ldots, i_r)s_1s_0s_1s_0.$

Case 3 $i_1 \in \{\widehat{1}, \widehat{2}, \ldots, \widehat{n}\}, i_2 \notin \{\widehat{1}, \widehat{2}, \ldots, \widehat{n}\}$;

as in case 1, but now $i_1 = \overline{i_1}$ so that we get

$(i_1, i_2, \ldots, i_r)s_1s_0s_1s_0 = (i_1, \overline{i_2}, \ldots, i_r) = (i_1, i_2, \ldots, i_r)s_1s_0s_1s_0.$

Case 4 $i_1 \notin \{\widehat{1}, \widehat{2}, \ldots, \widehat{n}\}, i_2 \in \{\widehat{1}, \widehat{2}, \ldots, \widehat{n}\}$;

as in case 1, but now $i_2 = \overline{i_2}$ so that we get

$(i_1, i_2, \ldots, i_r)s_1s_0s_1s_0 = (\overline{i_1}, i_2, \ldots, i_r) = (i_1, i_2, \ldots, i_r)s_1s_0s_1s_0.$

Therefore we have that $\mathbf{i}s_1s_0s_1s_0 = \mathbf{i}s_0s_1s_0s_1$ for all $\mathbf{i} \in I_B(n,r)$ and the proof is complete. $\square$

As before, we now transfer this action to a useful module. Let Y be the $3n$-dimensional vector space over k with basis $\{y_i \mid i \in Z(n)\}$, and let $Y^{\otimes r}$ be its r^{th} tensor power. Unsurprisingly, we now write a basis element in $Y^{\otimes r}$ as $y_{\mathbf{i}} = y_{i_1} \otimes y_{i_2} \otimes \cdots y_{i_r}$, for some $\mathbf{i} \in I_B(n,r)$. Then Hyp(r) acts on $Y^{\otimes r}$ via

$$y_{\mathbf{i}}.\sigma = y_{\mathbf{i}.\sigma} \text{ for all } \mathbf{i} \in I_B(n,r) \text{ and } \sigma \in \mathrm{Hyp}(r).$$

Extending this action linearly makes $Y^{\otimes r}$ into a $k\,\mathrm{Hyp}(r)$-module.

Young and younger subgroups

It is well known that the point stabilizers of the action of Sym(r) on the standard basis elements of the tensor space $E^{\otimes r}$ are the Young subgroups Sym(λ), which are defined in chapter 1. We can make a similar definition for Hyp(r).

Let $\lambda = (\mu;\nu)$ be a pair of partitions in $\Lambda_2^+(n,r)$, and put $L_\mu = \{\sum_{i=1}^{a}\mu_i \,|\, a \in \underline{n}\}$ and $L_\nu = \{\,|\mu| + \sum_{i=1}^{a}\nu_i \,|\, a \in \underline{n}\}$. Then let $A_\mu = [1,|\mu|]\setminus L_\mu$ and also let $A_\nu = [1+|\mu|,r]\setminus L_\mu$. Then the standard infant subgroup associated to λ, denoted Hyp(λ) is given by

$$\mathrm{Hyp}(\lambda) = \langle s_a, s_b, t_a \,|\, a \in A_\mu, b \in A_\nu\rangle.$$

An infant subgroup is any which is conjugate in Hyp(r) to a standard infant subgroup. Let $\mathcal{Y}$ denote the set of all infant subgroups of Hyp(r). Similarly to above we can associate an infant subgroup to each pair of compositions in $\Lambda_2(n,r)$. Then each coset Hyp(r)g, for $g \in$ Hyp(r), has a unique member of minimum length. We call these the distinguished coset representatives of Hyp(λ) in Hyp(r) and denote each set of these by Dist(λ), so that $|\,\mathrm{Dist}(\lambda)| = \frac{|\,\mathrm{Hyp}(r)|}{|\,\mathrm{Hyp}(\lambda)|}$. Also note that if $\lambda \in \Lambda_2^+(n,r)$ with $\lambda = (\mu_1,..,\mu_r;\nu_1,..,\nu_r)$, then

$$\begin{aligned}\mathrm{Hyp}(\lambda) \cong\ & \mathrm{Hyp}(\mu_1)\times\mathrm{Hyp}(\mu_2)\times\cdots\times\mathrm{Hyp}(\mu_r)\\ & \times S(\nu_1)\times S(\nu_2)\times\cdots\times S(\nu_r)\end{aligned}$$

The following lemma is useful.

LEMMA 2.2. *If X is a G-set, $x \in X$ and $g \in G$ then*

$$\mathrm{Stab}(xg) = g^{-1}\,\mathrm{Stab}(x)g.$$

PROOF. Let $h \in \mathrm{Stab}(x)$ so that $xh = x$. Then

$$(xg)g^{-1}hg = xhg = xg,$$

so that $g^{-1}hg \in \mathrm{Stab}(xg)$.

Conversely let $h \in \mathrm{Stab}(xg)$ so that $(xg)h = xg$. Then

$$x(ghg^{-1}) = (xg)hg^{-1} = xgg^{-1} = x,$$

so that $ghg^{-1} \in \mathrm{Stab}(x)$ and $h \in g^{-1}\,\mathrm{Stab}(x)g$. ☐

So from the above lemma for any $\lambda \in \Lambda_2^+(n,r)$, and $\sigma \in$ Hyp(r) we have the following.

COROLLARY 2.3. Hyp(λ) *is the point stabilizer of* $y_{\mathbf{i}} \iff$ Hyp$(\lambda)^\sigma$ *is the point stabilizer of* $y_{\mathbf{i}\sigma}$.

Now for a bit of notation. For any pair of compositions $\lambda \in \Lambda_2(n,r)$ let $\mathbf{i}(\lambda) = (\widehat{1},\widehat{1},\ldots,\widehat{1},\widehat{2},\ldots,\widehat{2},\widehat{n},\ldots,\widehat{n},1,1,\ldots,1,2,\ldots,2,n\ldots,n) \in I_B(n,r)$ with each $\widehat{j}$ occurring μ_j times, and each "naked" j occurring ν_j times, so that in particular Hyp(λ) is point stabilizer of $y_{\mathbf{i}(\lambda)}$. Also let $\mathcal{X}$ denote the set of all point stabilizers of standard basis elements in $Y^{\otimes r}$ under the action of Hyp(r).

LEMMA 2.4. $\mathcal{X} = \mathcal{Y}$, *i.e. the point stabilizers of basis elements in* $Y^{\otimes r}$ *are precisely the infant subgroups of* Hyp(r).

PROOF. Let $A = \text{Hyp}(\lambda) \in \mathcal{Y}$, so that A is a standard infant subgroup. Then A is the point stabilizer of $y_{\mathbf{i}(\lambda)}$, so that $A \in \mathcal{X}$. Now say $B = A^\sigma \in \mathcal{Y}$, so that B is an infant subgroup. Then, by previous lemma, we know that B is the point stabilizer of $y_{\mathbf{i}(\lambda)\sigma}$, so that B is also in $\mathcal{X}$ and $\mathcal{Y} \subseteq \mathcal{X}$.

Conversely let $y_{\mathbf{i}}$ be any standard basis element in $Y^{\otimes r}$, and let $X \in \mathcal{X}$ be its point stabilizer. Now any such $y_{\mathbf{i}}$ can be written as $y_{\mathbf{i}} = y_{\mathbf{i}(\lambda)\sigma}$ for suitable choice of $\lambda \in \Lambda_2^+(n,r)$ and $\sigma \in \text{Hyp}(r)$. Now $y_{\mathbf{i}(\lambda)}$ has point stabilizer Hyp(λ), so by the previous lemma we have that $y_{\mathbf{i}(\lambda)\sigma}$ has point stabilizer Hyp$(\lambda)^\sigma$. Therefore $X \in \mathcal{Y}$ also, and we have that $\mathcal{X} \subseteq \mathcal{Y}$.

$\square$

We now walk down the same road, but this time for Richard Green's space $V^{\otimes r}$. Let $\mathcal{X}_V$ denote the set of point stabilizers of the Hyp(r)-action on $V^{\otimes r}$, and also let

$$\mathcal{V} = \{\text{Hyp}(\lambda)^\sigma \in \mathcal{Y} \mid \sigma \in \text{Hyp}(r), \lambda = (\emptyset;\nu) \in \Lambda_2^+(n,r)\}$$

so that $\mathcal{V}$ consists of infant subgroups of Hyp(r) which are of the form $(S(\nu_1) \times S(\nu_2) \times \cdots \times S(\nu_n))^\sigma$ for some $\sigma \in \text{Hyp}(r)$, i.e. with all parts isomorphic to some smaller symmetric group inside the hyperoctahedral group. This means each member of $\mathcal{V}$ is isomorphic to some Young subgroup $S(\lambda)^\sigma$ inside Hyp(r), so that $\mathcal{V}$ is the set of all Young subgroups of the hyperoctahedral group. Similarly to above, if $\lambda = (\emptyset; \nu) \in \Lambda_2^+(n,r)$, $\sigma \in \text{Hyp}(r)$ and $\mathbf{i} \in I_{\mathcal{V}}(2n,r)$ then we have the following.

LEMMA 2.5. Hyp(λ) *is a point stabilizer of* $v_{\mathbf{i}} \iff$
Hyp$(\lambda)^\sigma$ *is a point stabilizer of* $v_{\mathbf{i}\sigma}$.

PROOF. This again follows from the above lemma about G-sets. □

COROLLARY 2.6. $\mathcal{X}_{\mathcal{V}} = \mathcal{V}$ *i.e. the set of point stabilizers of the* Hyp(r)*-action on standard basis elements of* $V^{\otimes r}$ *is precisely the set* $\mathcal{V}$.

For the time being we concentrate on $\mathcal{Y}$, and to that end we now look at an alternative description of it.

2.1. Schisms and Mackey systems

We recall the following definition of a schism from chapter 1. Let X be a set with a regular involution $^{-}$, called bar, so that X necessarily has $2r$ elements, for some natural number r. For the rest of this section X will always be such a set. Let Z be a subset of X such that $Z \sqcup \overline{Z} = X$, i.e. such that $Z \cup \overline{Z} = X$ and $Z \cap \overline{Z} = \emptyset$. Then we call Z a schism of X.

Now we can split up X even more. Let X be as above, and say that $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_a \sqcup X_{a+1} \sqcup \cdots \sqcup X_m$ where each X_i is a bar-stable subset, so has size $2b_i$ for some $1 \leqslant b_i \leqslant r$. In particular, each of $X_{a+1}, X_{a+2}, \ldots, X_m$ has bar acting on it as a regular involution, so we can choose $Z_1 \subset X_{a+1}, Z_2 \subset X_{a+2}, \ldots, Z_{m-a} \subset X_m$ so that $Z_i \sqcup \overline{Z_i} = X_{a+i}$ for each $1 \leqslant i \leqslant m-a$, i.e. for each $1 \leqslant i \leqslant m-a$, we have that Z_i is a schism of X_{a+i}.

Call such a sequence $(\underline{X}; \underline{Z}) = (X_1, \ldots, X_a; Z_1, \ldots, Z_b)$ of such a set X a schism sequence of X, and write $\Sigma(X)$ for the set of all schism sequences of a set X. We will call the X_i's and the Z_j's the parts of the schism sequence, and say that the X_i's belong to the first half of $(\underline{X}; \underline{Z})$, while Z_j's belong to the second half. (Of course, $b = m-a$.)

Now let Z be a schism of X, so that $\mathrm{Sym}(Z)$ is the symmetric group on Z. Again, recall from chapter 1 that we can identify $\mathrm{Sym}(Z)$ with a subgroup of $\mathrm{Sym}(X)$ via the following map. Let

$$\phi : \mathrm{Sym}(Z) \to \mathrm{Sym}(X) \text{ via } \phi(\sigma)(z) = \sigma z \text{ and } \phi(\sigma)(\overline{z}) = \overline{\sigma z}$$

for $\sigma \in \mathrm{Sym}(Z)$ and $z \in Z$. Now $\overline{\phi(\sigma)(z)} = \overline{\sigma z} = \phi(\sigma)(\overline{z})$ so each member of $\mathrm{Sym}(Z)$ is in fact identified with a member of $\mathrm{Hyp}(X) \subset \mathrm{Sym}(X)$ via this map. We denote the image of $\mathrm{Sym}(Z)$ under ϕ by $S(Z)$, which is a subgroup of $\mathrm{Hyp}(X)$. It is worth noting two things.

Firstly, for any schism Z of X, the subgroups $S(Z)$ and $S(\overline{Z})$ are the same.

Secondly, if $(\underline{X}; \underline{Z}) \in \Sigma(X)$ then $\mathrm{Hyp}(X_i)$ is the subgroup of $\mathrm{Hyp}(X)$ which acts as the hyperoctahedral group on X_i and fixes all other elements of $X \setminus X_i$. We also have for each Z_j a subgroup of $\mathrm{Hyp}(X)$ denoted by $S(Z_j)$, which acts on $X_{a+j} = Z_j \sqcup \overline{Z_j}$ as a subgroup isomorphic to $\mathrm{Sym}(Z)$ inside $\mathrm{Hyp}(X_j)$, via the map ϕ as described above. Also, $S(Z_j)$ fixes all elements of $X \setminus X_{a+j}$ and of course $S(Z_j) = S(\overline{Z_j})$.

Armed with these subgroups of $\mathrm{Hyp}(X)$ we can now make a definition.

Definition. Given a set X with a regular involution, and a schism sequence $(\underline{X}; \underline{Z}) \in \Sigma(X)$ of X, we define the schism subgroup $H(\underline{X}; \underline{Z})$

of $\mathrm{Hyp}(X)$ to be $H(\underline{X};\underline{Z}) =$

$$\mathrm{Hyp}(X_1) \times \mathrm{Hyp}(X_2) \times \cdots \times \mathrm{Hyp}(X_a) \times S(Z_1) \times \cdots \times S(Z_b).$$

Denote the set $\{H(\underline{X};\underline{Z}) | (\underline{X};\underline{Z}) \in \Sigma(X)\}$ of all schism subgroups of $\mathrm{Hyp}(X)$ by $\mathcal{Z}$.

Schism subgroups also come in another guise.

LEMMA 2.7. *For* $(\underline{X};\underline{Z}) \in \Sigma(X)$ *we have* $H(\underline{X};\underline{Z})$ $= \{\sigma \in \mathrm{Hyp}(X) | \sigma X_i = X_i$ *and* $\sigma Z_j = Z_j \ \forall 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b\}$.

PROOF. To make things a bit easier to read, we write $H' =$ $\{\sigma \in \mathrm{Hyp}(X) | \sigma X_i = X_i$ and $\sigma Z_j = Z_j \ \forall 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b\}$. Now each $\mathrm{Hyp}(X_i)$ permutes elements of X_i and fixes all elements of $X \setminus X_i$. Also each $S(Z_j)$ permutes elements of Z_j, and of $\overline{Z_j}$ correspondingly, and fixes all elements of X_{a+j}. Therefore $H(\underline{X};\underline{Z}) \subseteq H'$.

Conversely $\sigma X_i = X_i$ implies that $\sigma \in \mathrm{Hyp}(X_i)$, as $\sigma \in \mathrm{Hyp}(X)$. Also $\sigma Z_j = Z_j$ induces a $\pi \in \mathrm{Sym}(Z_j)$ but $\sigma(\overline{z}) = \overline{\sigma z} = \overline{\pi z}$. Therefore $\sigma = \phi(\pi)$ and $\sigma \in S(Z_j)$. So $H' \subseteq H(\underline{X};\underline{Z})$ and we are done. $\square$

We now prove 3 more lemmas about schism subgroups, which prove that $\mathcal{Y}$ is a p-Mackey system.

LEMMA 2.8. $\mathcal{Z}$ *is closed under conjugation.*

PROOF. Let $g \in \mathrm{Hyp}(X)$. Now we know that $S(Z_j)^g = S(gZ_j)$ and that $\mathrm{Hyp}(X_i)^g = \mathrm{Hyp}(gX_i)$ and therefore we get that $H(\underline{X};\underline{Z})^g$ $= H(X_1,\ldots,X_a;Z_1,\ldots,Z_b)^g = H(gX_1,\ldots,gX_a;gZ_1,\ldots,gZ_b)$ $= H(g\underline{X};g\underline{Z})$. So since $g \in \mathrm{Hyp}(X)$ and it maps X_i's to other X_i's and Z_j's to other Z_j's we get that $(g\underline{X};g\underline{Z}) \in \Sigma(X)$ and therefore $H(g\underline{X};g\underline{Z}) = H(\underline{X};\underline{Z})^g \in \mathcal{Z}$. $\square$

LEMMA 2.9. *Let* $X = \{1,2,\ldots,r,\overline{1},\overline{2},\ldots,\overline{r}\}$. *Then* $\mathcal{Z} = \mathcal{Y}$.

PROOF. Let $A \in \mathcal{Y}$ so that $A = \mathrm{Hyp}(\lambda)^\sigma$ for some $\lambda \in \Lambda_2^+(n,r)$ and $\sigma \in \mathrm{Hyp}(X)$. Choose $(\underline{X};\underline{Z}) \in \Sigma(X)$ with
$X_1 = \{1,2,\ldots,\mu_1,\overline{1},\overline{2},\ldots,\overline{\mu_1}\}$,
$X_2 = \{\mu_1+1,\mu_1+2,\ldots,\mu_1+\mu_2,\overline{\mu_1+1},\overline{\mu_1+2},\ldots,\overline{\mu_1+\mu_2}\}, \ldots$
$\ldots X_n = \{|\mu|-\mu_n+1,\ldots,|\mu|,\overline{|\mu|-\mu_n+1},\ldots,\overline{|\mu|}\}$, and
$Z_1 = \{|\mu|+1,|\mu|+2,\ldots,|\mu|+\nu_1\}$,
$Z_2 = \{|\mu|+\nu_1+1,|\mu|+\nu_1+2,\ldots,|\mu|+\nu_1+\nu_2\},\ldots$
$\ldots, Z_n = \{r-\nu_n+1,\ldots,r\}$.

Then $H(\underline{X};\underline{Z}) = \mathrm{Hyp}(\lambda)$ so that $A = \mathrm{Hyp}(\lambda)^\sigma = H(\underline{X};\underline{Z})^\sigma = H(\sigma\underline{X};\sigma\underline{Z}) \in \mathcal{Z}$.

Conversely, let $A \in \mathcal{Z}$, so that $A = H(\underline{X};\underline{Z})$, for some $(\underline{X};\underline{Z}) \in \Sigma(X)$, where $(\underline{X};\underline{Z}) = (X_1,X_2,\ldots,X_a;Z_1,\ldots,Z_b)$. Firstly we arrange the X_i's and Z_j's so that they are in descending size order to get $(\underline{X'};\underline{Z'}) = (X_1',X_2',\ldots,X_{a'}';Z_1',\ldots,Z_{b'}')$, where $|X_i'| \geqslant |X_{i+1}'|$ for all $1 \leqslant i \leqslant a'-1$ and $|Z_j'| \geqslant |Z_{j+1}'|$ for all $1 \leqslant j \leqslant b'-1$. This can be done by hitting $(\underline{X};\underline{Z})$ with a certain, but not usually unique, element $g \in \mathrm{Hyp}(X)$. This gives us the schism sequence $(\underline{X'};\underline{Z'}) = (g\underline{X};g\underline{Z})$.

Let $F = X_1' \cup X_2' \cup \cdots \cup X_n'$. Now we hit $(\underline{X'};\underline{Z'})$ with the (not necessarily unique) element $h \in \mathrm{Hyp}(X)$ so that
$hX_1' = \{1,2,\ldots,\frac{1}{2}|X_1'|,\overline{1},\overline{2},\ldots,\overline{\frac{1}{2}|X_1'|}\}$,
$hX_2' = \{\frac{1}{2}|X_1'|+1,\frac{1}{2}|X_1'|+2,\ldots,\frac{1}{2}|X_1'\cup X_2'|,\overline{\frac{1}{2}|X_1'|+1},\ldots,\overline{\frac{1}{2}|X_1'\cup X_2'|}\}$,
$\ldots$,etc, $\ldots$
$hX_n = \{\frac{1}{2}|X_1'\cup\cdots\cup X_{n-1}'|+1,\ldots,\frac{1}{2}|F|,\overline{\frac{1}{2}|X_1'\cup\cdots\cup X_{n-1}'|+1},\ldots,\overline{\frac{1}{2}|F|}\}$
and
$hZ_1' = \{\frac{1}{2}|F|+1,\frac{1}{2}|F|+2,\ldots,\frac{1}{2}|F|+|Z_1'|\}$,
$hZ_2' = \{\frac{1}{2}|F|+|Z_1'|+1,\frac{1}{2}|F|+|Z_1'|+2,\ldots,\frac{1}{2}|F|+|Z_1'\cup Z_2'|\}$, $\ldots$
$\ldots$, $hZ_n' = \{\frac{1}{2}|F|+|Z_1'\cup\cdots\cup Z_{n-1}'|+1,\ldots,r\}$.
Call this schism sequence $(\underline{X''};\underline{Z''})$ and notice that we have $(\underline{X''};\underline{Z''})$

$= (h\underline{X}'; h\underline{Z}') = (hg\underline{X}; hg\underline{Z})$. Also notice that $H(\underline{X}''; \underline{Z}'') = \mathrm{Hyp}(\lambda)$, for some suitable choice of $\lambda \in \Lambda_2^+(n,r)$.

But then $(\underline{X}; \underline{Z}) = (g^{-1}h^{-1}\underline{X}''; g^{-1}h^{-1}\underline{Z}'')$ so that $H(\underline{X}; \underline{Z})$ $= H(g^{-1}h^{-1}\underline{X}''; g^{-1}h^{-1}\underline{Z}'') = H(\underline{X}''; \underline{Z}'')^{h^{-1}g^{-1}} = \mathrm{Hyp}(\lambda)^{h^{-1}g^{-1}} \in \mathcal{Y}$. Therefore $\mathcal{Z} \subseteq \mathcal{Y}$ and we are done.

□

Now for the third and final lemma.

LEMMA 2.10. *$\mathcal{Z}$ is closed under intersection.*

PROOF. We claim that for $(\underline{X}; \underline{Z}), (\underline{X}'; \underline{Z}') \in \Sigma(X)$ we have that $H(\underline{X}; \underline{Z}) \cap H(\underline{X}'; \underline{Z}') =$

$$H(X_i \cap X_l'; X_i \cap Z_m', Z_j \cap X_l', Z_j \cap Z_m', Z_j \cap \overline{Z_m'}),$$

where $1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b, 1 \leqslant l \leqslant a', 1 \leqslant m \leqslant b'$. Call this intersection $H(\underline{X}''; \underline{Z}'')$. Since we want to use this name, we will first have to show that $(\underline{X}''; \underline{Z}'')$ is indeed a schism sequence of X. Here goes.

Now both $(\underline{X}; \underline{Z})$ and $(\underline{X}'; \underline{Z}')$ are schism sequences. Say A and C are both parts of $(\underline{X}; \underline{Z})$, and B and D are both parts of $(\underline{X}'; \underline{Z}')$, so that $A \cap B$ and $C \cap D$ are both parts of $(\underline{X}''; \underline{Z}'')$. Then $(A \cap B) \cap (C \cap D) = (A \cap C) \cap (B \cap D)$, but $A \cap C = \emptyset$ as $(\underline{X}; \underline{Z})$ is a schism sequence. Therefore $(A \cap B) \cap (C \cap D) = \emptyset$ and so any two parts of $(\underline{X}''; \underline{Z}'')$ are disjoint.

Also, if X_i is in the first part $(\underline{X}; \underline{Z})$, and X_j' is in the first part of $(\underline{X}'; \underline{Z}')$, then $X_i \cap X_j' = \overline{X_i} \cap \overline{X_j'} \subseteq \overline{X_i \cap X_j'}$. Therefore $X_i \cap X_j' = \overline{X_i \cap X_j'}$ as both sets have the same cardinality, so any part of the first half of $(\underline{X}''; \underline{Z}'')$ is bar stable.

Let A be any part of either of the two schism sequences $(\underline{X}; \underline{Z})$ and $(\underline{X}'; \underline{Z}')$. Then $Z_i \cap A \cap \overline{Z_i \cap A} \subseteq Z_i \cap \overline{Z_i} = \emptyset$. The same argument also shows that $\overline{Z_i} \cap A \cap \overline{\overline{Z_i} \cap A} = A \cap Z_j' \cap \overline{A \cap Z_j'} = A \cap \overline{Z_j'} \cap \overline{A \cap \overline{Z_j'}} = \emptyset$. This tells us that any part, B say, of the second half of $(\underline{X}''; \underline{Z}'')$ and its bar, $\overline{B}$ are disjoint.

Let U be the union of the following subsets of X; $X_i \cap X_l', X_i \cap Z_m', Z_j \cap X_l', Z_j \cap Z_m', Z_j \cap \overline{Z_m'}, \overline{X_i \cap Z_m'}, \overline{Z_j \cap X_l'}, \overline{Z_j \cap Z_m'}, \overline{Z_j \cap \overline{Z_m'}}$ where $1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b, 1 \leqslant l \leqslant a', 1 \leqslant m \leqslant b'$. Then we must show $U = X$. Now U is a union of subsets of X and so therefore $U \subseteq X$. Conversely, let $x \in X$. Then as $(\underline{X}; \underline{Z}) \in \Sigma(X)$ either $x \in X_i, x \in Z_j$, or $x \in \overline{Z_j}$, for some $1 \leqslant i \leqslant a$, $1 \leqslant j \leqslant b$, where the X_i and Z_j are the parts of $(\underline{X}; \underline{Z})$. Similarly for $(\underline{X}'; \underline{Z}')$, x must also lie in either X_l', Z_m' or $\overline{Z_m'}$, for some $1 \leqslant l \leqslant a'$, $1 \leqslant m \leqslant b'$, where the X_l' and Z_m' are the parts of $(\underline{X}'; \underline{Z}')$.

Now for all i, j, l, m as above we know that each of $X_i \cap X_l', X_i \cap Z_m', Z_j \cap X_l', Z_j \cap Z_m', Z_j \cap \overline{Z_m'}$ is in U. But also for all such i, j, l, m we have that $X_i \cap \overline{Z_j'} = \overline{X_i} \cap \overline{Z_j'} \subseteq \overline{X_i \cap Z_j'} \subseteq U$ and $\overline{Z_j} \cap Z_m' = \overline{Z_j \cap \overline{Z_m'}} \subseteq U$ and $\overline{Z_j} \cap X_l' = \overline{Z_j} \cap \overline{X_l'} \subseteq \overline{Z_j \cap X_l'} \subseteq U$ and finally $\overline{Z_j} \cap \overline{Z_m'} = \overline{Z_j \cap Z_m'} \subseteq U$. Then as x must lie in one of the above mentioned intersections we see that $x \in U$. Therefore $X \subseteq U$ so that $U = X$.

So we have shown that $(\underline{X}''; \underline{Z}'')$ is a schism sequence of X. Now we just need to show that $H(\underline{X}''; \underline{Z}'')$ is in fact equal to $H(\underline{X}; \underline{Z}) \cap H(\underline{X}'; \underline{Z}')$ as claimed. We will show this is true by showing that each set is contained in the other. Notice that for all i, j, l, m as above, we have that;

$$\text{if } x \in X_i \text{ and } X_l' \text{ then } x \in X_i \cap X_l',$$
$$\text{if } x \in X_i \text{ and } Z_m' \text{ then } x \in X_i \cap Z_m',$$
$$\text{if } x \in X_i \text{ and } \overline{Z_m'} \text{ then } x \in \overline{X_i} \cap \overline{Z_m'} \subseteq \overline{X_i \cap Z_m'},$$
$$\text{if } x \in Z_j \text{ and } X_l' \text{ then } x \in Z_j \cap X_l',$$
$$\text{if } x \in Z_j \text{ and } Z_m' \text{ then } x \in Z_j \cap Z_m',$$

$$\text{if } x \in Z_j \text{ and } \overline{Z'_m} \text{ then } x \in Z_j \cap \overline{Z'_m},$$
$$\text{if } x \in \overline{Z_j} \text{ and } X'_l \text{ then } x \in \overline{Z_j} \cap \overline{X'_l} \subseteq \overline{Z_j \cap X'_l},$$
$$\text{if } x \in \overline{Z_j} \text{ and } Z'_m \text{ then } x \in \overline{Z_j} \cap \overline{\overline{Z'_m}} \subseteq \overline{Z_j \cap \overline{Z'_m}},$$
$$\text{and finally if } x \in \overline{Z_j} \text{ and } \overline{Z'_m} \text{ then } x \in \overline{Z_j \cap Z'_m}.$$

So therefore each $g \in H(\underline{X};\underline{Z}) \cap H(\underline{X}';\underline{Z}')$ stabilizes each part of $H(\underline{X}'';\underline{Z}'')$, so therefore this tells us that as we have $H(\underline{X}'';\underline{Z}'') = \{\sigma \in \mathrm{Hyp}(X) \mid \sigma X''_u = X''_u \text{ and } \sigma Z''_v = Z''_v\}$ we have that $g \in H(\underline{X}'';\underline{Z}'')$ and so $H(\underline{X};\underline{Z}) \cap H(\underline{X}';\underline{Z}') \subseteq H(\underline{X}'';\underline{Z}'')$.

Now for the converse. Let i, j, l, m be as they consistently are above, and say that $\sigma \in \mathrm{Hyp}(X_i \cap X'_l) \subseteq H(\underline{X}'';\underline{Z}'')$. Then we can expand to get $X_i = (X_i \cap X'_1) \cup \cdots \cup (X_i \cap X'_l) \cup \cdots (X_i \cap X'_{a'}) \cup (X_i \cap Z'_1) \cup \cdots \cup (X_i \cap Z'_{b'}) \cup (X_i \cap \overline{Z'_1}) \cup \cdots \cup (X_i \cap \overline{Z'_{b'}})$, so that $\sigma \in \mathrm{Hyp}(X_i)$. But similarly we also get that $X'_l = (X_1 \cap X'_l) \cup \cdots \cup (X_i \cap X'_l) \cup \cdots \cup (X_a \cap X'_l) \cup (Z_1 \cap X'_l) \cup \cdots \cup (Z_b \cap X'_l) \cup (\overline{Z_1} \cap X'_l) \cup \cdots \cup (\overline{Z_b} \cap X'_l)$, and therefore $\sigma \in \mathrm{Hyp}(X'_l)$ also, so that $\sigma \in \mathrm{Hyp}(X_i) \cap \mathrm{Hyp}(X'_l) \subseteq H(\underline{X};\underline{Z}) \cap H(\underline{X}';\underline{Z}')$.

Next say that $\sigma \in S(X_i \cap Z'_m)$. Then from the expansion of X_i above we have see that $\sigma \in \mathrm{Hyp}(X_i)$. Also, we have that $Z'_m = (X_1 \cap Z'_m) \cup \cdots \cup (X_i \cap Z'_m) \cup \cdots \cup (X_a \cap Z'_m) \cup (Z_1 \cap Z'_m) \cup \cdots$ etc, so that $\sigma \in S(Z'_m)$ also. Therefore $\sigma \in \mathrm{Hyp}(X_i) \cap S(Z'_m) \subseteq H(\underline{X};\underline{Z}) \cap H(\underline{X}';\underline{Z}')$.

Similarly if $\sigma \in S(Z_j \cap X'_l)$ then $\sigma \in S(Z_j) \cap \mathrm{Hyp}(X'_l)$, if $\sigma \in S(Z_j \cap Z'_m)$ then $\sigma \in S(Z_j) \cap S(Z'_m)$, and finally if $\sigma \in S(Z_j \cap \overline{Z'_m})$ then $\sigma \in S(Z_j) \cap S(Z'_m)$. All these are contained in $H(\underline{X};\underline{Z}) \cap H(\underline{X}';\underline{Z}')$, and as each generator of $H(\underline{X}'';\underline{Z}'')$ must lie in one of the intersections we started with, we see that $H(\underline{X}'';\underline{Z}'') \subseteq H(\underline{X};\underline{Z}) \cap H(\underline{X}';\underline{Z}')$.

Therefore the two sets are equal and for $(\underline{X};\underline{Z})$ and $(\underline{X}';\underline{Z}') \in \Sigma(X)$ we now have that $H(\underline{X};\underline{Z}) \cap H(\underline{X}';\underline{Z}') = H(\underline{X}'';\underline{Z}'') \in \mathcal{Z}$, and so $\mathcal{Z}$ is closed under intersection. $\square$

We can now prove the following.

LEMMA 2.11. *If p is a prime dividing the order of* Hyp(r) *then $\mathcal{Y}$ is a p-Mackey system for* Hyp(r).

PROOF. (M1) If $\lambda = (\emptyset; 1^r) \in \Lambda_2^+(n,r)$ then $\mathrm{Hyp}(\lambda) = \{1\}$, so the identity subgroup is in $\mathcal{Y}$.
(M2)(i) and (ii) We know by the preceding lemmas that $\mathcal{Z}$ is closed under both conjugation and intersection. So as $\mathcal{Z} = \mathcal{Y}$, we also have that $\mathcal{Y}$ is closed under both conjugation and intersection.
(M3) If $\lambda = (r; \emptyset) \in \Lambda_2^+(n,r)$ then $\mathrm{Hyp}(\lambda)$ is just $\mathrm{Hyp}(r)$ itself and $\mathrm{Hyp}(r) \in \mathcal{Y}$. Therefore all Sylow p-subgroups are contained in a member of $\mathcal{Y}$ as they are all contained in $\mathrm{Hyp}(r)$.

□

2.2. The Count for $\mathcal{Y}$

Before we go on to count trivial source modules, we prove the following two useful lemmas concerning binomials. The first we call the first index lemma.

LEMMA 2.12. *For all primes p, and $i \geqslant 0$*

$$p \nmid x \Rightarrow p \nmid \binom{xp^i}{p^i}.$$

PROOF. Now $\binom{xp^i}{p^i} = \frac{(xp^i)!}{p^i!(xp^i-p^i)!} = \frac{((x-1)p^i+1)((x-1)p^i+2)\cdots(xp^i)}{1.2.3\ldots p^i}$

$$= \frac{((x-1)p^i+1)((x-1)p^i+2)\cdots((x-1)p^i+p)\cdots((x-1)p^i+2p)\cdots((x-1)p^i+p^i)}{1.2.3\ldots p^i}$$
$$= \frac{((x-1)p^i+1)((x-1)p^i+2)\cdots p((x-1)p^{i-1}+1)\cdots p((x-1)p^{i-1}+2)\cdots p^i x}{1.2.3\ldots p^i}.$$

Now each power of p in the denominator can be cancelled with the corresponding power of p in the numerator. This leaves us with a denominator coprime to p, and a numerator in which the only factors that could be divisible by p are those of the form

$(x-1)p^{i-j}+z$, where $i>j$ and $z \in [1, p-1]$, or just x itself.

But when $i>j$ we have that p divides $(x-1)p^{i-j}$ but does not divide z, as p and z are coprime. Therefore p does not divide $(x-1)p^{i-j}+z$ here. But also we have assumed that p does not divide x, so therefore the numerator is not divisible by p and the lemma follows. □

The next lemma is imaginatively called the second index lemma.

LEMMA 2.13. *For all primes p we have that p divides $\binom{p^n}{a}$ for all $0<a<p^n$.*

PROOF. It suffices to prove this for $0 \leqslant a \leqslant \frac{1}{2}p^n$, as $\binom{u}{v} = \binom{u}{u-v}$ for all $u \geqslant 1, v \geqslant 0, u \geqslant v$. Now

$\binom{p^n}{a} = \frac{p^n!}{(p^n-a)!a!} = \frac{(p^n-a+1)(p^n-a+2)\cdots(p^n)}{a!} = \frac{(p^n-(a-1))(p^n-(a-2))\cdots(p^n-1)(p^n)}{a(a-1)(a-2)\cdots 1}$.

So if for some $0<i<a-1$ we have that p divides a factor of the denominator $a-i$, then p also divides $p^n-(a-i)$ in the numerator, and hence these can be cancelled. This leaves us with a denominator which is coprime to p, apart from maybe the factor a, and a numerator with p^n as the only factor divisible by p.

So if p does not divide a, then the numerator is coprime to p, and the numerator has factor p^n and hence p divides $\binom{p^n}{a}$. If p divides a then $a = p^l m$, where $(p, m) = 1$ and $1 \leqslant l \leqslant n-1$, as $0 \leqslant a \leqslant \frac{1}{2}p^n$. So as the numerator has factor p^n, we see that in this case $\binom{p^n}{a}$ has a factor p^{n-l} where $n>l$, so that p divides $\binom{p^n}{a}$, and the lemma is proved.

□

(These two lemmas can also be proved using Lucas's Formula)

Our first count is of trivial source modules over a field k of odd characteristic. From now on in this section, p is an odd prime and is the characteristic of our field k. We now define a certain type of composition, which will be important for the rest of this chapter.

DEFINITION. Let p be any prime. Then even though it is not a partition, we call a composition $\lambda = (\mu, \nu) \in \Lambda_2(n,r)$ a p-power partition if it has $\mu = \emptyset$ and all parts of ν being powers of p. Denote the set of these by $\Pi_p(n,r)$, so that if $\lambda \in \Pi_p(n,r)$ then for some $a_i \geqslant 0$ we get that $\lambda = (\emptyset; 1^{a_0} p^{a_1} (p^2)^{a_2} \dots)$.

Recall that for a p-Mackey System $\mathcal{M}$, the vertices are precisely the set

$$\mathcal{M}_0 = \{P \in \mathcal{M} \mid p \text{ divides } |P : B| \, \forall B \in \mathcal{M} \text{ with } B < P\}.$$

We can now describe the vertices of our p-Mackey System $\mathcal{Y}$.

LEMMA 2.14. *For odd primes p, we have that $P \in \mathcal{Y}_0 \iff P = (\mathrm{Hyp}(\lambda))^{\sigma}$, where $\sigma \in \mathrm{Hyp}(r)$ and $\lambda \in \Pi_p(n,r)$.*

PROOF. (Note that all other cases follow from the case $\sigma = 1$.)

Let $P = (\mathrm{Hyp}(\lambda))^{\sigma}$ as above. Then to show P is in $\mathcal{Y}_0$ it is enough to show that p divides the index of $S(a) \times S(p^n - a)$ in $S(p^n)$ for all $n \geqslant 0$ and $0 < a < p^n$. This is because each P above is isomorphic to a product of symmetric groups of degree a power of p, and all subgroups of P will be isomorphic to a product of Young subgroups of these symmetric groups of degree a power of p. Now

$$|S(p^n) : S(p^n - a) \times S(a)| = \tfrac{p^n!}{(p^n-a)!a!} = \tbinom{p^n}{a}$$

But by second index lemma p divides $\binom{p^n}{a}$ for all $n \geqslant 0$ and $0 < a < p^n$ therefore p divides the index of $S(a) \times S(p^n - a)$ in $S(p^n)$ for all $n \geqslant 0$ and $0 < a < p^n$ as required, and $P \in \mathcal{Y}_0$.

Conversely, say that $\lambda \in \Lambda_2(n,r) \setminus \Pi_p(n,r)$. Then either $\mu \neq \emptyset$ or ν has some part which is not a power of p.

Firstly, suppose that $\mu \neq \emptyset$. Then $\mathrm{Hyp}(\lambda)$ is of the form $\mathrm{Hyp}(s) \times J$ for some non-negative integer s and some $J \leqslant \mathrm{Hyp}(r-s)$. Then $\mathrm{Hyp}(\lambda)$ has subgroup $S(s) \times J$ which is also in $\mathcal{Y}$ and has index 2^s in $\mathrm{Hyp}(s) \times J$. But $p \nmid 2^s$ so $\mathrm{Hyp}(\lambda) \notin \mathcal{Y}_0$.

Secondly, suppose $\mu = \emptyset$ so that ν has some part ν_i which is not a power of p. Then $\nu_i = xp^i$ for some $i \geqslant 0$ and $x \neq 0, 1$ which is coprime to p. Then $\mathrm{Hyp}(\lambda)$ is isomorphic to a subgroup of $\mathrm{Hyp}(r)$ of the form $S(xp^i) \times L$ for some $L \leqslant \mathrm{Hyp}(r - xp^i)$. Now $S(xp^i)$ has subgroup $S(p^i) \times S(xp^i - p^i)$ and this has index $\frac{(xp^i)!}{p^i!(xp^i-p^i)!} = \binom{xp^i}{p^i}$ in $S(xp^i)$. But from the first index lemma,

$$(p, x) = 1 \Rightarrow p \nmid \binom{xp^i}{p^i}.$$

So $S(xp^i) \times L$ has subgroup $(S(p^i) \times S(xp^i - p^i) \times L) \in \mathcal{Y}$ of index not divisible by p and we are done.

$\square$

This means that for $\mathcal{Y}_0'$, a set of representatives of conjugacy classes of subgroups in $\mathcal{Y}_0$, we can take the subgroups $\mathrm{Hyp}(\lambda)$, where $\lambda \in \Pi_p(n, r)$. We now take a look at normalizers.

LEMMA 2.15. *If* $\lambda = (\emptyset : 1^{a_0} p^{a_1} (p^2)^{a_2} \dots) \in \Pi_p(n, r)$ *then* $\frac{N_{\mathrm{Hyp}(r)} \mathrm{Hyp}(\lambda)}{\mathrm{Hyp}(\lambda)} \cong \mathrm{Hyp}(a_0) \times \mathrm{Hyp}(a_1) \times \mathrm{Hyp}(a_2) \times \cdots$.

PROOF. Let $\lambda \in \Pi_p(n, r)$, and let $H = \mathrm{Hyp}(\lambda)$. Also say that $X = \{1, 2, \dots, r, \overline{1}, \overline{2}, \dots, \overline{r}\}$, and let $A_1, A_2, \dots, A_m, \overline{A}_1, \overline{A}_2, \dots, \overline{A}_m$ be the H-orbits of X, so that if $A_i = \{a, a+1, \dots, b\}$ then we have that $\overline{A}_i = \{\overline{a}, \overline{a+1}, \dots, \overline{b}\}$. For $i \geqslant 0$ set

$$B_i = \{A_j, \overline{A}_j \text{ such that } |A_j| = p^i\}$$

so that $|B_i| = a_i$, and $\mathrm{Hyp}(B_i) \cong \mathrm{Hyp}(a_i)$.

Let $N(H) = N_{\mathrm{Hyp}(r)}(H)$. Now $N(H)$ permutes the H-orbits of X of the same size, and if $\sigma A_i = A_j$ for some $\sigma \in N(H)$, then $\sigma \overline{A_i} = \overline{A_j}$.

So $N(H)$ acts on each set B_i, and we can define the homomorphism

$$\phi : N(H) \mapsto \mathrm{Hyp}(B_0) \times \mathrm{Hyp}(B_1) \times \mathrm{Hyp}(B_2) \times \cdots$$

via

$$\phi(\sigma)(A_j) = \sigma(A_j), \text{ where } A_j \in B_i, \text{ and } \sigma \in N(H).$$

Now $\ker\phi = \{\sigma \in N(H) | \phi(\sigma)(A_j) = A_j, \forall A_j\}$ $= \{\sigma \in N(H) | \sigma(A_j) = A_j, \forall A_j\} = H$. So now we only need to worry about the image. Say $A_j, A_l \in B_i$ with $A_j \neq \overline{A}_l$, and say also that $A_j = \{x_1, x_2, \ldots, x_{p^i}\}$ and $A_l = \{z_1, z_2, \ldots, z_{p^i}\}$. Define $\sigma \in \mathrm{Hyp}(r)$ via $\sigma(x_s) = z_s$ and $\sigma(z_s) = x_s$ for $1 \leqslant s \leqslant p^i$, so of course we also have that $\overline{A}_j, \overline{A}_l \in B_i$ with $\sigma(\overline{x}_s) = \overline{z}_s$ and $\sigma(\overline{z}_s) = \overline{x}_s$. Let σ fix all other members of X, and let $h \in H$ then

$$(\sigma h \sigma^{-1})A_j = (\sigma h \sigma)A_j = \sigma h A_l = \sigma A_l = A_j$$

and similarly for $\overline{A}_j$, so $(\sigma h \sigma^{-1}) \in H$, so $\sigma \in N(H)$.

Now say $A_j \in B_i$ so $\overline{A}_j \in B_i$ too, and these sets are as above. Define $\pi \in \mathrm{Hyp}(r)$ via $\pi(x_s) = \overline{x}_s$ and $\pi(\overline{x}_s) = x_s$ for all $1 \leqslant s \leqslant p^i$, and let π fix all other elements. Again let $h \in H$ then

$$(\pi h \pi^{-1})A_j = (\pi h \pi)A_j = \pi h \overline{A}_j = \pi \overline{A}_j = A_j$$

and similarly for $\overline{A}_j$, so that $\pi \in N(H)$.

Now all standard generating involutions of each $\mathrm{Hyp}(B_i)$ can be realized as a $\phi(\sigma)$ or a $\phi(\pi)$ for some suitable choice of $\sigma, \pi \in N(H)$, so that ϕ is onto. So by the homomorphism theorem we have that

$$\begin{aligned} \tfrac{N(H)}{H} &= \mathrm{Hyp}(B_0) \times \mathrm{Hyp}(B_1) \times \mathrm{Hyp}(B_2) \times \cdots \\ &\cong \mathrm{Hyp}(a_0) \times \mathrm{Hyp}(a_1) \times \mathrm{Hyp}(a_2) \times \cdots. \end{aligned}$$

and the proof is complete.

□

Now for $\lambda \in \Pi_p(n, r)$, let $\frac{N_{\mathrm{Hyp}(r)}\mathrm{Hyp}(\lambda)}{\mathrm{Hyp}(\lambda)}$ be denoted by $N(\lambda)$, so that $N(\lambda) \cong \mathrm{Hyp}(a_0) \times \mathrm{Hyp}(a_1) \times \mathrm{Hyp}(a_2) \times \cdots$. Recall that for a finite group G, we let $\#_p(G)$ denote the number of p'-classes of G. Then applying Grabmeier's count, or theorem 1.6, we have that $\mathrm{TSM}(\mathcal{Y}) = \sum_{\lambda\in\Pi_p(n,r)} \#_p(N(\lambda))$.

Now [**1**, 2.5] tells us that when p is odd $\#_p(\mathrm{Hyp}(r))$ is equal to the number of bipartitions $(\mu;\nu) \in \Lambda_2^+(n,r)$ such that both μ and ν are column p-regular partitions. Therefore as $N(\lambda) \cong \mathrm{Hyp}(a_0)\times\mathrm{Hyp}(a_1)\times \mathrm{Hyp}(a_2)\times\cdots$ we get that $\#_p(N(\lambda))$ is equal to the number of pairs (ϕ,ψ) where $\phi = (\phi(0),\phi(1),\phi(2),\dots)$ and $\psi = (\psi(0),\psi(1),\psi(2),\dots)$ are sequences of column p-regular partitions with $|\phi(i)| + |\psi(i)| = a_i$ for all $i \geqslant 0$. So from Grabmeier's count, as above, we have that

$$\begin{aligned}\mathrm{TSM}(\mathcal{Y}) &= \textstyle\sum_{\lambda\in\Pi_p(n,r)} \#_p(N(\lambda))\\ &= \textstyle\sum_{\lambda\in\Pi_p(n,r)} \text{number of pairs } (\phi,\psi) \text{ as above}\end{aligned}$$

which means that $\mathrm{TSM}(\mathcal{Y})$ is the number of triples (λ,ϕ,ψ) where $\lambda \in \Pi_p(n,r)$ and ϕ and ψ are as above. Call the set of such triples Ψ.

Now recall that each partition μ in $\Lambda^+(n,r)$ has a unique p-adic expansion, that is to say that each is uniquely expressible as $\mu = \sum_{i\geqslant 0} p^i\mu(i)$ with each $\mu(i)$ a column p-regular partition. We define a map

$$f : \Psi \to \Lambda_2^+(n,r) \text{ via } (\lambda,\phi,\psi) \mapsto (\alpha;\beta)$$

where $\alpha = \sum_{i\geqslant 0} p^i\phi(i)$ and $\beta = \sum_{i\geqslant 0} p^i\psi(i)$.

LEMMA 2.16. *f is a bijection between Ψ and $\Lambda_2^+(n,r)$.*

PROOF. Due to the uniqueness of p-adic expansions, we know that f must be both well defined and injective.

Now say that $(\alpha;\beta) \in \Lambda_2^+(n,r)$. Then as above we can uniquely write $\alpha = \sum_{i\geqslant 0} p^i\xi(i)$ and $\beta = \sum_{i\geqslant 0} p^i\eta(i)$ where $\xi(i)$ and $\eta(i)$ are column p-regular partitions for all $i \geqslant 0$. Now $r = |\alpha| + |\beta| = \sum_{i\geqslant 0} p^i(|\xi(i)| + |\eta(i)|)$, so by putting $b_i = |\xi(i)| + |\eta(i)|$ for all $i \geqslant 0$ we retrieve a triple (λ', ξ, η) where $\lambda' = (\emptyset; 1^{b_0} p^{b_1} (p^2)^{b_2} \dots) \in \Pi_p(n,r)$ and $\xi = (\xi(0), \xi(1), \xi(2), \dots)$ and $\eta = (\eta(0), \eta(1), \eta(2), \dots)$ are sequences of column p-regular partitions with $|\xi(i)| + |\eta(i)| = b_i$. Therefore $(\lambda', \xi, \eta) \in \Psi$ with $f((\lambda', \xi, \eta)) = (\alpha; \beta)$, and f is surjective also. □

COROLLARY 2.17. *Over a field of odd characteristic* $\mathrm{TSM}(\mathcal{Y}) = |\Lambda_2^+(n,r)|$.

The case $p = 2$

Now we consider the case $p = 2$. We already have all the lemmas about binomials that we will need here, so we go straight into a slightly different version of p-power partitions, which are called double 2-power partitions.

DEFINITION. Even though it is not a partition, we call a composition $\lambda = (\mu; \nu) \in \Lambda_2(n,r)$ a double 2-power partition if it has all parts being powers of 2, written in ascending numerical order. Denote the set of these by $\Upsilon_2(n,r)$, so that if $\lambda \in \Upsilon_2(n,r)$ then for some $a_i, b_j \geqslant 0$ we get that $\lambda = (1^{a_0} 2^{a_1} 4^{a_2} \dots; 1^{b_0} 2^{b_1} 4^{b_2} \dots)$.

So for each $\lambda \in \Upsilon_2(n,r)$, we get an infant subgroup $\mathrm{Hyp}(\lambda) \in \mathcal{Y}$ with $\mathrm{Hyp}(\lambda) \cong \mathrm{Hyp}(1)^{a_0} \times \mathrm{Hyp}(2)^{a_1} \times \mathrm{Hyp}(4)^{a_2} \times \dots \times S(1)^{b_0} \times S(2)^{b_1} \times \dots$. We now prove a lemma concerning vertices similar to the one in the odd primes count.

LEMMA 2.18. *When $p = 2$ we have that $P \in \mathcal{Y}_0$ if and only if $P = \mathrm{Hyp}(\lambda)^\sigma$, for some $\lambda \in \Upsilon_2(n,r)$ and $\sigma \in \mathrm{Hyp}(r)$.*

PROOF. Again, all cases follow from the case $\sigma = 1$. Since we already know that 2 divides $|S(2^m) : S(2^m - a) \times S(a)|$ for all $m \geqslant 0$

and $0 < a < 2^m$, and also that $|\operatorname{Hyp}(2^m) : S(2^m)| = 2^{2m}$, to prove that subgroups of this form really are vertices it suffices to check that 2 divides $|\operatorname{Hyp}(2^m) : \operatorname{Hyp}(2^m - a) \times \operatorname{Hyp}(a)|$ for all $m \geqslant 0$ and $0 < a < 2^m$. Now we know that the order of the hyperoctahedral group is given by $|\operatorname{Hyp}(r)| = r!2^r$ so $|\operatorname{Hyp}(2^m) : \operatorname{Hyp}(2^m - a) \times \operatorname{Hyp}(a)| = \frac{2^m!2^{2m}}{((2^m-a)!2^{2m-a})(a!2^a)} = \frac{2^m!}{(2^m-a)!a!} = \binom{2^m}{a}$. But from the second index lemma we know that 2 divides $\binom{2^m}{a}$ for all $0 < a < 2^m$. Therefore if $\lambda \in \Upsilon_2(n,r)$ then $\operatorname{Hyp}(\lambda) \in \mathcal{Y}_0$.

Conversely we must now show that if λ is not in $\Upsilon_2(n,r)$ then $\operatorname{Hyp}(\lambda) \notin \mathcal{Y}_0$. If λ is not in $\Upsilon_2(n,r)$ then there are two possibilities;
1. λ has a part ν_j with $\nu_j = x2^i$, for some $i \geqslant 0$ and $x \neq 0,1$ with $(2,x) = 1$. But we already know that this implies that $\operatorname{Hyp}(\lambda) \notin \mathcal{Y}_0$ from the odd characteristic case (the same argument with symmetric groups works here).
2. The second possibility is that λ has a part μ_j with $\mu_j = x2^i$, for some $i \geqslant 0$ and $x \neq 0,1$ with $(2,x) = 1$. In this case $\operatorname{Hyp}(x2^i)$ has subgroup $\operatorname{Hyp}(2^i) \times \operatorname{Hyp}(2^i(x-1))$ whose index in $\operatorname{Hyp}(x2^i)$ is

$$\frac{2^{x2^i}(x2^i)!}{2^i!2^{2^i}(x2^i-2^i)!2^{x2^i-2^i}} = \frac{x2^i!}{2^i!(x2^i-2^i)!} = \binom{x2^i}{2^i}.$$

So from the first index lemma as 2 does not divide x, we have that 2 does not divide $\binom{x2^i}{2^i}$. Therefore $\operatorname{Hyp}(x2^i)$ has a subgroup whose index in $\operatorname{Hyp}(x2^i)$ is not divisible by 2 and so if λ is not in $\Upsilon_2(n,r)$ then $\operatorname{Hyp}(\lambda)$ is not a member of $\mathcal{Y}$.

□

So when $p = 2$, we can take the subgroups $\operatorname{Hyp}(\lambda)$, where $\lambda \in \Upsilon_2(n,r)$, as representatives of the conjugacy classes of subgroups in $\mathcal{Y}_0$ i.e. we can take $\mathcal{Y}'_0$ as $\{\operatorname{Hyp}(\lambda) | \lambda \in \Upsilon_2(n,r)\}$. The next step in Grabmeier's method is to calculate the normalizers of these subgroups, modulo the subgroups themselves.

LEMMA 2.19. *If* $\lambda \in \Upsilon_2(n,r)$ *then* $\frac{N_{\mathrm{Hyp}(r)}(\mathrm{Hyp}(\lambda))}{\mathrm{Hyp}(\lambda)} \cong$ $S(a_0) \times S(a_1) \times S(a_2) \times \cdots \times \mathrm{Hyp}(b_0) \times \mathrm{Hyp}(b_1) \times \cdots$. *For ease we refer to this group as* $N(\lambda)$.

PROOF. Now $\lambda \in \Upsilon_2(n,r)$ so $\lambda = (1^{a_0}2^{a_1}4^{a_2}\dots;1^{b_0}2^{b_1}\dots)$ for some $a_i, b_j \geqslant 0$. To keep things tidy we give some subgroups shorter names, so that $H = S(1)^{a_0} \times S(2)^{a_1} \times S(4)^{a_2} \times \cdots \times S(1)^{b_0} \times S(2)^{b_1} \times \cdots$, $H' = \mathrm{Hyp}(\lambda)$, and $N(H') = N_{\mathrm{Hyp}(r)}(H')$. As usual we also put $X = \{1,2,\dots,r,\overline{1},\overline{2},\dots,\overline{r}\}$, so that $A_1, A_2, \dots, A_t, \overline{A}_1, \overline{A}_2, \dots, \overline{A}_t$, say, are the H-orbits of X, in a similar manner to the odd characteristic case. Then $C_1, C_2, \dots, C_s, A_{s+1}, A_{s+2}, \dots, A_t, \overline{A}_{s+1}, \dots, \overline{A}_t$ are the H'-orbits of X, where $s \leqslant t$ and for $j \leqslant s$ we put $C_j = A_j \sqcup \overline{A}_j$. Now for all $i \geqslant 0$ we set

$$B_i = \{A_j, \overline{A}_j \mid |A_j| = 2^i\} \text{ and } D_i = \{C_j \mid |C_j| = 2^{i+1}\},$$

so that $|B_i| = 2b_i$ and $|D_i| = a_i$, with $\mathrm{Hyp}(B_i) \cong \mathrm{Hyp}(b_i)$ and $\mathrm{Sym}(D_i) \cong S(a_i)$. We know from the odd characteristic case that $N(H')$ acts on each set B_i, and for each $g \in N(H')$ and $C_j = A_j \sqcup \overline{A}_j \in B_i$ we have

$$g(C_j) = g(A_j \sqcup \overline{A}_j) = g(A_j) \sqcup g(\overline{A}_j) = A_l \sqcup \overline{A}_l$$

for some $A_l, \overline{A}_l \in B_i$. Then as $C_l = A_l \sqcup \overline{A}_l \in D_i$ we have that $N(H')$ acts on each D_i too. Then letting $H'' = \mathrm{Sym}(D_0) \times \mathrm{Sym}(D_1) \times \mathrm{Sym}(D_2) \times \cdots \times \mathrm{Hyp}(B_0) \times \mathrm{Hyp}(B_1) \times \cdots$, we can now define the map

$$\phi : N(H') \to H'' \text{ via } \phi(\sigma)(A_j) = \sigma(A_j) \text{ for all } 1 \leqslant j \leqslant t.$$

It is clear that $\ker\phi = H'$. We must now find the image of ϕ. Say $C_j, C_l \in D_i$ with $C_j = \{x_1, x_2, \dots, x_{2^i}, \overline{x}_1, \overline{x}_2, \cdots, \overline{x}_{2^i}\}$, and $C_l = \{z_1, z_2, \dots, z_{2^i}, \overline{z}_1, \overline{z}_2, \cdots, \overline{z}_{2^i}\}$. Then we define an element $\sigma \in \mathrm{Hyp}(r)$ via $\sigma(x_u) = z_u$ and $\sigma(z_u) = x_u$ for each $1 \leqslant u \leqslant 2^i$, so that also $\sigma(\overline{x}_u) = \overline{z}_u$ and $\sigma(\overline{z}_u) = \overline{x}_u$, as $\sigma \in \mathrm{Hyp}(r)$. Let σ fix all other elements of X, and let $h \in H'$. Then

$$\sigma^{-1}h\sigma C_j = \sigma^{-1}hC_l = \sigma^{-1}C_l = C_j.$$

So $\sigma^{-1}h\sigma \in H'$ and $\sigma \in N(H')$, and all standard generating involutions of $\mathrm{Sym}(D_0) \times \mathrm{Sym}(D_1) \times \mathrm{Sym}(D_2) \times \cdots$ can be realized as a $\phi(\sigma)$ for suitably chosen C_j and C_l in D_i.

Also, all standard generating involutions of $\mathrm{Hyp}(B_0) \times \mathrm{Hyp}(B_1) \times \mathrm{Hyp}(B_2) \times \cdots$ occur in $\phi(N(H'))$ by identical arguments to those given in the odd characteristic case. Therefore $\mathrm{im}\phi = H''$, and so by the homomorphism theorem we have that $\frac{N(H')}{H'} \cong H''$, and since $H'' \cong N(\lambda)$ the lemma follows. □

Now by [**1**, 2.5] we have that the number of $2'$-classes in $\mathrm{Hyp}(r)$, denoted $\#_2(\mathrm{Hyp}(r))$, is equal to the number of column 2-regular partitions of r i.e. $|\Lambda^+(n,r)_{col}|$. Also, by [**22**] for example, the number of $2'$-classes in $\mathrm{Sym}(r)$, denoted $\#_2(\mathrm{Sym}(r))$, is equal to the number of 2-regular partitions in $\Lambda^+(n,r)$, or $|\Lambda^+(n,r)_{col}|$.

So therefore if $\lambda \in \Upsilon_2(n,r)$, we see that $\#_2(N(\lambda))$ is equal to the number of pairs (ϕ,ψ) where $\phi = (\phi(0),\phi(1),\phi(2),\dots)$ and $\psi = (\psi(0),\psi(1),\psi(2),\dots)$ are sequences of column 2-regular partitions with each $\phi(i) \in \Lambda^+(a_i,a_i)$, and each $\psi(i) \in \Lambda^+(b_i,b_i)$ for each $i \geqslant 0$. Then applying theorem 1.6, Grabmeier's method gives us that when $p = 2$

$$\mathrm{TSM}(\mathcal{Y}) = \textstyle\sum_{\lambda\in\Upsilon_2(n,r)} \text{number of pairs } (\phi,\psi) \text{ as above}$$

which means that $\mathrm{TSM}(\mathcal{Y})$ is the number of triples (λ,ϕ,ψ), where $\lambda \in \Upsilon_2(n,r)$ and ϕ and ψ are as above. Call the set of such triples $\mathcal{T}$, and then define the map

$$F : \mathcal{T} \to \Lambda_2^+(n,r) \text{ via } F((\lambda,\phi,\psi)) = (\alpha;\beta)$$

where $\alpha = \sum_{i\geqslant 0} 2^i\phi(i)$ and $\beta = \sum_{i\geqslant 0} 2^i\psi(i)$.

LEMMA 2.20. *F is a bijection between $\mathcal{T}$ and $\Lambda_2^+(n,r)$.*

PROOF. By uniqueness of p-adic expansions of partitions F is well defined and distinct triples in $\mathcal{T}$ will map to distinct elements of $\Lambda_2^+(n,r)$ under F. Therefore F is injective.

Say that $(\delta;\epsilon) \in \Lambda_2^+(n,r)$. Then δ and ϵ both have unique 2-adic expansions $\delta = \sum_{i\geqslant 0} 2^i\xi(i)$ and $\epsilon = \sum_{i\geqslant 0} 2^i\eta(i)$, with each $\xi(i)$ and $\eta(i)$ a 2-regular partition for each $i \geqslant 0$. Now put $\xi = (\xi(0),\xi(1),\xi(2),\dots)$ and $\eta = (\eta(0),\eta(1),\eta(2),\dots)$. From these we can recover a double 2-power partition $\gamma = (1^{|\xi(0)|}2^{|\xi(1)|}4^{|\xi(2)|}\dots;1^{|\eta(0)|}2^{|\eta(1)|}4^{|\eta(2)|}\dots) \in \Upsilon_2(n,r)$, and therefore we have constructed a triple $(\gamma,\xi,\eta) \in \mathcal{T}$ with $(\delta;\epsilon) = F((\gamma,\xi,\eta))$.

So every element of $\Lambda_2^+(n,r)$ is the image under F of some element of $\mathcal{T}$, and F is surjective. $\square$

COROLLARY 2.21. *When $p = 2$ we have that* $\mathrm{TSM}(\mathcal{Y}) = |\Lambda_2^+(n,r)|$. *Therefore we have now shown this holds over any suitable field of prime characteristic.*

2.3. Counting trivial source modules for $\mathcal{V}$

We now repeat this method with $\mathcal{V}$, which we recall is the set of Young subgroups viewed as subgroups of the hyperoctahedral group.

Firstly, we again go back to schisms to give us an alternative description of $\mathcal{V}$. Recall that for a set X with a regular involution $^-$, we have the set $\Sigma(X)$ of schism sequences of X, and the Mackey system $\mathcal{Z}$ of schism subgroups of $\mathrm{Hyp}(X)$. Let $\Sigma(X)_{res}$ be the subset of $\Sigma(X)$ consisting of all schism sequences of X with $\underline{X} = \emptyset$. Then put

$$\mathcal{Z}_{res} = \{H(\underline{X};\underline{Z}) \mid (\underline{X};\underline{Z}) \in \Sigma(X)_{res}\} = \{H(\emptyset;\underline{Z}) \mid (\emptyset;\underline{Z}) \in \Sigma(X)\}.$$

So if $A \in \mathcal{Z}_{res}$ then A is $S(Z_1) \times S(Z_2) \times \cdots \times S(Z_m)$ as described earlier. We now prove many lemmas.

LEMMA 2.22. $H(\emptyset;\underline{Z}) = \{\sigma \in \mathrm{Hyp}(X) \mid \sigma Z_j = Z_j \ \forall 1 \leqslant j \leqslant m\}$

PROOF. $H(\emptyset;\underline{Z}) = \{\sigma \in \mathrm{Hyp}(X) \mid \sigma\emptyset = \emptyset, \sigma Z_j = Z_j \ \forall 1 \leqslant j \leqslant m\}$
$= \{\sigma \in \mathrm{Hyp}(X) \mid \sigma Z_j = Z_j \ \forall 1 \leqslant j \leqslant m\}$. □

LEMMA 2.23. *$\mathcal{Z}_{res}$ is closed under conjugation.*

PROOF. Let $H(\emptyset;\underline{Z}) \in \mathcal{Z}_{res}$ and $g \in \mathrm{Hyp}(X)$, then $H(\emptyset;\underline{Z})^g = H(g\emptyset; g\underline{Z}) = H(\emptyset; g\underline{Z}) \in \mathcal{Z}_{res}$. □

LEMMA 2.24. $\mathcal{Z}_{res} = \mathcal{V}$.

LEMMA 2.25. *$\mathcal{Z}_{res}$ is closed under intersection.*

PROOF. Similarly, these two lemma are proved by putting $\underline{X} = \emptyset$, and correspondingly $\mu = \emptyset$, into our proofs that $\mathcal{Z} = \mathcal{Y}$, and that $\mathcal{Z}$ is closed under intersection. □

To summarize, we have that $\mathcal{Z}_{res}$ is closed under conjugation and intersection, and is equal to $\mathcal{V}$. Now we come to Mackey systems and counting. As before, we deal with the odd characteristic case first. We can now prove the following.

LEMMA 2.26. *If p is an odd prime then $\mathcal{V}$ is a p-Mackey system of* $\mathrm{Hyp}(r)$.

PROOF. (M1) If $\lambda = (\emptyset; 1^r) \in \Lambda_2^+(n,r)$ then $\mathrm{Hyp}(\lambda) = \{1\}$, so the identity subgroup is in $\mathcal{V}$.
(M2) (i) and (ii) We now know that $\mathcal{Z}_{res}$ is closed under both intersection and conjugation, and is equal to $\mathcal{V}$. Therefore $\mathcal{V}$ is closed under conjugation and intersection too.
(M3) If p is odd then all Sylow p-subgroups of $\mathrm{Hyp}(r)$ are in fact subgroups of $S(r) < \mathrm{Hyp}(r)$. Now if $\lambda = (\emptyset; r) \in \Lambda_2^+(n,r)$ then $\mathrm{Hyp}(\lambda) = S(r)$ so that $S(r) \in \mathcal{V}$ and all Sylow p-subgroups of $\mathrm{Hyp}(r)$ are contained in a member of $\mathcal{V}$. □

Our first job in the count is to find the vertices $\mathcal{V}_0$.

LEMMA 2.27. *When p is odd we have that $P \in \mathcal{V}_0$ if and only if $P = \mathrm{Hyp}(\lambda)^\sigma$ for some $\lambda \in \Pi_p(n,r)$, and $\sigma \in \mathrm{Hyp}(r)$.*

PROOF. This is proved using an identical method to that of finding $\mathcal{Y}_0$ for odd p, except in the converse the case $\mu \neq \emptyset$ is not applicable here, and need not be considered. □

This lemma tells us that when p is odd $\mathcal{V}_0$ is equal to $\mathcal{Y}_0$, so applying Grabmeier's count to $\mathcal{V}$ will give us the following.

COROLLARY 2.28. *When p is odd* $\mathrm{TSM}(\mathcal{V}) = \mathrm{TSM}(\mathcal{Y}) = |\Lambda_2^+(n,r)|$.

We now must consider the case $p = 2$. In this case it is no longer true that $\mathcal{V}$ is a p-Mackey system for $\mathrm{Hyp}(r)$. For example, when $r = n = 2$, we see that $\mathrm{Hyp}(2)$ has Sylow 2-subgroup $\mathrm{Hyp}(2)$, which is not a subgroup of any member of $\mathcal{V}$, as subgroups in $\mathcal{V}$ are either conjugate to $S(2)$ or are the identity subgroup. However, we can get around this problem. We let $\mathcal{V}^*$ denote $\mathcal{V} \cup \{\mathrm{Hyp}(r)\}$.

LEMMA 2.29. *If $p = 2$ then $\mathcal{V}^*$ is a p-Mackey system for* $\mathrm{Hyp}(r)$.

PROOF. (M1) We know that $\{1\} \in \mathcal{V}$ so therefore $\{1\} \in \mathcal{V}^*$ also.
(M2)(i) We know that $\mathcal{V}$ is closed under conjugation, and also that $\mathrm{Hyp}(r)$ is conjugate to itself under conjugation by any member of itself. Therefore $\mathcal{V}^*$ is closed under conjugation.
(M2)(ii) $\mathcal{V}$ itself is closed under intersection, and for all $A \in \mathcal{V}$ we have $A \cap \mathrm{Hyp}(r) = A$. Therefore $\mathcal{V}^*$ is closed under intersection.
(M3) $\mathrm{Hyp}(r) \in \mathcal{V}^*$ so all Sylow 2-subgroups of $\mathrm{Hyp}(r)$ are trivially contained in a member of $\mathcal{V}^*$. □

Now say that $\mathcal{M}$ is a p-Mackey system for a finite group G, such that G is an $\mathcal{M}$-vertex. Then G must be a vertex of some indecomposable kG-module M, and here M must have trivial source, so that M must be a direct summand of $k\uparrow_G^G$, which is just k itself. Therefore $M = k$, so that G is the vertex of the trivial kG-module k. From the definition of $\mathcal{M}$-vertices, we get that k is not a direct summand of any $k\uparrow_H^G$, where $H < G$. Hence G as a vertex contributes the trivial source module k to our count, and only this module as $\#_2(\frac{N_{\mathrm{Hyp}(r)}(\mathrm{Hyp}(r))}{\mathrm{Hyp}(r)}) = 1$, and no other $\mathcal{M}$-vertex contributes k to our count as above. Therefore, in this situation, $\mathrm{TSM}(\mathcal{M}) = \mathrm{TSM}(\mathcal{M}\setminus\{G\}) + 1$, or

$$\mathrm{TSM}(\mathcal{M}\setminus\{G\}) = \mathrm{TSM}(\mathcal{M}) - 1.$$

So as this is exactly the set up we have in the case of $\mathcal{V}^*$, we have that

$$\mathrm{TSM}(\mathcal{V}) = \mathrm{TSM}(\mathcal{V}^*) - 1.$$

So the method we will use to count trivial source modules over $\mathcal{V}$ is to count them over $\mathcal{V}^*$ and subtract 1 from the result. Here goes.

LEMMA 2.30. $\mathcal{V}_0^* = \mathcal{V}_0 \cup \{\mathrm{Hyp}(r)\}$.

PROOF. In $\mathcal{V}^*$, the group $\mathrm{Hyp}(r)$ has subgroups $\mathrm{Hyp}(\lambda)^\sigma$, where $\lambda \in \Pi_2(n,r)$, and $\sigma \in \mathrm{Hyp}(r)$. But these are all contained in some $\mathrm{Hyp}(r)$-conjugate of $S(r)$, which has index 2^r in $\mathrm{Hyp}(r)$. Hence all proper subgroups of $\mathrm{Hyp}(r)$ in $\mathcal{V}^*$ have 2 dividing their index in $\mathrm{Hyp}(r)$. Therefore $\mathrm{Hyp}(r) \in \mathcal{V}_0^*$, and so $\mathcal{V}_0^* = \mathcal{V}_0 \cup \{\mathrm{Hyp}(r)\}$. $\square$

So for representatives of conjugacy classes in $\mathcal{V}_0^*$ we can take the subgroups $\mathrm{Hyp}(r)$ and $\mathrm{Hyp}(\lambda)$ where $\lambda \in \Pi_2(n,r)$. Now applying Grabmeier's count we have that $\mathrm{TSM}(\mathcal{V}^*)$ is equal to

$$\textstyle\sum_{P\in\mathcal{V}_0^*} \#_2(\frac{N_{\mathrm{Hyp}(r)}(P)}{P}) =$$
$$\#_2(\tfrac{N_{\mathrm{Hyp}(r)}(\mathrm{Hyp}(r))}{\mathrm{Hyp}(r)}) + \textstyle\sum_{\lambda\in\Pi_2(n,r)} \#_2(\frac{N_{\mathrm{Hyp}(r)}(\mathrm{Hyp}(\lambda))}{\mathrm{Hyp}(\lambda)})$$

$$= 1 + \textstyle\sum_{\lambda \in \Pi_2(n,r)} \#_2(\frac{N_{\mathrm{Hyp}(r)}(\mathrm{Hyp}(\lambda))}{\mathrm{Hyp}(\lambda)})$$

so that by our previous discussions

$$\mathrm{TSM}(\mathcal{V}) = \textstyle\sum_{\lambda \in \Pi_2(n,r)} \#_2(\frac{N_{\mathrm{Hyp}(r)}(\mathrm{Hyp}(\lambda))}{\mathrm{Hyp}(\lambda)}).$$

Of course, we also remember that for $\lambda \in \Pi_2(n,r)$ we have that

$$\frac{N_{\mathrm{Hyp}(r)}(\mathrm{Hyp}(\lambda))}{\mathrm{Hyp}(\lambda)} \cong \mathrm{Hyp}(a_0) \times \mathrm{Hyp}(a_1) \times \mathrm{Hyp}(a_2) \times \cdots$$

and that the number of $2'$-classes of $\mathrm{Hyp}(r)$ is equal to the number of column 2-regular partitions of r. Therefore $\#_2(\frac{N_{\mathrm{Hyp}(r)}(\mathrm{Hyp}(\lambda))}{\mathrm{Hyp}(\lambda)})$ is equal to the number of sequences $\phi = (\phi(0), \phi(1), \phi(2), \dots)$ of 2-regular partitions $\phi(i)$, where for each $i \geqslant 0$ we have $\phi(i) \in \Lambda^+(a_i, a_i)$. So $\mathrm{TSM}(\mathcal{V})$ is equal to the number of pairs (λ, ϕ), where $\lambda \in \Pi_2(n,r)$ and ϕ is as above. Call the set of such pairs $\gimel$. Then we define a map

$$\mathcal{F} : \gimel \to \Lambda^+(n,r) \text{ via } (\lambda, \phi) \mapsto \textstyle\sum_{i \geqslant 0} 2^i \phi(i).$$

LEMMA 2.31. *$\mathcal{F}$ is a bijection between $\gimel$ and $\Lambda^+(n,r)$.*

PROOF. $\mathcal{F}$ is well defined and injective by uniqueness of 2-adic expansions.

Now say $\alpha \in \Lambda^+(n,r)$ then α has a unique 2-adic expansion $\alpha = \sum_{i \geqslant 0} 2^i \xi(i)$, with $\xi(i)$ a column 2-regular partition for each $i \geqslant 0$. So by setting $b_i = |\xi(i)|$ we get a $\lambda' = (\emptyset; 1^{b_0} 2^{b_1} 4^{b_2} \dots) \in \Pi_2(n,r)$, and also if $\xi = (\xi(0), \xi(1), \xi(2), \dots)$ we have a pair $(\lambda', \xi) \in \gimel$ with $\mathcal{F}((\lambda', \xi)) = \alpha$. So $\mathcal{F}$ is also surjective. $\square$

COROLLARY 2.32. *When $p = 2$ we have that* $\mathrm{TSM}(\mathcal{V}) = |\Lambda^+(n,r)|$.

In summary, we have the following theorem.

THEOREM 2.33. *Over a field of odd characteristic we have that*

$$\mathrm{TSM}(\mathcal{Y}) = \mathrm{TSM}(\mathcal{V}) = |\Lambda_2^+(n,r)|.$$

However, when the characteristic of k is 2, we have that $\mathrm{TSM}(\mathcal{Y}) = |\Lambda_2^+(n,r)|$ *and* $\mathrm{TSM}(\mathcal{V}) = |\Lambda^+(n,r)|$, *so that* $\mathrm{TSM}(\mathcal{Y}) \neq \mathrm{TSM}(\mathcal{V})$.

CHAPTER 3

$\bigoplus M^\lambda$ in Disguise

We now make our $k\,\mathrm{Hyp}(r)$-module $Y^{\otimes r}$ into a module for the type B Hecke algebra $\mathcal{H}$. Recall that $\mathrm{Hyp}(r)$ acts on the generators of $Y^{\otimes r}$ via $y_{\mathbf{i}}s_j = y_{\mathbf{i}s_j}$ and that the set $Z(n)$ is ordered via $\overline{n} < \cdots < \overline{1} < \widehat{1} < \cdots < \widehat{n} < 1 < \cdots < n$. Then we have the following.

LEMMA 3.1. *$\mathcal{H}$ acts on $Y^{\otimes r}$ via*

$$y_{\mathbf{i}}T_{s_a} = \begin{cases} q y_{\mathbf{i}s_a} & \textit{if } a > 0 \textit{ and } i_a \leqslant i_{a+1} \\ y_{\mathbf{i}s_a} + (q-1)y_{\mathbf{i}} & \textit{if } a > 0 \textit{ and } i_a > i_{a+1} \\ Q y_{\mathbf{i}s_a} & \textit{if } a = 0 \textit{ and } i_1 \notin \{\overline{1}, \overline{2}, \dots, \overline{n}\} \\ y_{\mathbf{i}s_a} + (Q-1)y_{\mathbf{i}} & \textit{if } a = 0 \textit{ and } i_1 \in \{\overline{1}, \overline{2}, \dots, \overline{n}\} \end{cases}$$

Extending linearly makes $Y^{\otimes r}$ into a right $\mathcal{H}$-module.

PROOF. It suffices to check the case $r = 2$, as the action of each generator affects at most two tensors. Note that all relations not involving T_{s_0} follow from [**3**, 3.1.4], swapping the left action onto the right. Also note that $T_{s_j}T_{s_0} = T_{s_0}T_{s_j}$ is obvious for any $j \geqslant 2$, as the two generators act on separate coordinates of the tensor. This just leaves us with two relations to check.

Now if $a \in \{\widehat{1}, \widehat{2}, \dots, \widehat{n}\}$ then

$$(y_a \otimes y_b)(T_{s_0} - 1)(T_{s_0} - Q) = (Q(y_a \otimes y_b) - (y_a \otimes y_b))(T_{s_0} - Q) =$$
$$= Q^2(y_a \otimes y_b) - Q(y_a \otimes y_b) - Q(Q(y_a \otimes y_b) + Q(y_a \otimes y_b)) = 0$$

and if $a \notin \{\widehat{1},\widehat{2},\ldots,\widehat{n}\}$ the same result follows from [**19**].

So $(T_{s_0}-1)(T_{s_0}-Q)=0$ holds. Also if $a,b \notin \{\widehat{1},\widehat{2},\ldots,\widehat{n}\}$ then

$$(y_a\otimes y_b)T_{s_1}T_{s_0}T_{s_1}T_{s_0}=(y_a\otimes y_b)T_{s_0}T_{s_1}T_{s_0}T_{s_1}$$

follows, again from [**19**].

This just leaves another 7 cases to check.

Case 1 $a\in\{\widehat{1},\widehat{2},\ldots,\widehat{n}\}$, $b\in\{\overline{1},\overline{2},\ldots,\overline{n}\}$.

Now $(y_a\otimes y_b)T_{s_0}T_{s_1}T_{s_0}T_{s_1}=Q(y_a\otimes y_b)T_{s_1}T_{s_0}T_{s_1}$

$=Q((y_b\otimes y_a)+(q-1)(y_a\otimes y_b))T_{s_0}T_{s_1}$

$=Q((y_{\bar b}\otimes y_a)+(Q-1)(y_b\otimes y_a)+Q(q-1)(y_a\otimes y_b))T_{s_1}$

$=Q((y_a\otimes y_{\bar b})+(q-1)(y_{\bar b}\otimes y_a)+q(Q-1)(y_a\otimes y_b)$

$+Q(q-1)(y_b\otimes y_a)+Q(q-1)^2(y_a\otimes y_b))$,

and also

$(y_a\otimes y_b)T_{s_1}T_{s_0}T_{s_1}T_{s_0}=((y_b\otimes y_a)+(q-1)(y_a\otimes y_b))T_{s_0}T_{s_1}T_{s_0}$

$=((y_{\bar b}\otimes y_a)+(Q-1)(y_b\otimes y_a)+Q(q-1)(y_a\otimes y_b))T_{s_1}T_{s_0}$

$=((y_a\otimes y_{\bar b})+(q-1)(y_{\bar b}\otimes y_a)$

$+q(Q-1)(y_a\otimes y_b)+Q(q-1)(y_b\otimes y_a)+Q(q-1)^2(y_a\otimes y_b))T_{s_0}$

$=Q((y_a\otimes y_{\bar b})+(q-1)(y_b\otimes y_a)+q(Q-1)(y_a\otimes y_b)$

$+Q(q-1)^2(y_a\otimes y_b)+(q-1)(y_{\bar b}\otimes y_a)+(Q-1)(q-1)(y_b\otimes y_a))$

and the two expressions are equal as $Q(q-1)+Q(Q-1)(q-1)=Q^2(q-1)$.

Case 2 $a\in\{\widehat{1},\widehat{2},\ldots,\widehat{n}\}$, $b\in\{1,2,\ldots,n\}$.

Now $(y_a\otimes y_b)T_{s_1}T_{s_0}T_{s_1}T_{s_0}=q(y_b\otimes y_a)T_{s_0}T_{s_1}T_{s_0}=qQ(y_{\bar b}\otimes y_a)T_{s_1}T_{s_0}$

$=q^2Q(y_a\otimes y_{\bar b})T_{s_0}=q^2Q^2(y_a\otimes y_{\bar b})$,

and also

$(y_a\otimes y_b)T_{s_0}T_{s_1}T_{s_0}T_{s_1}=Q(y_a\otimes y_b)T_{s_1}T_{s_0}T_{s_1}=qQ(y_b\otimes y_a)T_{s_0}T_{s_1}$

$=qQ^2(y_{\bar b}\otimes y_a)T_{s_1}=q^2Q^2(y_a\otimes y_{\bar b})$

and the two expressions are equal.

Case 3 $a \in \{\overline{1},\overline{2},\ldots,\overline{n}\}$, $b \in \{\widehat{1},\widehat{2},\ldots,\widehat{n}\}$.

Now $(y_a \otimes y_b)T_{s_0}T_{s_1}T_{s_0}T_{s_1} = ((y_{\overline{a}} \otimes y_b) + (Q-1)(y_a \otimes y_b))T_{s_1}T_{s_0}T_{s_1}$

$= ((y_b \otimes y_{\overline{a}}) + (q-1)(y_{\overline{a}} \otimes y_b) + q(Q-1)(y_b \otimes y_a))T_{s_0}T_{s_1}$

$= (Q(y_b \otimes y_{\overline{a}}) + Qq(q-1)(y_a \otimes y_b) + Qq(Q-1)(y_b \otimes y_a))T_{s_1}$

$= Qq(y_{\overline{a}} \otimes y_b) + Q\ \ (q-1)(y_b \otimes y_a) + Qq(Q-1)(y_a \otimes y_b)$

$+Qq(q-1)(Q-1)((y_b \otimes y_a)$

$= Qq(y_{\overline{a}} \otimes y_b) + Qq(Q-1)(y_a \otimes y_b) + Q^2q(q-1)(y_b \otimes y_a)$

and also

$(y_a \otimes y_b)T_{s_1}T_{s_0}T_{s_1}T_{s_0} = q(y_b \otimes y_a)T_{s_0}T_{s_1}T_{s_0} = Qq(y_b \otimes y_a)T_{s_1}T_{s_0}$

$= Qq((y_a \otimes y_b) + (q-1)(y_b \otimes y_a))T_{s_0}$

$= Qq(y_{\overline{a}} \otimes y_b) + Qq(Q-1)(y_a \otimes y_b) + Q^2q(q-1)(y_b \otimes y_a)$

and the two expressions are equal.

Case 4 $a \in \{1,2,\ldots,n\}$, $b \in \{\widehat{1},\widehat{2},\ldots,\widehat{n}\}$.

Now $(y_a \otimes y_b)T_{s_0}T_{s_1}T_{s_0}T_{s_1} = Q(y_{\overline{a}} \otimes y_b)T_{s_1}T_{s_0}T_{s_1}$

$= Qq(y_b \otimes y_{\overline{a}})T_{s_0}T_{s_1} = Q^2q(y_b \otimes y_{\overline{a}})T_{s_1}$

$= Q^2q((y_{\overline{a}} \otimes y_b) + (q-1)(y_b \otimes y_{\overline{a}}))$

and also

$(y_a \otimes y_b)T_{s_1}T_{s_0}T_{s_1}T_{s_0} = ((y_b \otimes y_a) + (q-1)(y_a \otimes y_b))T_{s_0}T_{s_1}T_{s_0}$

$= Q((y_b \otimes y_a) + (q-1)(y_{\overline{a}} \otimes y_b))T_{s_1}T_{s_0}$

$= Qq((y_a \otimes y_b) + (q-1)(y_b \otimes y_{\overline{a}}))T_{s_0}$

$= Q^2q((y_{\overline{a}} \otimes y_b) + (q-1)(y_b \otimes y_{\overline{a}}))$

and the two expressions are equal.

Case 5 $a,b \in \{\widehat{1},\widehat{2},\ldots,\widehat{n}\}$, $a<b$.

Now $(y_a \otimes y_b)T_{s_0}T_{s_1}T_{s_0}T_{s_1} = Q(y_a \otimes y_b)T_{s_1}T_{s_0}T_{s_1} = qQ(y_b \otimes y_a)T_{s_0}T_{s_1}$

$= qQ^2(y_b \otimes y_a)T_{s_1} = qQ^2((y_a \otimes y_b) + (q-1)(y_b \otimes y_a))$

and also

$(y_a \otimes y_b)T_{s_1}T_{s_0}T_{s_1}T_{s_0} = q(y_b \otimes y_a)T_{s_0}T_{s_1}T_{s_0} = qQ(y_b \otimes y_a)T_{s_1}T_{s_0}$

$= qQ((y_a \otimes y_b) + (q-1)(y_b \otimes y_a))T_{s_0} = qQ^2((y_a \otimes y_b) + (q-1)(y_b \otimes y_a))$
and the two expressions are equal.

Case 6 $a, b \in \{\widehat{1}, \widehat{2}, \ldots, \widehat{n}\}$, $a = b$.
Now $(y_a \otimes y_a)T_{s_0}T_{s_1}T_{s_0}T_{s_1} = Q(y_a \otimes y_a)T_{s_1}T_{s_0}T_{s_1}$
$= qQ(y_a \otimes y_a)T_{s_0}T_{s_1} = qQ^2(y_a \otimes y_a)T_{s_0} = q^2Q^2(y_a \otimes y_a)$
and also
$(y_a \otimes y_a)T_{s_1}T_{s_0}T_{s_1}T_{s_0} = q(y_a \otimes y_a)T_{s_0}T_{s_1}T_{s_0} = qQ(y_a \otimes y_a)T_{s_1}T_{s_0}$
$= q^2Q(y_a \otimes y_a)T_{s_0} = q^2Q^2(y_a \otimes y_a)$
and the two expressions are equal.

Case 7 $a, b \in \{\widehat{1}, \widehat{2}, \ldots, \widehat{n}\}$, $a > b$.
Now $(y_a \otimes y_b)T_{s_0}T_{s_1}T_{s_0}T_{s_1} = Q(y_a \otimes y_b)T_{s_1}T_{s_0}T_{s_1} = Q((y_b \otimes y_a)$
$+(q-1)(y_a \otimes y_b))T_{s_0}T_{s_1} = Q^2((y_b \otimes y_a) + (q-1)(y_a \otimes y_b))T_{s_1}$
$= Q^2(q(y_a \otimes y_b) + (q-1)(y_b \otimes y_a) + (q-1)^2(y_a \otimes y_b))$
and also
$(y_a \otimes y_b)T_{s_1}T_{s_0}T_{s_1}T_{s_0} = ((y_b \otimes y_a) + (q-1)(y_a \otimes y_b))T_{s_0}T_{s_1}T_{s_0}$
$= Q((y_b \otimes y_a) + (q-1)(y_a \otimes y_b))T_{s_1}T_{s_0}$
$= Q(q(y_a \otimes y_b) + (q-1)(y_b \otimes y_a) + (q-1)^2(y_a \otimes y_b))T_{s_0}$
$= Q^2(q(y_a \otimes y_b) + (q-1)(y_b \otimes y_a) + (q-1)^2(y_a \otimes y_b))$
and the two expressions are equal.

□

Now let $\mathbf{i} \in I_B(n,r)$, so that $y_{\mathbf{i}}$ is a basis element in $Y^{\otimes r}$. Then for $\lambda = (\mu;\nu) \in \Lambda_2(n,r)$ we say that $\mathbf{i} = (i_1, i_2, \ldots, i_r)$ has content λ if $\mathbf{i}$ consists of precisely μ_j $\widehat{j}$'s and precisely ν_j of either j's, $\bar{j}$'s, or a combination of the two, so that $\mathbf{i}(\lambda)$ has content λ. Then for any $\lambda \in \Lambda_2(n,r)$ define

$$Y_{\lambda}^{\otimes r} = k - \mathrm{span}\{y_{\mathbf{i}} \in Y^{\otimes r} \mid \mathbf{i} \text{ has content } \lambda\}.$$

As each element in $I_B(n,r)$ has unique content λ for some $\lambda \in \Lambda_2(n,r)$

this gives us a direct sum decomposition

$$Y^{\otimes r} = \bigoplus\nolimits_{\lambda \in \Lambda_2(n,r)} Y_\lambda^{\otimes r}.$$

LEMMA 3.2. $Y_\lambda^{\otimes r} = y_{\mathbf{i}(\lambda)}\mathcal{H}$.

PROOF. If $\mathbf{j} \in I_{\mathbf{B}}(n,r)$ has content λ then $\mathbf{j} = \mathbf{i}(\lambda)w$ for some $w \in \mathrm{Hyp}(r)$. But of course $\{\mathbf{i}(\lambda)w \,|\, w \in \mathrm{Hyp}(r)\} = \{\mathbf{i}(\lambda)d \,|\, d \in \mathrm{Dist}(\lambda)\}$. Hence $Y_\lambda^{\otimes r} = k - \mathrm{span}\{y_{\mathbf{i}(\lambda)w} \,|\, w \in \mathrm{Dist}(\lambda)\}$, and $\dim Y_\lambda^{\otimes r} = \frac{|\mathrm{Hyp}(r)|}{|\mathrm{Hyp}(\lambda)|}$.

Now it is easy to see from the action of $\mathcal{H}$ on $Y^{\otimes r}$ that each member of $y_{\mathbf{i}(\lambda)}\mathcal{H}$ is a linear combination of elements indexed by r-tuples of weight λ. Therefore $y_{\mathbf{i}(\lambda)}\mathcal{H} \subseteq Y_\lambda^{\otimes r}$. Now let $d \in \mathrm{Dist}(\lambda)$. Then again from the action of $\mathcal{H}$ on $Y^{\otimes r}$ it is easy to see that

$$y_{\mathbf{i}(\lambda)}T_d = a_d y_{\mathbf{i}(\lambda)d} + \sum\nolimits_{l(w)<l(d)} a_w y_{\mathbf{i}(\lambda)w}$$

where $a_d \in k^\times$ and $a_w \in k$. Of course, each of these elements $y_{\mathbf{i}(\lambda)w}$ can be written as $y_{\mathbf{i}(\lambda)w} = y_{\mathbf{i}(\lambda)d'}$ where $d' \in \mathrm{Dist}(\lambda)$ and $l(d') \leqslant l(w)$. Therefore

$$y_{\mathbf{i}(\lambda)}T_d = a_d y_{\mathbf{i}(\lambda)d} + \sum\nolimits_{l(d')<l(d)} a_{d'} y_{\mathbf{i}(\lambda)d'}$$

for $a_d \in k^\times$, $a_{d'} \in k$ and $d' \in \mathrm{Dist}(\lambda)$.

Now from the above expression it is easy to see that the elements $y_{\mathbf{i}(\lambda)}T_d$, where $d \in \mathrm{Dist}(\lambda)$, are all linearly independent. Then since there are $|\,\mathrm{Dist}(\lambda)| = \frac{|\mathrm{Hyp}(r)|}{|\mathrm{Hyp}(\lambda)|}$ of them we get that $y_{\mathbf{i}(\lambda)}\mathcal{H}$ is a right $\mathcal{H}$-submodule of $Y_\lambda^{\otimes r}$ of dimension at least $\frac{|\mathrm{Hyp}(r)|}{|\mathrm{Hyp}(\lambda)|}$ and so $y_{\mathbf{i}(\lambda)}\mathcal{H} = Y_\lambda^{\otimes r}$. □

Therefore we now have a direct sum decomposition of our tensor space as $Y^{\otimes r} = \bigoplus_{\lambda \in \Lambda_2(n,r)} Y_\lambda^{\otimes r} = \bigoplus_{\lambda \in \Lambda_2(n,r)} y_{\mathbf{i}(\lambda)}\mathcal{H}$.

We now recall two completely unrelated facts. Firstly we remember from Chapter 1 that Du and Scott (and similarly Dipper, James and Mathas) associate to each $\lambda \in \Lambda_2(n,r)$ a permutation module $M^\lambda = m_\lambda \mathcal{H}$. By [5, 3.3] this module has k-basis $\{m_\lambda T_d \mid d \in \mathrm{Dist}(\lambda)\}$, so that in particular M^λ and $Y_\lambda^{\otimes r}$ have the same dimension. Secondly we recall that if R is a ring and M is a right R-module then for any $m \in M$ the right annihilator $\mathrm{ann}_R(m)$ of m is the set $\mathrm{ann}_R(m) = \{r \in R \mid mr = 0\}$. We now define two homomorphisms ϕ and ψ. Let $\phi : \mathcal{H} \to m_\lambda\mathcal{H}$ and $\psi : \mathcal{H} \to y_{\mathrm{i}(\lambda)}\mathcal{H}$ via $\phi(T_h) = m_\lambda T_h$ and $\psi(T_h) = y_{\mathrm{i}(\lambda)}T_h$. Then both ϕ and ψ are onto so that by the homomorphism theorem for modules we have that

$$m_\lambda\mathcal{H} \cong \frac{\mathcal{H}}{\ker\phi} = \frac{\mathcal{H}}{\mathrm{ann}_R(m_\lambda)} \text{ and } y_{\mathrm{i}(\lambda)}\mathcal{H} \cong \frac{\mathcal{H}}{\ker\psi} = \frac{\mathcal{H}}{\mathrm{ann}_R(y_{\mathrm{i}(\lambda)})}.$$

Note that since $\dim m_\lambda\mathcal{H} = \dim y_{\mathrm{i}(\lambda)}\mathcal{H} = \frac{|\,\mathrm{Hyp}(r)|}{|\,\mathrm{Hyp}(\lambda)|}$ we have that

$$\dim \mathrm{ann}_R(m_\lambda) = \dim \mathrm{ann}_R(y_{\mathrm{i}(\lambda)}) = |\,\mathrm{Hyp}(r)| - \frac{|\,\mathrm{Hyp}(r)|}{|\,\mathrm{Hyp}(\lambda)|}.$$

We now wish to calculate $\mathrm{ann}_R(m_\lambda)$ and $\mathrm{ann}_R(y_{\mathrm{i}(\lambda)})$. To this end we make a definition.

DEFINITION. Recall from chapter 1 that for $\lambda \in \Lambda_2(n,r)$ we have the Young subgroup $S(\lambda)$ of $\mathrm{Hyp}(r)$. Now for $\lambda \in \Lambda_2(n,r)$ let A_λ be the set $\{\sum_{i=1}^a \mu_i\,,\, |\mu| + \sum_{i=1}^a \nu_i \mid a \in [1,n]\}$ and put $A'_\lambda = [1,r] \setminus A_\lambda$. Then $S(\lambda)$ is generated by the set $H_1(\lambda) = \{s_a \mid a \in A'_\lambda\}$, so that $S(\lambda) \cong S(\mu_1) \times S(\mu_2) \times \cdots \times S(\mu_n) \times S(\nu_1) \times \cdots \times S(\nu_n)$.

LEMMA 3.3. *If $h \in S(\lambda)$ then*

(i) $m_\lambda T_h = q^{l(h)} m_\lambda$ and

(ii) $y_{\mathrm{i}(\lambda)}T_h = q^{l(h)} y_{\mathrm{i}(\lambda)}$.

PROOF. It suffices to check both statements when h is a standard generator of $S(\lambda)$ i.e. when $h \in H_1(\lambda)$.

(i) Recall that $m_\lambda = x_\lambda\pi_\lambda = \pi_\lambda x_\lambda$, by [**6**, 3.7] for example. Now also, it follows from [**28**, 3.3] that if s_i is a generator of $S(\lambda)$ then $x_\lambda T_{s_i} = qx_\lambda$. Therefore, for $s_i \in H_1(\lambda)$ we have that

$$m_\lambda T_{s_i} = x_\lambda\pi_\lambda T_{s_i} = \pi_\lambda x_\lambda T_{s_i} = q\pi_\lambda x_\lambda = qm_\lambda,$$

and part (i) follows by induction.

(ii) Let $s_a \in H_1(\lambda)$. Then $s_a \in \mathrm{Hyp}(\lambda)$ so that s_a stabilizes $y_{\mathbf{i}(\lambda)}$ and also we must have that $\mathbf{i}(\lambda)_a = \mathbf{i}(\lambda)_{a+1}$. Then

$$y_{\mathbf{i}(\lambda)}T_{s_a} = qy_{\mathbf{i}(\lambda)s_a} = qy_{\mathbf{i}(\lambda)},$$

and by induction part (ii) follows too. $\square$

COROLLARY 3.4. *The elements* **of the set** $\{(T_w - q^{l(w)})T_d \mid w \in S(\lambda) \setminus \{1\}, d \in \mathrm{Dist}(\lambda)\}$ *are all members of both* $\mathrm{ann}_R(m_\lambda)$ *and* $\mathrm{ann}_R(y_{\mathbf{i}(\lambda)})$.

Recall that for each $1 \leqslant i \leqslant r$ we have the element $t_i = s_{i-1}s_{i-2}\dots \dots s_0s_1\dots s_{i-1}$. Then we let $C_\lambda(i) = \langle t_i, t_{i+1}, \dots, t_{|\mu|}\rangle$, so that $C_\lambda(1)$ is Du and Scott's group C_λ (see [**11**, 2.2]). Let $u_i = s_{i-1}s_{i-2}\dots s_1$ and $v_i = s_1s_2\dots s_{i-1}$, so that $t_i = u_is_0v_i$. Then by [**11**, proof of 3.2.2] we have that if $t_i \in C_\lambda(1)$ then

$$m_\lambda T_{t_i} = Qm_\lambda T_{u_i}T_{v_i}.$$

We can now tell a similar tale for $y_{\mathbf{i}(\lambda)}$.

LEMMA 3.5. *If* $t_i \in C_\lambda(1)$ *then* $y_{\mathbf{i}(\lambda)}T_{t_i} = Qy_{\mathbf{i}(\lambda)}T_{u_i}T_{v_i}$.

PROOF. $y_{\mathbf{i}(\lambda)}T_{t_i} = y_{\mathbf{i}(\lambda)}T_{u_i}T_{s_0}T_{v_i} = q^{i-1}y_{\mathbf{i}(\lambda)u_i}T_{s_0}T_{v_i}$
$= Qq^{i-1}y_{\mathbf{i}(\lambda)u_is_0}T_{v_i} = Qq^{i-1}y_{\mathbf{i}(\lambda)u_i}T_{v_i} = Qy_{\mathbf{i}(\lambda)}T_{u_i}T_{v_i}$. $\square$

Now each non-identity element w of $C_\lambda(1)$ can be uniquely written as $w = t_iw'$ for some $1 \leqslant i \leqslant |\mu|$ and $w' \in C_\lambda(i+1)$ so that each $(T_{t_i} - QT_{u_i}T_{v_i})T_{w'}$ annihilates both m_λ and $y_{\mathbf{i}(\lambda)}$. Now $C_\lambda(1)$ forms a set of distinguished right coset representatives of $S(\lambda)$ in $\mathrm{Hyp}(\lambda)$. Therefore each element of $\mathrm{Hyp}(r)$ can be uniquely written as

$$ht_iw'd \text{ for } h \in S(\lambda), t_i \in C_\lambda(1), w' \in C_\lambda(i+1), d \in \mathrm{Dist}(\lambda).$$

So the elements $(T_h - q^{l(h)})T_{t_i}T_{w'}T_d$ with $h \in S(\lambda) \setminus \{1\}, d, w', t_i$ as above, and $(T_{t_i} - QT_{u_i}T_{v_i})T_{w'}T_d$ with $t_i \in C_\lambda(1) \setminus \{1\}, w', d$ as above all annihilate both m_λ and $y_{\mathrm{i}(\lambda)}$, and are all linearly independent. Note that $|\operatorname{Hyp}(\lambda) : S(\lambda)| = 2^{|\mu|}$. Then the fact that there are

$$\begin{aligned}
&(|S(\lambda)| - 1)2^{|\mu|}\tfrac{|\operatorname{Hyp}(r)|}{|\operatorname{Hyp}(\lambda)|} + (2^{|\mu|} - 1)\tfrac{|\operatorname{Hyp}(r)|}{|\operatorname{Hyp}(\lambda)|} \\
&= (|S(\lambda)| - 1)\tfrac{|\operatorname{Hyp}(\lambda)|}{|S(\lambda)|}\tfrac{|\operatorname{Hyp}(r)|}{|\operatorname{Hyp}(\lambda)|} + (\tfrac{|\operatorname{Hyp}(\lambda)|}{|S(\lambda)|} - 1)\tfrac{|\operatorname{Hyp}(r)|}{|\operatorname{Hyp}(\lambda)|} \\
&= |\operatorname{Hyp}(r)| - \tfrac{|\operatorname{Hyp}(r)|}{|S(\lambda)|} + \tfrac{|\operatorname{Hyp}(r)|}{|S(\lambda)|} - \tfrac{|\operatorname{Hyp}(r)|}{|\operatorname{Hyp}(\lambda)|} = |\operatorname{Hyp}(r)| - \tfrac{|\operatorname{Hyp}(r)|}{|\operatorname{Hyp}(\lambda)|} \\
&= \dim \ker\phi = \dim \ker\psi
\end{aligned}$$

of these independent elements means that they must form a k-basis of both $\operatorname{ann}_R(m_\lambda)$ and $\operatorname{ann}_R(y_{\mathrm{i}(\lambda)})$ so therefore $\operatorname{ann}_R(m_\lambda) = \operatorname{ann}_R(y_{\mathrm{i}(\lambda)})$ and we have the following.

THEOREM 3.6. *$Y_\lambda^{\otimes r} \cong M^\lambda$.*

PROOF. $Y_\lambda^{\otimes r} = y_{\mathrm{i}(\lambda)}\mathcal{H} \cong \frac{\mathcal{H}}{\operatorname{ann}_R(y_{\mathrm{i}(\lambda)})} = \frac{\mathcal{H}}{\operatorname{ann}_R(y_{m_\lambda})} \cong m_\lambda\mathcal{H} = M^\lambda$. $\square$

COROLLARY 3.7. *$Y^{\otimes r} = \bigoplus_{\lambda\in\Lambda_2(n,r)} Y_\lambda^{\otimes r} \cong \bigoplus_{\lambda\in\Lambda_2(n,r)} M^\lambda$ and $\mathcal{S}_q^2(n,r) \cong \operatorname{End}_{\mathcal{H}}(Y^{\otimes r})$.*

CHAPTER 4

Morita Equivalences and More

In the words of Benson [**2**, 3.11.2] :

LEMMA 4.1. *An indecomposable kG-module has trivial source if and only if it is a direct summand of a permutation module.*

We now see how this relates to our situation. Let Ω be a finite set on which a finite group G acts, so that $M = k - \mathrm{span}\Omega$ is a permutation module for G. All G-permutation modules arise in this way. Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$ be the orbits of Ω with respect to our action of G, so each $k\mathcal{O}_i$ is a kG-submodule of M, and

$$M = k\mathcal{O}_1 \oplus k\mathcal{O}_2 \oplus \cdots \oplus k\mathcal{O}_m.$$

Let $x \in \mathcal{O}_i$ and say x has stabilizer $H = \mathrm{stab}_G(x)$. If k also denotes the trivial kH-module, we have the following.

LEMMA 4.2. *Let $x \in \mathcal{O}_i$,* so that $\mathrm{stab}_G(x) = H_i$. Then

$$k \uparrow_{H_i}^{G} \cong k\mathcal{O}_i \text{ as } kG\text{-modules.}$$

PROOF. We know the dimensions are the same from the orbit-stabilizer theorem. Now define a map $\phi : k \uparrow_{H_i}^{G} \to k\mathcal{O}_i$ via $gH_i \mapsto g.x$. This is onto as $\mathcal{O}_i = \{g.x | g \in G\}$, and $\ker\phi = H_i$, so that ϕ is an isomorphism, by the homomorphism theorem.

□

So we have that $M = \bigoplus_i k\mathcal{O}_i = \bigoplus_i k \uparrow_{H_i}^{G}$, where $x_i \in \mathcal{O}_i$, and $\mathrm{stab}_G(x_i) = H_i$. So if M is a permutation module for G with set of point stabilizers $\mathcal{M}$ then counting the number of G-components of

M, over a field of characteristic $p > 0$, amounts to counting the number of isomorphism classes of indecomposable trivial source modules over $\mathcal{M}$. If we assume that $\mathcal{M}$ is a p-Mackey system for G, which we will from now on, we can use Grabmeier's methods to do the count. In other words if $\mathrm{Comp}(M)$ denotes the number of components of a kG-permutation module M, with notation as above we have the following.

LEMMA 4.3. $\mathrm{Comp}(M) = \mathrm{TSM}(\mathcal{M})$.

We can now apply this to our earlier results.

THEOREM 4.4. *When p, the characteristic of k, is odd we have $\mathrm{Comp}(Y^{\otimes r}) = \mathrm{Comp}(V^{\otimes r}) = |\Lambda_2^+(n,r)|$. However if $p = 2$ then $\mathrm{Comp}(Y^{\otimes r}) = |\Lambda_2^+(n,r)|$ and $\mathrm{Comp}(V^{\otimes r}) = |\Lambda^+(n,r)|$, so the number of components of our two tensor spaces are not equal.*

We can now prove the main theorems of this chapter.

THEOREM 4.5. *Let $q = Q = 1$, and let p, the characteristic of k, be odd. Then $\mathrm{End}_{\mathcal{H}}(Y^{\otimes r})$ and $\mathrm{End}_{\mathcal{H}}(V^{\otimes r})$ are Morita equivalent i.e. in the group case Du-Scott's q-Schur^2 algebra, Richard Green's hyperoctahedral Schur algebra, and Dipper, James and Mathas's (Q,q)-Schur algebra are all Morita equivalent. However when $p = 2$ the hyperoctahedral Schur algebra is not Morita equivalent to the other two.*

PROOF. When p is odd, we have shown that $\mathrm{Comp}(Y^{\otimes r})$ is equal to $\mathrm{Comp}(V^{\otimes r})$, and therefore since $V^{\otimes r}$ is a $k\,\mathrm{Hyp}(r)$-submodule of $Y^{\otimes r}$, we see that $V^{\otimes r}$ and $Y^{\otimes r}$ have the same components, but with different multiplicities. So applying theorem 1.3 from the introduction, we see that when $q = Q = 1$, $\mathrm{End}_{\mathcal{H}}(Y^{\otimes r})$ and $\mathrm{End}_{\mathcal{H}}(V^{\otimes r})$ are Morita equivalent. Then of course $\mathrm{End}_{\mathcal{H}}(V^{\otimes r}) = \mathcal{S}^{\mathrm{Hyp}}(n,r)$ and $\mathrm{End}_{\mathcal{H}}(Y^{\otimes r}) =$

$\mathcal{S}^2(n,r)$, and it is already known that $\mathcal{S}^2_q(n,r)$ and $\mathcal{S}_{Q,q}(n,r)$ are Morita equivalent when $q, Q \neq 0$, so the main part of the theorem follows. The second part follows as when $p = 2$ we have already seen that $\mathrm{Comp}(Y^{\otimes r}) < \mathrm{Comp}(V^{\otimes r})$.

□

Now we can apply Fitting's Theorem, or Theorem 1.2, to give the following.

THEOREM 4.6. *Over a field of odd characteristic, the Schur algebras $\mathcal{S}^{\mathrm{Hyp}}(n,r)$, $\mathcal{S}^2(n,r)$ and $\mathcal{S}_{1,1}(n,r)$ all have precisely $|\Lambda_2^+(n,r)|$ inequivalent irreducible modules.*

In characteristic 2, the algebra $\mathcal{S}^{\mathrm{Hyp}}(n,r)$ has $|\Lambda^+(n,r)|$ irreducibles, whilst $\mathcal{S}^2(n,r)$ and $\mathcal{S}_{1,1}(n,r)$ both have $|\Lambda_2^+(n,r)|$ inequivalent irreducible modules.

CHAPTER 5

Coalgebras

In this chapter our main aim is to follow methods of [**3**] to construct a graded bialgebra, denoted $C_{q,Q}(3n)$, which has the dual of its r^{th} homogeneous part isomorphic to the q-Schur^2 algebra.

Now, in [**3**], Dipper and Donkin construct a graded bialgebra $A_q(n)$, with r^{th} homogeneous part $A_q(n,r)$. This has the property that its dual, $A_q(n,r)^*$, is isomorphic to the q-Schur algebra $\mathcal{S}_q(n,r)$. We quickly review this construction, which is given in Chapter 1. Start with $F(n)$, the free k-algebra in non-commuting indeterminates $x_{i,j}$, where i and j run over $\underline{n}$. This is naturally a bialgebra. Now let J be the ideal of $F(n)$ generated by elements of the form

$$x_{i,l}x_{j,m} - qx_{j,m}x_{i,l} \quad \text{for } i > j \text{ and } l \leqslant m,$$
$$x_{i,l}x_{j,m} - x_{j,m}x_{i,l} - (q-1)x_{j,l}x_{i,m} \quad \text{for } i > j \text{ and } l > m,$$
$$x_{i,l}x_{i,m} - x_{i,m}x_{i,l} \quad \text{for all } i, l, m,$$

where $i, j, l, m \in \underline{n}$. Then let $A_q(n) = F(n)/J$, so that writing $X_{i,j}$ for the canonical image $x_{i,j} + J$ of $x_{i,j}$ in $A_q(n)$ we get that in $A_q(n)$, and its r^{th} homogeneous part $A_q(n,r)$, we have the following relations

$$X_{i,l}X_{j,m} = qX_{j,m}X_{i,l} \quad \text{for } i > j \text{ and } l \leqslant m,$$
$$X_{i,l}X_{j,m} = X_{j,m}X_{i,l} + (q-1)X_{j,l}X_{i,m} \quad \text{for } i > j \text{ and } l > m,$$
$$X_{i,l}X_{i,m} = X_{i,m}X_{i,l} \quad \text{for all } i, l, m,$$

where $i, j, l, m \in \underline{n}$. Then $E^{\otimes r}$ is an $A_q(n,r)$ comodule and the relations above ensure that multiplication by elements of $\mathrm{Hec}(r)$ is a comodule map, and this induces an isomorphism

$$\theta : A_q(n,r)^* \to \mathrm{End}_{\mathrm{Hec}(r)}(E^{\otimes r}) = \mathcal{S}_q(n,r).$$

In this chapter we follow these methods to construct a bialgebra in type B. We begin with the free k-algebra $F_B(3n)$ in the non-commuting indeterminants $x_{i,j}$, where $i, j \in Z(n)$, which is naturally a bialgebra. We now determine what relations we are required to factor out from $F_B(3n)$ to give our new bialgebra for which $Y^{\otimes r}$ is a comodule and multiplication by elements of $\mathcal{H}(r)$ is a comodule map.

Recall that $Z(n)$ is the set $\{\overline{n}, \dots, \overline{1}, \widehat{1}, \dots, \widehat{n}, 1, \dots, n\}$, with ordering $\overline{n} < \cdots < \overline{1} < \widehat{1} < \cdots < \widehat{n} < 1 < \cdots < n$. Let J be the ideal of $F_B(3n)$ generated by the elements

$$\begin{gathered} x_{i,l}x_{j,m} - qx_{j,m}x_{i,l} \qquad \text{for } i > j \text{ and } l \leqslant m, \\ x_{i,l}x_{j,m} - x_{j,m}x_{i,l} - (q-1)x_{j,l}x_{i,m} \qquad \text{for } i > j \text{ and } l > m, \\ x_{i,l}x_{i,m} - x_{i,m}x_{i,l} \qquad \text{for all } i, l, m, \end{gathered}$$

where $i, j, l, m \in Z(n)$, and put $A_q(3n) = F_B(3n)/J$. Then if we write $X_{i,j} = x_{i,j} + J$ for the canonical image of $x_{i,j}$ in $A_q(3n)$ we get that $A_q(3n)$ is the k-algebra given by generators $\{X_{i,j} \mid i, j \in Z(n)\}$, and relations

$$\begin{gathered} X_{i,l}X_{j,m} = qX_{j,m}X_{i,l} \qquad \text{for } i > j \text{ and } l \leqslant m, \\ X_{i,l}X_{j,m} = X_{j,m}X_{i,l} + (q-1)X_{j,l}X_{i,m} \qquad \text{for } i > j \text{ and } l > m, \\ X_{i,l}X_{i,m} = X_{i,m}X_{i,l} \qquad \text{for all } i, l, m, \end{gathered}$$

where $i, j, l, m \in Z(n)$. Let $A_q(3n, r)$ be the r^{th} homogeneous part of this. By [**3**, 1.4.2, 1.4.3], $A_q(3n)$ is a bialgebra, so that each $A_q(3n, r)$ is also a coalgebra.

Then, also by [**3**, 3.1.6], $Y^{\otimes r}$ is already a right comodule for $A_q(3n, r)$, with structure map $\tau : Y^{\otimes r} \to Y^{\otimes r} \otimes A_q(3n, r)$ given by $\tau(y_{\mathbf{i}}) = \sum_{\mathbf{j} \in I_B(n,r)} y_{\mathbf{i}} \otimes X_{\mathbf{i},\mathbf{j}}$, and multiplication by an element of the algebra generated by the elements $T_{s_1}, \dots, T_{s_{r-1}}$ is a comodule map. This means that all the relations in our new algebra that come from the action of $T_{s_1}, \ldots, T_{s_{r-1}}$ on $Y^{\otimes r}$ are already accounted for, and we just need to find the ones coming from the action of T_{s_0}. We can then factor out the ideal, I, generated by these relations from our existing algebra $A_q(3n)$ to give us our new algebra $C_{q,Q}(3n) = A_q(3n)/I \cong F_B(3n)/(I + J)$.

Now since the action of T_{s_0} on $Y^{\otimes r}$ only sees the first place in the tensor, it suffices to consider the case $r = 1$. Let $\phi : Y \to Y$ be given by $\phi(y) = yT_{s_0}$ for all $y \in Y^{\otimes r}$, so that for $i \in \underline{n}$ the map ϕ acts on Y via

$$\phi(y_i) = Qy_{\bar{i}}, \phi(y_{\hat{i}}) = Qy_{\hat{i}} \text{ and } \phi(y_{\bar{i}}) = y_i + (Q-1)y_{\bar{i}}.$$

Also for $l \in Z(n)$ the structure map on our new coalgebra will be inherited from $A_q(3n)$ so that $\tau : Y \to Y \otimes C_{q,Q}(3n)$ is given by $\tau(y_l) = \sum_{m \in Z(n)} y_m \otimes c_{m,l}$.

Now for multiplication by elements of $\mathcal{H}$ to be a comodule map we require that $\tau(\phi y_j) = (\phi \otimes 1)\tau(y_j)$ where y_j is a basis element of Y and 1 is the identity map on $A_q(3n, 1)$. So we equate the two sides of this identity and read off the consequences for our new coalgebra. Letting $j \in \underline{n}$ we have

$\tau(\phi y_j) = Q\tau y_{\bar{j}} = Q \sum_{i \in Z(n)} y_i \otimes c_{i,\bar{j}}$
$= Q \sum_{i \in \underline{n}} (y_{\bar{i}} \otimes c_{\bar{i},\bar{j}} + y_{\hat{i}} \otimes c_{\hat{i},\bar{j}} + y_i \otimes c_{i,\bar{j}})$

and also that

$(\phi \otimes 1)\tau(y_j) = (\phi \otimes 1)(\sum_{i \in Z(n)} y_i \otimes c_{i,j})$
$= \sum_{i \in \underline{n}} (\phi \otimes 1)(y_{\bar{i}} \otimes c_{\bar{i},j} + y_{\hat{i}} \otimes c_{\hat{i},j} + y_i \otimes c_{i,j})$
$= \sum_{i \in \underline{n}} (Qy_{\bar{i}} \otimes c_{i,j} + Qy_{\hat{i}} \otimes c_{\hat{i},j} + y_i \otimes c_{\bar{i},j} + (Q-1)y_{\bar{i}} \otimes c_{\bar{i},j})$
$= \sum_{i \in \underline{n}} (y_{\bar{i}} \otimes (Qc_{i,j} + (Q-1)c_{\bar{i},j}) + Qy_{\hat{i}} \otimes c_{\hat{i},j} + y_i \otimes c_{\bar{i},j})$.

Equating coefficients gives that for all $i, j \in \underline{n}$ we have

$$c_{\bar{i},j} = Qc_{i,\bar{j}}$$

$$c_{\hat{i},\bar{j}} = c_{\hat{i},j}$$

and $Qc_{\bar{i},\bar{j}} = Qc_{i,j} + (Q-1)c_{\bar{i},j}$, which using the first relation above gives $Qc_{\bar{i},\bar{j}} = Qc_{i,j} + Q(Q-1)c_{i,\bar{j}}$ or, for $Q \neq 0$, it gives

$$c_{\bar{i},\bar{j}} = c_{i,j} + (Q-1)c_{i,\bar{j}}.$$

Now for the hatted case. Let $j \in \underline{n}$ then
$\tau(\phi y_{\hat{j}}) = Q\tau(y_{\hat{j}}) = Q(\sum_{i \in Z(n)} y_i \otimes c_{i,\hat{j}})$
$= Q(\sum_{i \in \underline{n}} y_{\bar{i}} \otimes c_{\bar{i},\hat{j}} + \sum_{i \in \underline{n}} y_{\hat{i}} \otimes c_{\hat{i},\hat{j}} + \sum_{i \in \underline{n}} y_i \otimes c_{i,\hat{j}})$
and $(\phi \otimes 1)\tau(y_{\hat{j}})$
$= (\phi \otimes 1)(\sum_{i \in \underline{n}} y_{\bar{i}} \otimes c_{\bar{i},\hat{j}} + \sum_{i \in \underline{n}} y_{\hat{i}} \otimes c_{\hat{i},\hat{j}} + \sum_{i \in \underline{n}} y_i \otimes c_{i,\hat{j}})$
$= \sum_{i \in \underline{n}} (y_i + (Q-1)y_{\bar{i}}) \otimes c_{\bar{i},\hat{j}} + Q\sum_{i \in \underline{n}} y_{\hat{i}} \otimes c_{\hat{i},\hat{j}} + Q\sum_{i \in \underline{n}} y_{\bar{i}} \otimes c_{i,\hat{j}}$
$= Q\sum_{i \in \underline{n}} y_{\hat{i}} \otimes c_{\hat{i},\hat{j}} + \sum_{i \in \underline{n}} y_i \otimes c_{\bar{i},\hat{j}} + \sum_{i \in \underline{n}} y_{\bar{i}} \otimes ((Q-1)c_{\bar{i},\hat{j}} + Qc_{i,\hat{j}})$
which equating coefficients gives that for all $i, j \in \underline{n}$ we have

$$c_{\bar{i},\hat{j}} = Qc_{i,\hat{j}}.$$

It also gives us that $c_{\hat{i},\hat{j}} = c_{\hat{i},\hat{j}}$, which should come as no surprise, and lastly that $Qc_{\bar{i},\hat{j}} = (Q-1)c_{\bar{i},\hat{j}} + Qc_{i,\hat{j}}$, which is just a less economical way of saying that $c_{\bar{i},\hat{j}} = Qc_{i,\hat{j}}$.

Now asking that $\tau(\phi y_{\bar{i}}) = (\phi \otimes 1)\tau(y_{\bar{i}})$ only gives us the relations we have already seen, so we have now found all the relations coming from the action of the element T_{s_0} on the k-space Y.

We can now build ourselves an algebra.

Definition. Let $Q \neq 0$. Let I be the ideal of $A_q(3n)$ generated by linear relations of the form

$$X_{\bar{i},j} - QX_{i,\bar{j}}$$
$$X_{\bar{i},\bar{j}} - X_{i,j} - (Q-1)X_{i,\bar{j}}$$
$$X_{\bar{i},\hat{j}} - QX_{i,\hat{j}}$$
$$X_{\hat{i},\bar{j}} - X_{\hat{i},j}$$

where $i, j \in \underline{n}$. Then we denote the k-algebra $A_q(3n)/I$ by $C_{q,Q}(3n)$, and we also denote the canonical image $X_{i,j} + I$ of $X_{i,j}$ in $C_{q,Q}(3n)$ by

$c_{i,j}$ for all $i,j \in Z(n)$. Therefore $C_{q,Q}(3n)$ is the k-algebra given by generators $\{c_{i,j} \mid i,j \in Z(n)\}$ subject to quadratic relations

$$c_{i,l}c_{j,m} = qc_{j,m}c_{i,l} \qquad \text{for } i > j \text{ and } l \leqslant m,$$
$$c_{i,l}c_{j,m} = c_{j,m}c_{i,l} + (q-1)c_{j,l}c_{i,m} \qquad \text{for } i > j \text{ and } l > m,$$
$$c_{i,l}c_{i,m} = c_{i,m}c_{i,l} \qquad \text{for all } i,l,m,$$

where $i,j,l,m \in Z(n)$, and linear relations

$$c_{\bar{i},j} = Qc_{i,\bar{j}}$$
$$c_{\bar{i},\bar{j}} = c_{i,j} + (Q-1)c_{i,\bar{j}}$$
$$c_{\bar{i},\hat{j}} = Qc_{i,\hat{j}}$$
$$c_{\hat{i},\bar{j}} = c_{\hat{i},j}$$

where $i,j \in \underline{n}$.

Recall that the hyperoctahedral group $\mathrm{Hyp}(r)$ acts on $I_B(n,r)$. Then we can also make $\mathrm{Hyp}(r)$ act on $I_B^2(n,r) = I_B(n,r) \times I_B(n,r)$ via $(\mathbf{i},\mathbf{j})\pi = (\mathbf{i}\pi, \mathbf{j}\pi)$, where $\pi \in \mathrm{Hyp}(r)$. Using the notation of [**18**, 2.1] and [**3**, 1.1], for $\mathbf{i} = (i_1, i_2, \dots, i_r)$ and $\mathbf{j} = (j_1, j_2, \dots, j_r)$ in $I_B(n,r)$ we let $X_{\mathbf{i},\mathbf{j}} = X_{i_1,j_1}X_{i_2,j_2}\dots X_{i_r,j_r} \in A_q(3n)$. Then $A_q(3n)$ is a graded algebra, given by $A_q(3n) = \bigoplus_{r\geqslant 0} A_q(3n,r)$, where $A_q(3n,r)$ is spanned by monomials $X_{\mathbf{i},\mathbf{j}}$ where $\mathbf{i},\mathbf{j} \in I_B(3n,r)$ for $r \geqslant 1$, and $A_q(3n,0) = k$ by definition. The generators of I are all of degree one, so that I is a homogeneous ideal and $C_{q,Q}(3n)$ inherits the grading from $A_q(3n)$. Similarly, the subspace $C_{q,Q}(3n,r)$ is spanned by monomials $c_{\mathbf{i},\mathbf{j}}$ where $\mathbf{i},\mathbf{j} \in I_B(3n,r)$ for $r \geqslant 1$, and $C_{q,Q}(3n,0) = k$ by definition. Then the above grading is given by $C_{q,Q}(3n) = \bigoplus_{r\geqslant 0} C_{q,Q}(3n,r)$.

We now want give $C_{q,Q}(3n)$ the structure of a bialgebra. Recall that bialgebra structure is given on $A_q(3n)$ by

$$\Delta(X_{i,j}) = \textstyle\sum_{a\in Z(n)} X_{i,a} \otimes X_{a,j} \quad \text{and} \quad \varepsilon(X_{i,j}) = \delta_{ij}.$$

Now if we can show that I is a biideal of $A_q(3n)$ then we will have proved that $C_{q,Q}(3n)$ is a bialgebra with inherited comultiplication and counit. Now to show that I is a biideal of $A_q(3n)$ we need to prove that:

(i) $\varepsilon(g) = 0$; and

(ii) $\Delta(g) \subseteq I \otimes A_q(3n) + A_q(3n) \otimes I$

for all generators g of our ideal I.

LEMMA 5.1. *I is a biideal of $A_q(3n)$.*

PROOF. Let B denote $I \otimes A_q(3n) + A_q(3n) \otimes I$.

(i) We have

$$\varepsilon(X_{\bar{i},j} - QX_{i,\bar{j}}) = \varepsilon(X_{\bar{i},j}) - Q\varepsilon(X_{i,\bar{j}}) = \delta_{\bar{i}j} - Q\delta_{i\bar{j}} = 0 - Q.0 = 0,$$
$$\varepsilon(X_{\bar{i},\hat{j}} - QX_{i,\hat{j}}) = \varepsilon(X_{\bar{i},\hat{j}}) - Q\varepsilon(X_{i,\hat{j}}) = \delta_{\bar{i}\hat{j}} - Q\delta_{i\hat{j}} = 0 - Q.0 = 0,$$
$$\varepsilon(X_{\hat{i},\bar{j}} - X_{\hat{i},j}) = \varepsilon(X_{\hat{i},\bar{j}}) - \varepsilon(X_{\hat{i},j}) = \delta_{\hat{i}\bar{j}} - \delta_{\hat{i}j} = 0 - 0 = 0,$$

and finally

$$\varepsilon(X_{\bar{i},\bar{j}} - X_{i,j} - (Q-1)X_{i,\bar{j}}) = \varepsilon(X_{\bar{i},\bar{j}}) - \varepsilon(X_{i,j}) - (Q-1)\varepsilon(X_{i,\bar{j}}) =$$
$$\delta_{\bar{i}\bar{j}} - \delta_{ij} - (Q-1)\delta_{i\bar{j}} = \delta_{ij} - \delta_{ij} - (Q-1).0 = 0$$

so that $\varepsilon(g) = 0$ for all generators g of I.

(ii) Now for comultiplication. We have

$$\begin{aligned}
&\Delta(X_{\bar{i},j} - QX_{i,\bar{j}}) = \Delta(X_{\bar{i},j}) - Q\Delta(X_{i,\bar{j}}) \\
&= \textstyle\sum_{a\in Z(n)} X_{\bar{i},a} \otimes X_{a,j} - Q\sum_{a\in Z(n)} X_{i,a} \otimes X_{a,\bar{j}} \\
&= \textstyle\sum_{a\in\underline{n}} X_{\bar{i},\bar{a}} \otimes X_{\bar{a},j} + \sum_{a\in\underline{n}} X_{\bar{i},\hat{a}} \otimes X_{\hat{a},j} + \sum_{a\in\underline{n}} X_{\bar{i},a} \otimes X_{a,j} \\
&-Q\textstyle\sum_{a\in\underline{n}} X_{i,\bar{a}} \otimes X_{\bar{a},\bar{j}} - Q\sum_{a\in\underline{n}} X_{i,\hat{a}} \otimes X_{\hat{a},\bar{j}} - Q\sum_{a\in\underline{n}} X_{i,a}X_{a,\bar{j}} \\
&\equiv Q\textstyle\sum_{a\in\underline{n}} X_{\bar{i},\bar{a}} \otimes X_{a,\bar{j}} + \sum_{a\in\underline{n}} X_{\bar{i},\hat{a}} \otimes X_{\hat{a},j} + \sum_{a\in\underline{n}} X_{\bar{i},a} \otimes X_{a,j} \\
&-Q\textstyle\sum_{a\in\underline{n}} X_{i,\bar{a}} \otimes X_{a,j} - Q(Q-1)\sum_{a\in\underline{n}} X_{i,\bar{a}} \otimes X_{a,\bar{j}} - Q\sum_{a\in\underline{n}} X_{i,\hat{a}} \otimes X_{\hat{a},j} \\
&-Q\textstyle\sum_{a\in\underline{n}} X_{i,a}X_{a,\bar{j}} \quad , \qquad\qquad \text{(modulo } B\text{)} \\
&= Q\textstyle\sum_{a\in\underline{n}}(X_{\bar{i},\bar{a}} - X_{i,a} - (Q-1)X_{i,\bar{a}}) \otimes X_{a,\bar{j}} \\
&+\textstyle\sum_{a\in\underline{n}}(X_{\bar{i},\hat{a}} - QX_{i,\hat{a}}) \otimes X_{\hat{a},j} + \sum_{a\in\underline{n}}(X_{\bar{i},a} - QX_{i,\bar{a}}) \otimes X_{a,j}
\end{aligned}$$

$\equiv 0.$ (modulo B)

Therefore $\Delta(X_{\bar{i},j} - QX_{i,\bar{j}}) \subseteq I \otimes A_q(3n) + A_q(3n) \otimes I)$.

Also $\Delta(X_{\bar{i},\bar{j}} - X_{i,j} - (Q-1)X_{i,\bar{j}}) = \Delta(X_{\bar{i},\bar{j}}) - \Delta(X_{i,j}) - (Q-1)\Delta(X_{i,\bar{j}})$

$= \sum_{a\in Z(n)} X_{\bar{i},a} \otimes X_{a,\bar{j}} - \sum_{a\in Z(n)} X_{i,a} \otimes X_{a,j} - (Q-1)\sum_{a\in Z(n)} X_{i,a} \otimes X_{a,\bar{j}}$

$= \sum_{a\in \underline{n}} X_{\bar{i},\bar{a}} \otimes X_{\bar{a},\bar{j}} + \sum_{a\in \underline{n}} X_{\bar{i},\widehat{a}} \otimes X_{\widehat{a},\bar{j}} + \sum_{a\in \underline{n}} X_{\bar{i},a} \otimes X_{a,\bar{j}}$

$- \sum_{a\in \underline{n}} X_{i,\bar{a}} \otimes X_{\bar{a},j} - \sum_{a\in \underline{n}} X_{i,\widehat{a}} \otimes X_{\widehat{a},j} - \sum_{a\in \underline{n}} X_{i,a} \otimes X_{a,j}$

$-(Q-1)\sum_{a\in \underline{n}} X_{i,\bar{a}} \otimes X_{\bar{a},\bar{j}} - (Q-1)\sum_{a\in \underline{n}} X_{i,\widehat{a}} \otimes X_{\widehat{a},\bar{j}}$

$-(Q-1)\sum_{a\in \underline{n}} X_{i,a} \otimes X_{a,\bar{j}}$

$\equiv \sum_{a\in \underline{n}} X_{\bar{i},\bar{a}} \otimes X_{a,j} + (Q-1)\sum_{a\in \underline{n}} X_{\bar{i},\bar{a}} \otimes X_{a,\bar{j}} - Q\sum_{a\in \underline{n}} X_{i,\bar{a}} \otimes X_{a,\bar{j}}$

$-(Q-1)\sum_{a\in \underline{n}} X_{i,\bar{a}} \otimes X_{a,j} + (Q-1)^2\sum_{a\in \underline{n}} X_{i,\bar{a}} \otimes X_{a,\bar{j}}$

$+\sum_{a\in \underline{n}} X_{\bar{i},\widehat{a}} \otimes X_{\widehat{a},j} - \sum_{a\in \underline{n}} X_{i,\widehat{a}} \otimes X_{\widehat{a},j} - (Q-1)\sum_{a\in \underline{n}} X_{i,\widehat{a}} \otimes X_{\widehat{a},j}$

$+\sum_{a\in \underline{n}} X_{\bar{i},a} \otimes X_{a,\bar{j}} - \sum_{a\in \underline{n}} X_{i,a} \otimes X_{a,j}$

$-(Q-1)\sum_{a\in \underline{n}} X_{i,a} \otimes X_{a,\bar{j}}$, (modulo B)

$\equiv \sum_{a\in \underline{n}}(X_{\bar{i},\bar{a}} - X_{i,a} - (Q-1)X_{i,\bar{a}}) \otimes X_{a,j}$

$+(Q-1)\sum_{a\in \underline{n}}(X_{\bar{i},\bar{a}} - X_{i,a} - (Q-1)X_{i,\bar{a}}) \otimes X_{a,\bar{j}}$

$+\sum_{a\in \underline{n}}(X_{\bar{i},\widehat{a}} - QX_{i,\widehat{a}}) \otimes X_{\widehat{a},j}$

$\equiv 0$. (modulo B)

Now $\Delta(X_{\bar{i},\widehat{j}} - QX_{i,\widehat{j}}) = \Delta(X_{\bar{i},\widehat{j}}) - \Delta(QX_{i,\widehat{j}})$

$= \sum_{a\in Z(n)} X_{\bar{i},a} \otimes X_{a,\widehat{j}} - Q\sum_{a\in Z(n)} X_{i,a} \otimes X_{a,\widehat{j}}$

$= \sum_{a\in \underline{n}} X_{\bar{i},\bar{a}} \otimes X_{\bar{a},\widehat{j}} + \sum_{a\in \underline{n}} X_{\bar{i},\widehat{a}} \otimes X_{\widehat{a},\widehat{j}} + \sum_{a\in \underline{n}} X_{\bar{i},a} \otimes X_{a,\widehat{j}}$

$-Q\sum_{a\in \underline{n}} X_{i,\bar{a}} \otimes X_{\bar{a},\widehat{j}} - Q\sum_{a\in \underline{n}} X_{i,\widehat{a}} \otimes X_{\widehat{a},\widehat{j}} - Q\sum_{a\in \underline{n}} X_{i,a} \otimes X_{a,\widehat{j}}$

$\equiv \sum_{a\in \underline{n}} X_{\bar{i},\bar{a}} \otimes X_{\bar{a},\widehat{j}} + \sum_{a\in \underline{n}} X_{\bar{i},a} \otimes X_{a,\widehat{j}} - Q\sum_{a\in \underline{n}} X_{i,\bar{a}} \otimes X_{\bar{a},\widehat{j}}$

$-Q\sum_{a\in \underline{n}} X_{i,a} \otimes X_{a,\widehat{j}}$, (modulo B)

$\equiv Q\sum_{a\in\underline{n}} X_{\bar{i},\bar{a}}\otimes X_{a,\hat{j}} + \sum_{a\in\underline{n}} X_{\bar{i},a}\otimes X_{a,\hat{j}} - Q^2\sum_{a\in\underline{n}} X_{i,\bar{a}}\otimes X_{a,\hat{j}}$
$-Q\sum_{a\in\underline{n}} X_{i,a}\otimes X_{a,\hat{j}}$, (modulo B)
$\equiv Q\sum_{a\in\underline{n}} X_{i,a}\otimes X_{a,\hat{j}} + Q(Q-1)\sum_{a\in\underline{n}} X_{i,\bar{a}}\otimes X_{a,\hat{j}} + Q\sum_{a\in\underline{n}} X_{i,\bar{a}}\otimes X_{a,\hat{j}}$
$-Q^2\sum_{a\in\underline{n}} X_{i,\bar{a}}\otimes X_{a,\hat{j}} - Q\sum_{a\in\underline{n}} X_{i,a}\otimes X_{a,\hat{j}}$, (modulo B)
$= 0.$

Therefore $\Delta(X_{\bar{i},\hat{j}} - X_{i,\hat{j}}) \subseteq I\otimes A_q(3n) + A_q(3n)\otimes I)$.

Finally, we also have that $\Delta(X_{\hat{i},\bar{j}} - X_{\hat{i},j}) = \Delta(X_{\hat{i},\bar{j}}) - \Delta(X_{\hat{i},j})$
$= \sum_{a\in Z(n)} X_{\hat{i},a}\otimes X_{a,\bar{j}} - \sum_{a\in Z(n)} X_{\hat{i},a}\otimes X_{a,j}$
$= \sum_{a\in\underline{n}} X_{\hat{i},\bar{a}}\otimes X_{\bar{a},\bar{j}} + \sum_{a\in\underline{n}} X_{\hat{i},\hat{a}}\otimes X_{\hat{a},\bar{j}} + \sum_{a\in\underline{n}} X_{\hat{i},a}\otimes X_{a,\bar{j}}$
$- \sum_{a\in\underline{n}} X_{\hat{i},\bar{a}}\otimes X_{\bar{a},j} - \sum_{a\in\underline{n}} X_{\hat{i},\hat{a}}\otimes X_{\hat{a},j} - \sum_{a\in\underline{n}} X_{\hat{i},a}\otimes X_{a,j}$
$\equiv \sum_{a\in\underline{n}} X_{\hat{i},\bar{a}}\otimes X_{\bar{a},\bar{j}} + \sum_{a\in\underline{n}} X_{\hat{i},a}\otimes X_{a,\bar{j}}$
$- \sum_{a\in\underline{n}} X_{\hat{i},\bar{a}}\otimes X_{\bar{a},j} - \sum_{a\in\underline{n}} X_{\hat{i},a}\otimes X_{a,j}$, (modulo B)
$= \sum_{i\in\underline{n}} X_{\hat{i},a}\otimes (X_{a,j} + (Q-1)X_{a,\bar{j}} + X_{a,\bar{j}} - QX_{a,\bar{j}} - X_{a,j}) = 0,$
and this completes the proof. $\square$

COROLLARY 5.2. *$C_{q,Q}(3n)$ is bialgebra with comultiplication and counit given by*

$$\Delta(c_{i,j}) = \sum_{a\in Z(n)} c_{i,a}\otimes c_{a,j} \quad \textit{and} \quad \varepsilon(c_{i,j}) = \delta_{ij}.$$

We also have, for each $r \geqslant 0$, that $C_{q,Q}(3n,r)$ is a subcoalgebra of $C_{q,Q}(3n)$.

We now have a detailed look at what happens for the hyperoctahedral group.

5.1. The group case

Putting $q = Q = 1$ we are back in the group case, and we have built a bialgebra $C_{1,1}(3n)$ with r^{th} homogeneous part $C_{1,1}(3n,r)$ from the action of $k\,\mathrm{Hyp}(r)$ on $Y^{\otimes r}$. In this section we will always assume that

$q = Q = 1$, and write $C(3n)$ and $C(3n, r)$ for $C_{1,1}(3n)$ and $C_{1,1}(3n, r)$ respectively. Then $C(3n)$ is the k-algebra generated by the elements $\{c_{i,j} \mid i, j \in Z(n)\}$ with quadratic relation

$$c_{i,j}c_{l,m} = c_{l,m}c_{i,j} \text{ where } i, j, l, m \in Z(n),$$

so all elements $c_{i,j}$ and $c_{l,m}$ commute, and linear relations

$$\begin{aligned} c_{\bar{i},j} &= c_{i,\bar{j}} \\ c_{\bar{i},\bar{j}} &= c_{i,j} \\ c_{\bar{i},\hat{j}} &= c_{i,\hat{j}} \\ c_{\hat{i},\bar{j}} &= c_{\hat{i},j} \end{aligned}$$

where $i, j \in \underline{n}$. We now write $\mathbf{i} \sim \mathbf{j}$ to denote that $\mathbf{i}$ and $\mathbf{j}$ are in the same $\mathrm{Hyp}(r)$-orbit of $I_B(n, r)$, and similarly $(\mathbf{i}, \mathbf{j}) \sim (\mathbf{i}', \mathbf{j}')$ to denote that $(\mathbf{i}, \mathbf{j})$ and $(\mathbf{i}', \mathbf{j}')$ are in the same $\mathrm{Hyp}(r)$-orbit of $I_B(n, r) \times I_B(n, r)$.

Our next job is to construct a k-basis for $C(3n, r)$. The following is useful.

THEOREM 5.3. *For $\mathbf{i}, \mathbf{j}, \mathbf{i}', \mathbf{j}' \in I_B(n, r)$, we have*

$$c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}',\mathbf{j}'} \Leftrightarrow (\mathbf{i}, \mathbf{j}) \sim (\mathbf{i}', \mathbf{j}') \qquad .$$

PROOF. The relation $c_{i,j}c_{l,m} = c_{l,m}c_{i,j}$ gives us that for all elements $x \in C(3n, a-1), y \in C(3n, r-(a+1))$ where $1 \leqslant a \leqslant r-1$, and $i, j, l, m \in \underline{3n}$ we have that

$$xc_{i,j}c_{l,m}y = xc_{l,m}c_{i,j}y.$$

Then since $(xc_{i,j}c_{l,m}y).s_a = xc_{l,m}c_{i,j}y$ we can re-write this relation in $C(3n, r)$ as

$$c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}s_a,\mathbf{j}s_a} \text{ for all } \mathbf{i}, \mathbf{j} \in I_B(n, r) \text{ and } 1 \leqslant a \leqslant r-1.$$

Similarly, with x as above, $y \in C(3n, r-a)$, and $i, j \in \underline{n}$, the relation $c_{\bar{i},j} = c_{i,\bar{j}}$ tells us that

$$xc_{\bar{i},j}y = xc_{i,\bar{j}}y \text{ for all such } i, j, x \text{ and } y \text{ as above.}$$

Now recalling that for all $1 \leqslant a \leqslant r$ we have $t_a = s_{a-1} \cdots s_0 \cdots s_{a-1}$, we have that $(xc_{\bar{i},j}y).t_a = xc_{i,\bar{j}}y$, so that we can rewrite our original relation in $C(3n,r)$ as

$$c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}.t_a,\mathbf{i}.t_a} \text{ if } i_a \text{ is barred and } j_a \text{ is unbarred.}$$

Applying similar arguments to the other 3 linear relations in $C(3n,r)$ we get

$$c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}t_a,\mathbf{j}t_a} \text{ for all } \mathbf{i},\mathbf{j} \in I_B(n,r) \text{ and } 1 \leqslant a \leqslant r.$$

So for any relation $c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}',\mathbf{j}'}$ in $C(3n,r)$ we must have that $(\mathbf{i},\mathbf{j}).\pi = (\mathbf{i}',\mathbf{j}')$ for some $\pi \in \mathrm{Hyp}(r)$, so that $(\mathbf{i},\mathbf{j}) \sim (\mathbf{i}',\mathbf{j}')$.

Conversely, let s_a be a generator of $\mathrm{Hyp}(r)$. Then since the elements $c_{i,j}$ in $C(3n,r)$ all commute, and for $1 \leqslant a \leqslant r-1$ the generators s_a just transpose c_{i_a,j_a} and $c_{i_{a+1},j_{a+1}}$ we have that if $(\mathbf{i}',\mathbf{j}') = (\mathbf{i},\mathbf{j})s_a = (\mathbf{i}s_a,\mathbf{j}s_a)$ for some $1 \leqslant a \leqslant r-1$, then $c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}',\mathbf{j}'}$. We now just need to show this is true for s_0.

As s_0 only sees the first coordinate of elements in $I_B(n,r)$, we can just consider the case $r = 1$, so we can write $c_{\mathbf{i},\mathbf{j}}$ simply as $c_{i,j}$. Then, for $i, j \in \underline{n}$ we have

$$c_{is_0,\bar{j}s_0} = c_{\bar{i},j} = c_{i,\bar{j}},$$

by the first linear relation. Similarly we also have

$$\begin{aligned}
c_{is_0,\hat{j}s_0} &= c_{\bar{i},\hat{j}} = c_{i,\hat{j}}, \\
c_{is_0,js_0} &= c_{\bar{i},\bar{j}} = c_{i,j} \\
c_{\hat{i}s_0,\bar{j}s_0} &= c_{\hat{i},j} = c_{\hat{i},\bar{j}} \\
c_{\hat{i}s_0,js_0} &= c_{\hat{i},\bar{j}} = c_{\hat{i},j} \\
c_{\bar{i}s_0,\bar{j}s_0} &= c_{i,j} = c_{\bar{i},\bar{j}} \\
c_{\bar{i}s_0,\hat{j}s_0} &= c_{i,\hat{j}} = c_{\bar{i},\hat{j}} \\
c_{\bar{i}s_0,js_0} &= c_{i,\bar{j}} = c_{\bar{i},j}
\end{aligned}$$

and of course

$$c_{\hat{i}s_0,\hat{j}s_0} = c_{\hat{i},\hat{j}}.$$

Therefore $(\mathbf{i},\mathbf{j})s_0 = (\mathbf{i}',\mathbf{j}') \Rightarrow c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}',\mathbf{j}'}$ and so we have that

$$(\mathbf{i},\mathbf{j}) \sim (\mathbf{i}',\mathbf{j}') \Rightarrow c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}',\mathbf{j}'},$$

and the proof is complete. □

Let $\mho(n,r)$ be a set of representatives of the orbits of the $\mathrm{Hyp}(r)$-action on $I_B(n,r) \times I_B(n,r)$. Then the above theorem gives us the following.

COROLLARY 5.4. *The set* $\{c_{\mathbf{i},\mathbf{j}} \mid (\mathbf{i},\mathbf{j}) \in \mho(n,r)\}$ *is a k-basis for* $C(3n,r)$.

Now recall that $Z(n)$ is an ordered set with ordering $\overline{n} < \cdots < \overline{1} < \widehat{1} < \cdots < \widehat{n} < 1 < \cdots < n$. Then we let $\mathfrak{B}(n,r)$ be the subset of $I_B(n,r) \times I_B(n,r)$ consisting of elements $(\mathbf{i},\mathbf{j})$ where:

(B1) All elements of $\mathbf{i}$ are not barred, and if i_a is hatted then j_a is not barred;

(B2) We have that $i_1 \leqslant \ldots \leqslant i_r$ and if $i_a = i_{a+1}$ then $j_a \leqslant j_{a+1}$.

Then $\mathfrak{B}(n,r)$ is a transversal for the action of $\mathrm{Hyp}(r)$ on $I_B(n,r) \times I_B(n,r)$, so in the corollary above we could take $\mho(n,r) = \mathfrak{B}(n,r)$. Since $\mathfrak{B}(n,r)$ has cardinality $\binom{5n^2+r-1}{r}$, we know that $C(3n,r)$ has dimension $\binom{5n^2+r-1}{r}$.

From Chapter 1, as in [**17**, 1.1, Remark 2], we know that since $C(3n,r)$ is a coalgebra, its dual $C(3n,r)^* = \mathrm{Hom}_k(C(3n,r),k)$ is an associative k-algebra. We can now mimic [**18**, 2.3] to give a k-basis for $C(3n,r)^*$, and then follow the methods of [**18**, 2.6] to show that $C(3n,r)^*$ is isomorphic to the Schur^2 algebra. Firstly we make a definition.

DEFINITION. For $\mathbf{a}, \mathbf{b}, \mathbf{i}, \mathbf{j} \in I_B(n,r)$ we define the element $\eta_{\mathbf{a},\mathbf{b}} \in C(3n,r)^*$ via

$$\eta_{\mathbf{a},\mathbf{b}}(c_{\mathbf{i},\mathbf{j}}) = \begin{cases} 1 & \text{if } (\mathbf{a},\mathbf{b}) \sim (\mathbf{i},\mathbf{j}) \\ 0 & \text{otherwise} \end{cases}$$

It is now clear that $\eta_{\mathbf{i},\mathbf{j}} = \eta_{\mathbf{i}',\mathbf{j}'} \Leftrightarrow (\mathbf{i},\mathbf{j}) \sim (\mathbf{i}',\mathbf{j}')$ which gives us the following.

THEOREM 5.5. *The set* $\{\eta_{\mathbf{i},\mathbf{j}} \mid (\mathbf{i},\mathbf{j}) \in (n,r)\}$ *is a k-basis for* $C(3n,r)^*$.

Of course the dimension of $C(3n,r)^*$ is the same as that of $C(3n,r)$, namely $\binom{5n^2+r-1}{r}$.

We now derive an action of $C(3n,r)^*$ on $Y^{\otimes r}$, making $Y^{\otimes r}$ into a left $C(3n,r)^*$-module. Say we have a bialgebra C, which has right comodules V, with structure map $\tau : V \to V \otimes C$ such that for $v \in V$ we have $\tau(v) = \sum_i v_i \otimes a_i$, and W with structure map $\kappa : W \to W \otimes C$ such that for $w \in W$ we have $\kappa(w) = \sum_j w_j \otimes a'_j$. Then $V \otimes W$ is a right C-comodule with structure map $\varphi : V \otimes W \to V \otimes W \otimes C$ defined by $\varphi(v \otimes w) = \sum_{i,j} v_i \otimes w_j \otimes a_i a'_j$, with v_i, w_j, a_i and a'_j as above.

Now $C(3n)$ is a bialgebra, and it comes equipped with a right comodule Y with structure map $\tau : Y \to Y \otimes C(3n)$ given by $\tau(y_j) = \sum_{i \in Z(n)} y_i \otimes c_{ij}$. So by repeatedly applying what is said above about the tensor product of two comodules for a bialgebra we see that $Y^{\otimes r}$ is a right comdule with structure map $\tau : Y^{\otimes r} \to Y^{\otimes r} \otimes C(3n)$ given by $\tau(y_{\mathbf{j}}) = \sum_{\mathbf{i} \in Z(n)} y_{\mathbf{i}} \otimes c_{\mathbf{i},\mathbf{j}}$.

Now say V is a right comodule for a coalgebra C, with basis $\{v_j \mid j \in J\}$, and structure map $\tau(v_j) = \sum_{i \in J} v_i \otimes x_{ij}$. Then the k-span of the elements x_{ij} as above is called the coefficient space of V, and is denoted $\mathrm{cf}(V)$. Now the $C(3n)$-comodule Y is homogeneous of degree 1, i.e. we have that $\mathrm{cf}(V) \leqslant C(3n,1)$, and therefore $Y^{\otimes r}$ is homogeneous of degree r, so that the structure map $\tau : Y^{\otimes r} \to Y^{\otimes r} \otimes C(3n)$ of V is actually a map from $Y^{\otimes r}$ into $Y^{\otimes r} \otimes C(3n, r)$, so that $Y^{\otimes r}$ is a right $C(3n, r)$-comodule. We recall the following from chapter 1 (and also from in [**17**, 1.1,Remark 2]). Let (V, τ) be a right C-comodule. Then the k-space V can be given the structure of a left C^*-module via the product $\alpha v = (1 \otimes \alpha)\tau(v)$, where $\alpha \in C^*$ and $v \in V$. Now from above we know that $Y^{\otimes r}$ is a right $C(3n, r)$-comodule with structure map $\tau(y_{\mathbf{j}}) = \sum_{\mathbf{i} \in I_B(n,r)} y_{\mathbf{i}} \otimes c_{\mathbf{i},\mathbf{j}}$, so $Y^{\otimes r}$ is a left $C(3n, r)^*$-module, and for $\eta \in C(3n, r)^*$ we have the product

$\eta y_{\mathbf{j}} = (1 \otimes \eta)\tau(y_{\mathbf{j}}) = (1 \otimes \eta)(\sum_{\mathbf{i} \in I_B(n,r)} y_{\mathbf{i}} \otimes c_{\mathbf{i},\mathbf{j}}) = \sum_{\mathbf{i} \in I_B(n,r)} y_{\mathbf{i}} \otimes \eta(c_{\mathbf{i},\mathbf{j}})$.

So identifying $\eta(c_{\mathbf{i},\mathbf{j}}) \otimes 1$ and $\eta(c_{\mathbf{i},\mathbf{j}})$, we have that the following is true.

LEMMA 5.6. *$Y^{\otimes r}$ is a left $C(3n, r)^*$-module, and for $\eta_{\mathbf{a},\mathbf{b}}$ a basis elements of $C(3n, r)^*$, and $y_{\mathbf{j}}$ a basis element of $Y^{\otimes r}$ we have the product*

$$\eta_{\mathbf{a},\mathbf{b}} y_{\mathbf{j}} = \sum_{\mathbf{i} \in I_B(n,r)} \eta_{\mathbf{a},\mathbf{b}}(c_{\mathbf{i},\mathbf{j}}) y_{\mathbf{i}}.$$

We can now use this action to show the following.

LEMMA 5.7. *The actions of $C(3n, r)^*$ and* $\mathrm{Hyp}(r)$ *on $Y^{\otimes r}$ centralize each other.*

Note this lemma is a consequence of the way in which we constructed $C_{q,Q}(3n, r)$, but the proof given here is straightforward.

PROOF. For each of our basis elements $\eta_{\mathbf{a},\mathbf{b}} \in C(3n, r)^*$, each $\pi \in \mathrm{Hyp}(r)$ and each basis element $y_{\mathbf{i}}$ of $Y^{\otimes r}$ we have

$(\eta_{\mathbf{a},\mathbf{b}} y_{\mathbf{i}})\pi = (\sum_{\mathbf{i} \in I_B(n,r)} \eta_{\mathbf{a},\mathbf{b}}(c_{\mathbf{i},\mathbf{j}}) y_{\mathbf{i}})\pi = \sum_{\mathbf{i} \in I_B(n,r)} \eta_{\mathbf{a},\mathbf{b}}(c_{\mathbf{i},\mathbf{j}}) y_{\mathbf{i}}\pi$

$= \sum_{\mathbf{i} \in I_B(n,r)} \eta_{\mathbf{a},\mathbf{b}}(c_{\mathbf{i},\mathbf{j}}) y_{\mathbf{i}\pi} = \sum_{\mathbf{i} \in I_B(n,r)} \eta_{\mathbf{a},\mathbf{b}}(c_{\mathbf{i}\pi,\mathbf{j}}) y_{\mathbf{i}\pi}$

$= \eta_{\mathbf{a},\mathbf{b}}(y_{\mathbf{i}\pi}) = \eta_{\mathbf{a},\mathbf{b}}(y_{\mathbf{i}}\pi)$,

i.e. we have that $(\eta_{\mathbf{a},\mathbf{b}} y_{\mathbf{i}})\pi = \eta_{\mathbf{a},\mathbf{b}}(y_{\mathbf{i}}\pi)$. This then extends linearly to give the theorem. $\square$

We can now prove one of the main theorems of this chapter.

THEOREM 5.8. *As k-algebras we have that, for all $n, r \geqslant 1$,*

$$C(3n, r)^* \cong \mathrm{End}_{k\,\mathrm{Hyp}(r)}(Y^{\otimes r}) \cong \mathcal{S}^2(n, r).$$

PROOF. Now $Y^{\otimes r}$ is a left $C(3n,r)^*$-module, which affords a representation

$$\rho : C(3n, r)^* \to \mathrm{End}_k(Y^{\otimes r}) \text{ where } \rho(\eta)(y) = \eta y$$

for all $\eta \in C(3n,r)^*$ and $y \in Y^{\otimes r}$. Since by the previous lemma we have that $(\eta y)\pi = \eta(y\pi)$ for all $\eta \in C(3n,r)^*$, $y \in Y^{\otimes r}$ and $\pi \in \mathrm{Hyp}(r)$ we see that $\rho(\eta)$ is actually a $k\,\mathrm{Hyp}(r)$-endomorphism of $Y^{\otimes r}$ so we in fact have that

$$\rho : C(3n, r)^* \to \mathrm{End}_{k\,\mathrm{Hyp}(r)}(Y^{\otimes r}).$$

We now show that ρ is an isomorphism. Each $\vartheta \in \mathrm{End}_k(Y^{\otimes r})$ can be written as a matrix $\theta = (\theta_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j} \in I_B(n,r)}$, for some $\theta_{\mathbf{i},\mathbf{j}} \in k$, with respect to our basis $\Xi = \{y_{\mathbf{j}} \mid \mathbf{j} \in I_B(n,r)\}$ of $Y^{\otimes r}$. Now we want ϑ to be a $k\,\mathrm{Hyp}(r)$-endomorphism, so that for all $y_{\mathbf{j}} \in \Xi$ and $\pi \in \mathrm{Hyp}(r)$, we have

$$\vartheta(y_{\mathbf{j}}\pi) = (\vartheta y_{\mathbf{j}})\pi,$$

or, what is the same thing, that

$$\theta(y_{\mathbf{j}}\pi) = (\theta y_{\mathbf{j}})\pi.$$

Of course, the action of θ on $Y^{\otimes r}$ is given by $\theta(y_{\mathbf{j}}) = \sum_{\mathbf{i} \in I_B(n,r)} \theta_{\mathbf{i},\mathbf{j}} y_{\mathbf{i}}$. Then we have that

$\theta(y_{\mathbf{j}}\pi) = \theta(y_{\mathbf{j}\pi}) = \sum_{\mathbf{i} \in I_B(n,r)} \theta_{\mathbf{i},\mathbf{j}\pi} y_{\mathbf{i}} = \sum_{\mathbf{i} \in I_B(n,r)} \theta_{\mathbf{i}\pi,\mathbf{j}\pi} y_{\mathbf{i}\pi}$

but also

$(\theta y_{\mathbf{j}})\pi = (\sum_{\mathbf{i} \in I_B(n,r)} \theta_{\mathbf{i},\mathbf{j}} y_{\mathbf{i}})\pi = \sum_{\mathbf{i} \in I_B(n,r)} \theta_{\mathbf{i},\mathbf{j}} y_{\mathbf{i}\pi}$.

Therefore, we see that $\vartheta \in \mathrm{End}_{k\,\mathrm{Hyp}(r)}(Y^{\otimes r})$ if and only if $\theta_{\mathbf{i},\mathbf{j}} = \theta_{\mathbf{i}\pi,\mathbf{j}\pi}$ for all $\mathbf{i}, \mathbf{j} \in I_B(n,r)$ and $\pi \in \mathrm{Hyp}(r)$. Consequently $\mathrm{End}_{k\,\mathrm{Hyp}(r)}(Y^{\otimes r}) = \mathcal{S}^2(n,r)$ has k-basis

$$\{\vartheta_{(\mathbf{a},\mathbf{b})} \mid (\mathbf{a},\mathbf{b}) \in \mathfrak{B}(n,r)\}$$

where for $(\mathbf{a},\mathbf{b}) \in \mathfrak{B}(n,r)$ we define the basis element $\vartheta_{(\mathbf{a},\mathbf{b})}$ to be that whose matrix $(\Theta_{\mathbf{i},\mathbf{j}})$ is such that

$$\Theta_{\mathbf{i},\mathbf{j}} = \begin{cases} 1 & \text{if } (\mathbf{a},\mathbf{b}) \sim (\mathbf{i},\mathbf{j}) \\ 0 & \text{otherwise} \end{cases}$$

Now

$$\rho(\eta_{\mathbf{a},\mathbf{b}})(y_{\mathbf{j}}) = \eta_{\mathbf{a},\mathbf{b}} y_{\mathbf{j}} = \textstyle\sum_{\mathbf{i}\in I_B(n,r)} \eta_{\mathbf{a},\mathbf{b}}(c_{\mathbf{i},\mathbf{j}}) y_{\mathbf{i}} = \sum_{\mathbf{i}\in I_B(n,r)} \Theta_{\mathbf{i},\mathbf{j}} y_{\mathbf{i}}$$

by definition of the element $\eta_{\mathbf{a},\mathbf{b}} \in C(3n,r)^*$. Therefore

$$\rho(\eta_{\mathbf{a},\mathbf{b}}) y_{\mathbf{i}} = \textstyle\sum_{\mathbf{i}\in I_B(n,r)} \Theta_{\mathbf{i},\mathbf{j}} y_{\mathbf{i}} = \Theta_{(\mathbf{a},\mathbf{b})} y_{\mathbf{i}} = \vartheta_{(\mathbf{a},\mathbf{b})} y_{\mathbf{i}}$$

so that

$$\rho(\eta_{\mathbf{a},\mathbf{b}}) = \vartheta_{(\mathbf{a},\mathbf{b})}$$

and ρ induces an isomorphism $C(3n,r)^* \cong \mathrm{End}_{k\,\mathrm{Hyp}(r)}(Y^{\otimes r})$.

□

Now from [**11**, 6.1.1], for fixed $n, r \geqslant 1$, the q-Schur^2 algebra has a k-basis which is indexed by the same set for all values of q and Q, and hence the dimension of the q-Schur^2 algebra, $\mathcal{S}^2_q(n,r)$, is the same for all values of q and Q. So since the above theorem gives us that $\dim \mathcal{S}^2(n,r) = \binom{5n^2+r-1}{r}$ for all $r, n \geqslant 1$ we have the following.

COROLLARY 5.9. *For all $q, Q \in k$, and $n, r \geqslant 1$ we have*

$$\dim \mathcal{S}^2_q(n,r) = \binom{5n^2+r-1}{r}.$$

5. 2. The non-commuting case

We now return to the general setting of $C_{q,Q}(3n,r)$. We can show the following.

LEMMA 5.10. *$\mathfrak{B}(n,r)$ is a spanning set for $C_{q,Q}(3n,r)$.*

PROOF. Consider a general element $c_{\mathbf{l},\mathbf{m}}$. Using the linear relations in $C_{q,Q}(3n,r)$ we can write each c_{l_a,m_a} as a linear combination of elements satisfying (B1) as above. This means we can write $c_{\mathbf{l},\mathbf{m}}$ as a linear combination of elements satisfying (B1). We can now use the quadratic relations and the fact that if $i,j,l \in \underline{n}$ and $m \in \{\widehat{1},\ldots,\widehat{n},1,\ldots,n\}$ then

$$c_{i,\bar{l}}c_{\widehat{j},m} = c_{\widehat{j},m}c_{i,\bar{l}} + (q-1)c_{\widehat{j},\bar{l}}c_{i,m} = c_{\widehat{j},m}c_{i,\bar{l}} + (q-1)c_{\widehat{j},l}c_{i,m}$$

(using the second quadratic and fourth linear relations) to write each part of this linear combination as a further linear combination of elements satisfying both (B1) and (B2). Therefore $\mathfrak{B}(n,r)$ is a spanning set for $C_{q,Q}(3n,r)$. □

Then as $|\mathfrak{B}(n,r)| = \binom{5n^2+r-1}{r}$ we can make the following conclusion.

COROLLARY 5.11. $\dim C_{q,Q}(3n,r) \leqslant \binom{5n^2+r-1}{r}$.

We can now make a stronger statement.

THEOREM 5.12. *If $q \neq 1$ and $r,n > 1$ then*

$$\dim C_{q,Q}(3n,r) < \binom{5n^2+r-1}{r}.$$

PROOF. It is enough to prove this for the case $r = 2, n > 1$, then the general case $r,n > 1$ of the theorem comes for free. Consider the element $c_{1,2}c_{\bar{1},1} \in C_{q,Q}(3n,2)$. Then using the relations we have that

$c_{1,2}c_{\bar{1},1} = Qc_{1,2}c_{1,\bar{1}} = Qc_{1,\bar{1}}c_{1,2}$

but also

$c_{1,2}c_{\bar{1},1} = c_{\bar{1},1}c_{1,2} + (q-1)c_{\bar{1},2}c_{1,1} = Qc_{1,\bar{1}}c_{1,2} + Q(q-1)c_{1,\bar{2}}c_{1,1}$

which as $Q \neq 0$ implies that

$$c_{1,\bar{2}}c_{1,1} = 0, \text{ unless } q = 1.$$

Therefore if $q \neq 1$ and $n > 1$ then $C_{q,Q}(3n,2)$ is spanned as a k-space by the set $\{c_{\mathbf{i},\mathbf{j}} \mid (\mathbf{i},\mathbf{j}) \in \mathfrak{B}(n,2)\} \setminus \{c_{1,\bar{2}}\, c_{1,1}\}$, so that when $n > 1$

$$\dim C_{q,Q}(3n,2) < \binom{5n^2+1}{2}.$$

The general case now follows.

□

COROLLARY 5.13. *If $q \neq 1$ and $r, n > 1$ then*

$$\dim C_{q,Q}(3n,r)^* < \binom{5n^2+r-1}{r} = \dim \mathcal{S}_q^2(n,r),$$

so that in particular $C_{q,Q}(3n,r)^*$ is not isomorphic to $\mathcal{S}_q^2(n,r)$ in these cases .

We therefore conclude that it is not possible for all values of q and Q to use the $\mathcal{H}$-action on $Y^{\otimes r}$ to build a bialgebra having the dual of its r^{th} homogeneous part isomorphic to $\mathcal{S}_q^2(n,r)$ using this method. It is however certainly possible in the group case, but not in general for $\mathcal{H}$.

5. 3. Associated Linear groups

We close this chapter on a more positive note, by defining a subgroup of $\mathrm{GL}_{3n}(k)$ whose action on $Y^{\otimes r}$ commutes with that of $\mathrm{Hyp}(r)$. This is an analogue of the fact that in type A the actions of $\mathrm{Sym}(r)$ and $\mathrm{GL}_n(k)$ on $E^{\otimes r}$ commute, so we have introduced a type B linear group into our situation.

Let S be the member of GL_n with S_{ij} equal to 1 if $i+j=n+1$ and 0 otherwise. We can now make the following definition.

DEFINITION. For all $i \geqslant 1$, define $\Gamma = \Gamma_{3n}(k)$ to be the subgroup of $\mathrm{GL}_{3n}(k)$ consisting of invertible matrices of the form

$$\begin{pmatrix} A & B & C \\ D & E & DS \\ SCS & SB & SAS \end{pmatrix},$$ where A, B, C, D and E are $n \times n$ matrices.

We index the rows and columns of these matrices by the set $Z(n)$, with respect to the usual ordering $\overline{n} < \cdots < \overline{1} < \widehat{1} < \cdots < \widehat{n} < 1 < \cdots < n$.

We now derive an action of our new group Γ on $Y^{\otimes r}$. The k-space Y is a left $k\Gamma$-module, with $g \in \Gamma$ acting via

$$gy_b = \textstyle\sum_{a\in Z(n)} g_{a,b}y_a.$$

Now for $a, b \in Z(n)$ and $g \in \Gamma$ we put $c_{a,b}(g) = g_{a,b}$, and define $c_{\mathbf{i},\mathbf{j}}(g) = c_{i_1,j_1}(g)\cdots c_{i_r,j_r}(g)$ for all $g \in \Gamma$ and $\mathbf{i},\mathbf{j} \in I_B(n,r)$. Note that from the definition of Γ we have that $c_{a,b}(g) = c_{as_0,bs_0}(g)$ for all $a, b \in Z(n)$.

Then Γ acts diagonally on $Y^{\otimes r}$. Relative to our basis Ξ of $Y^{\otimes r}$ the action is given by

$$\begin{aligned} g.y_{\mathbf{j}} = gy_{j_1} \otimes \cdots \otimes gy_{j_r} &= \textstyle\sum_{\mathbf{i}\in I_B(n,r)} g_{i_1,j_1}\cdots g_{i_r,j_r}y_{\mathbf{i}} \\ &= \textstyle\sum_{\mathbf{i}\in I_B(n,r)} c_{\mathbf{i},\mathbf{j}}(g)y_{\mathbf{i}}. \end{aligned}$$

PROPOSITION 5.14. *The actions of* Γ *and* $\mathrm{Hyp}(r)$ *on* $Y^{\otimes r}$ *centralize each other.*

PROOF. We know from earlier that $c_{a,b}(g) = c_{as_0,bs_0}(g)$ for all $a, b \in Z(n)$, and also it is trivial to see that $c_{\mathbf{i},\mathbf{j}}(g) = c_{\mathbf{i}s_a,\mathbf{j}s_a}(g)$ for all $1 \leqslant a \leqslant r-1$. Putting these together we get that

$$c_{\mathbf{i},\mathbf{j}}(g) = c_{\mathbf{i}\pi,\mathbf{j}\pi}(g)$$

for all $\mathbf{i},\mathbf{j} \in I_B(n,r)$, $g \in \Gamma$ and $\pi \in \mathrm{Hyp}(r)$.

Now, to prove the theorem it suffices to prove that

$$(gy_{\mathbf{j}})\pi = g(y_{\mathbf{j}}\pi)$$

for all $g \in \Gamma$, $y_{\mathbf{j}} \in \Xi$ and $\pi \in \mathrm{Hyp}(r)$. This then extends linearly. Now $(gy_{\mathbf{j}})\pi = (\sum_{\mathbf{i}\in I_B(n,r)} c_{\mathbf{i},\mathbf{j}}(g)y_{\mathbf{i}})\pi = \sum_{\mathbf{i}\in I_B(n,r)} c_{\mathbf{i},\mathbf{j}}(g)y_{\mathbf{i}\pi}$

and

$g(y_{\mathbf{j}}\pi) = g(y_{\mathbf{j}\pi}) = \sum_{\mathbf{i}\in I_B(n,r)} c_{\mathbf{i},\mathbf{j}\pi}(g)y_{\mathbf{i}} = \sum_{\mathbf{i}\in I_B(n,r)} c_{\mathbf{i}\pi,\mathbf{j}\pi}(g)y_{\mathbf{i}\pi}.$

But $c_{\mathbf{i},\mathbf{j}}(g) = c_{\mathbf{i}\pi,\mathbf{j}\pi}(g)$, which tells us that

$$(gy_{\mathbf{j}})\pi = g(y_{\mathbf{j}}\pi)$$

and we are done. $\square$

CHAPTER 6

Wreath products and Mackey Systems

Our results on trivial source modules and Mackey systems from previous chapters lead us to make the following definition.

DEFINITION. Let G be a finite group, p a prime, and $\mathcal{M}$ a p-Mackey system for G. Then we call $\mathcal{M}$ balanced if $\mathrm{TSM}(\mathcal{M}) = \#_0(G)$ i.e. if the number of isomorphism types of indecomposable trivial source modules over $\mathcal{M}$ is equal to the number of conjugacy classes of G, and therefore to the number of ordinary irreducible representations of G.

Note that our model Mackey system, that of Young subgroups of the symmetric group, is balanced for all p. Also, in the characteristic zero case all 0-Mackey systems for a finite group G have $\mathrm{TSM}(G) = \#_0(G)$, so the above definition in some way extends what is happening in this case. Given a balanced p-Mackey system $\mathcal{B}$ for a finite group G we define the Mackey algebra $\mathrm{Mac}(G, \mathcal{B})$ as follows. Let $N = \bigoplus_{A \in \mathcal{B}} \mathrm{ind}_A^G k$. Then $\mathrm{Mac}(G, \mathcal{B}) := \mathrm{End}_{kG}(N)$, is the Mackey algebra associated to G and $\mathcal{B}$. This gives us an analogue of the Schur algebra for each finite group G with balanced Mackey system $\mathcal{B}$. In fact when $\mathcal{B}$ is the Mackey system of Young subgroups of the symmetric group, $\mathrm{Mac}(\mathrm{Sym}(r), \mathcal{B})$ is Morita equivalent to $\mathcal{S}(n, r)$, and when $\mathcal{B}$ is the Mackey system of infant subgroups $\mathrm{Mac}(\mathrm{Hyp}(r), \mathcal{B})$ is Morita equivalent to $\mathcal{S}^2(n, r)$, both by Theorem 1.3.

In this chapter we show that the Mackey system of Young subgroups inside certain wreath products, called complete monomial groups, is a balanced Mackey system for certain primes p. Throughout this section G is a finite group, and unless otherwise stated we work over an algebraically closed field whose characteristic p does not divide the order of G.

6.1. Wreath products and representations

It will be useful to consider $G \wr \mathrm{Sym}(r)$ as a permutation group. Now G is a finite group, so Cayley's subgroup theorem [**21**, 9.24] tells us that G is a subgroup of some symmetric group, which we may as well assume is $\mathrm{Sym}(\Gamma)$, where $\Gamma = \{1, 2, \ldots, m\}$, for some $m \in \mathbb{Z}_{\geqslant 1}$. Also let $\Omega = \{1, 2, \ldots, r\}$, so that $\mathrm{Sym}(\Omega)$ is $\mathrm{Sym}(r)$. We write i^j for the ordered pair $(i, j) \in \Omega \times \Gamma$, and for $i \in [1, r-1]$ let

$$s_i = (i^1\,(i+1)^1)(i^2\,(i+1)^2)\cdots(i^m\,(i+1)^m) \in \mathrm{Sym}(\Omega \times \Gamma).$$

Then $\langle s_1, s_2, \ldots, s_{r-1}\rangle \cong \mathrm{Sym}(r)$ and we write $S(r)$ for this subgroup of $G \wr \mathrm{Sym}(r)$, analogously to in the previous chapters . Now, assume G has t generators $g_1, g_2, \ldots, g_t$. Then each g_i can be written as a product of l distinct cycles of elements of Γ, say $g_i = g_{i,1}g_{i,2}\cdots g_{i,l}$, so that for each $j \in \underline{l}$ we have $g_{i,j} = (g_{i,j}(1)g_{i,j}(2)\cdots g_{i,j}(k(j)))$, for some particular $g_{i,j}(z) \in \Gamma$ and $k(j) \in \mathbb{Z}_{\geqslant 0}$. Putting these together we have that

$$g_i = \textstyle\prod_{j=1}^{l}(g_{i,j}(1)g_{i,j}(2)\cdots g_{i,j}(k(j))),$$

explicitly as a product of distinct cycles of elements of Γ. For $b \in \underline{r}$ put

$$\gamma_{i,b} = \textstyle\prod_{j=1}^{l}(b^{g_{i,j}(1)}b^{g_{i,j}(2)}\ldots b^{g_{i,j}(k(j))}) \in \mathrm{Sym}(\Omega \times \Gamma).$$

Then $\langle \gamma_{i,b} \,|\, i \in [1, t], b \in [1, m]\rangle$ is isomorphic to G^r and we also have that $\gamma_{i,b}^{s_b} = \gamma_{i,b+1}$, so that the conjugation action of $S(r)$ simply permutes the factors of G^r, and thus $G^r \lhd G \wr \mathrm{Sym}(r)$. Hence the subgroup of $\mathrm{Sym}(\Omega \times \Gamma)$ generated by $\{s_1, s_2, \cdots, s_{r-1}, \gamma_{1,1}, \gamma_{2,1}, \ldots, \gamma_{t,1}\}$ is a semidirect product and is isomorphic to $G \wr \mathrm{Sym}(r)$.

As an example, we look at the complete monomial group $\mathrm{Sym}(3) \wr \mathrm{Sym}(2)$. Here we have that $\Omega = \{1, 2\}$ and $\Gamma = \{1, 2, 3\}$, so all elements of this group are permutations of the set $\{1^1, 1^2, 1^3, 2^1, 2^2, 2^3\}$. Firstly we look at the top part. Here we have just one generator, namely

$$s_1 = (1^1 2^1)(1^2 2^2)(1^3 2^3)$$

and the group generated by the single element s_1 is isomorphic to $\mathrm{Sym}(2)$ as a subgroup of $\mathrm{Sym}(3) \wr \mathrm{Sym}(2)$. Now $\mathrm{Sym}(3)$ has generators $g_1 = (12)$ and $g_2 = (23)$ which give us corresponding elements

$$\gamma_{1,1} = (1^1 1^2) \text{ and } \gamma_{2,1} = (1^2 1^3)$$

with $\langle \gamma_{1,1}, \gamma_{2,1} \rangle$ isomorphic to $\mathrm{Sym}(3)$. Now $s_1 \gamma_{1,1} s_1 = (2^1 2^2) = \gamma_{1,2}$ and $s_1 \gamma_{2,1} s_1 = (2^2 2^3) = \gamma_{2,2}$, so that $\langle \gamma_{1,1}, \gamma_{1,2}, \gamma_{2,1}, \gamma_{2,2} \rangle$ is isomorphic to $\mathrm{Sym}(3)^2$. This is the base group of $\mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ and is a normal subgroup. Hence $\langle s_1, \gamma_{1,1}, \gamma_{2,1} \rangle$ is isomorphic to $\mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ as required.

$G \wr \mathrm{Sym}(r)$ can also be written as linear group. We start by looking at a way of viewing the symmetric group (which trivially is $\{1\} \wr \mathrm{Sym}(r)$) as a linear group. Each $\sigma \in \mathrm{Sym}(r)$ can be represented as an $r \times r$ permutation matrix $P(\sigma)$ with (i, j) entry

$$P(\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

Then if $\sigma, \sigma', \pi \in \mathrm{Sym}(r)$ with $\sigma\sigma' = \pi$ then $P(\sigma)P(\sigma') = P(\pi)$ and the group $\{P(\sigma) \mid \sigma \in \mathrm{Sym}(r)\}$ is a linear group isomorphic to $\mathrm{Sym}(r)$.

Now allow the non-zero entries in our matrices to be any member of the finite group G, and call this group $\mathrm{Mono}(G)$. Then the normal subgroup of $\mathrm{Mono}(G)$ consisting of matrices where all non-zero entries occur on the diagonal is isomorphic to G^r, and also the subgroup whose non-zero entries are all equal to 1 is isomorphic to $\mathrm{Sym}(r)$. From this description it is easy to see that $\mathrm{Mono}(G)$ is the semidirect product of the above two subgroups, and that it is indeed isomorphic to $G \wr \mathrm{Sym}(r)$.

Again suppose G has t generators $g_1, g_2, \ldots, g_t$. Let

$$\gamma_{i,b} = \mathrm{diag}(1, \ldots, 1, g_i, 1, \ldots, 1) \in \mathrm{Mono}(G),$$

the g_i being in the b^{th} place, and for $i \in [1, r-1]$, let s_i be the $r \times r$ identity matrix with the i^{th} and $i+1^{th}$ columns switched. Then $\mathrm{Mono}(G)$ is generated by the matrices $\{\gamma_{1,1}, \gamma_{2,1}, \ldots, \gamma_{t,1}, s_1, s_2, \ldots, s_{r-1}\}$.

This gives us two useful descriptions of $G \wr \mathrm{Sym}(r)$. Note both are generated by elements $S = \{s_1, \ldots, s_{r-1}\}$, which generate the top group, $S(r)$, and by elements $B = \{\gamma_{1,1}, \gamma_{2,1}, \ldots, \gamma_{t,1}\}$, which together with their conjugates in $\mathrm{Sym}(r)$ generate the base group G^r.

It will be useful later on to know the number of both ordinary and modular representations of $G \wr \mathrm{Sym}(r)$, and we review these now.

Let k be an algebraically closed field of characteristic $p \neq 0$, where p is coprime to $|G \wr \mathrm{Sym}(r)|$. If G has m conjugacy classes (i.e. G has m inequivalent ordinary irreducibles), then by [**25**, 5.21] we have that $G \wr \mathrm{Sym}(r)$ has $|\Lambda_m^+(n, r)|$ ordinary irreducible representations. Note that this agrees with the particular case $\mathrm{Hyp}(r) = C_2 \wr \mathrm{Sym}(r)$ that we considered earlier.

Now say char $k = p$ divides $|G \wr \mathrm{Sym}(r)|$. Then [**25**, 5.22] says that if t is the number of p-regular classes of G then $G \wr \mathrm{Sym}(r)$ has $|\Lambda_t^+(n, r)_{col}|$ irreducibles over k.

In particular, if p does not divide $|G|$ but does divide $|\,\mathrm{Sym}(r)|$ then $G \wr \mathrm{Sym}(r)$ has $|\Lambda_m^+(n, r)_{col}|$ irreducibles over k, as G has m conjugacy classes.

6.2. A Mackey system

Unsurprisingly, we want to consider the Mackey system of Young subgroups inside our wreath product. So firstly we need to define this.

DEFINITION. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition in $\Lambda^+(n, r)$, and put $L = \{\sum_{i=1}^{a} \lambda_i \mid a \in \underline{r}\}$. Put $A = \underline{r} \setminus L$. Then the standard Young subgroup of $G \wr \mathrm{Sym}(r)$ associated to λ, denoted $S(\lambda)$ is given by

$$S(\lambda) = \langle s_a \mid a \in A \rangle.$$

A Young subgroup is then any which is conjugate in $G \wr \mathrm{Sym}(r)$ to a standard Young subgroup. Let $\mathcal{W}$ denote the set of all Young subgroups of $G \wr \mathrm{Sym}(r)$.

We also have the following useful description of Young subgroups. Let $\Delta(n, r)$ be the set of all dissections of $\underline{r}$ i.e.

$$\Delta(n, r) = \{\delta = (\delta_1, \delta_2, \ldots, \delta_n) \mid \delta_1 \sqcup \delta_2 \sqcup \cdots \sqcup \delta_n = \underline{r}\}$$

where any of the δ_i may be empty. If $1 \leqslant j \leqslant r$ and $\delta_j = \{j_1, j_2, \ldots, j_d\}$, where $d = |\delta_j|$, then we put

$$S(\delta_j) = \langle (j_i^1 j_{i+1}^1) \cdots (j_i^m j_{i+1}^m) \mid 1 \leqslant i \leqslant d-1 \rangle$$

and $S(\delta) = S(\delta_1) \times S(\delta_2) \times \cdots \times S(\delta_n)$, a subgroup of $G \wr \mathrm{Sym}(r)$.

We can make similar definitions for $S(\delta) \leqslant \mathrm{Mono}(G)$. Again, for $j \in \underline{r}$ let $\delta_j = \{j_1, j_2, \ldots, j_d\}$, where $d = |\delta_j|$. Then for $i \in [1, d-1]$ let M_i be the $r \times r$ identity matrix, but with the j_i^{th} and the j_{i+1}^{th} columns switched. Our group $S(\delta_j)$ is the linear group generated by $M_1, M_2, \ldots, M_{d-1}$, and we similarly have $S(\delta) = S(\delta_1) \times S(\delta_2) \times \cdots \times S(\delta_n)$.

If we now take an $\delta^s \in \Delta(n, r)$ with $\delta_1^s = \{1, 2, \ldots, |\delta_1|\}$, $\delta_2^s = \{|\delta_1| + 1, |\delta_1| + 2, \ldots, |\delta_1| + |\delta_2|\}$, $\delta_3^s = \ldots$ etc., and with

$|\delta_i^s| \geqslant |\delta_{i+1}^s|$ for all $i \in \underline{r}$, then we have that $S(\delta^s) = S(\lambda)$ for some $\lambda \in \Lambda^+(n,r)$, so our $S(\delta^s)$ is a standard Young subgroup. Call such a dissection $\delta^s \in \Delta(n,r)$ a standard dissection, and let $\Delta^+(n,r)$ denote the set of all standard dissections of $\underline{r}$. Then for each $\delta \in \Delta(n,r)$, there exists a $\delta^s \in \Delta^+(n,r)$ and a $\pi \in S(r)$ such that

$$S(\delta) = S(\delta^s)^\pi. \quad (*)$$

Then the following lemma brings together dissections and partitions.

LEMMA 6.1. $\mathcal{W} = \{S(\lambda)^\omega \mid \lambda \in \Lambda^+(n,r), \omega \in G \wr \mathrm{Sym}(r)\}$
$= \{S(\delta^s)^\omega \mid \delta^s \in \Delta^+(n,r), \omega \in G \wr \mathrm{Sym}(r)\}$,
$= \{S(\delta)^\pi \mid \delta \in \Delta(n,r), \pi \in G \wr \mathrm{Sym}(r)\}$.

PROOF. The first equality is trivial. That the second set is contained in the third is trivial, as each $\delta^s \in \Delta^+(n,r)$ is in $\Delta(n,r)$ also, and conversely $(*)$ gives us that $S(\delta)^\omega = (S(\delta^s)^\pi)^\omega$, which is $S(\delta^s)^\tau$, for some $\tau \in G \wr \mathrm{Sym}(r)$, and the proof is complete. $\square$

We take the following definition from the world of Association Schemes.

DEFINITION. Let $\delta, \delta' \in \Delta(n,r)$. We write $\delta \preceq \delta'$ if every part of δ is contained in a part of δ'. We then say that δ is finer than δ'. We denote by $\delta \wedge \delta'$ the coarsest (least fine) dissection in $\Delta(n,r)$ such that both $\delta \wedge \delta' \preceq \delta$ and $\delta \wedge \delta' \preceq \delta'$.

It is then easy to see the following.

LEMMA 6.2. $S(\delta) \cap S(\delta') = S(\delta \wedge \delta')$.

From this it follows that the set of Young subgroups of the symmetric group is closed under intersection. We can now show the same is true for Young subgroups of $G \wr \mathrm{Sym}(r)$.

LEMMA 6.3. *$\mathcal{W}$ is closed under conjugation.*

PROOF. Throughout this proof, we will have in mind the linear representation of $G \wr \mathrm{Sym}(r)$, and $\mathcal{W}$ in its third incarnation in the above lemma, i.e. indexed by dissections, and members of $G \wr \mathrm{Sym}(r)$.

It suffices to prove that $S(\delta)^{\omega} \cap S(\delta') \in \mathcal{W}$, for $\delta, \delta' \in \Delta(n, r)$ and $w \in G \wr \mathrm{Sym}(r)$. Moreover, since each element $\omega \in G \wr \mathrm{Sym}(r)$ can be written uniquely as πg with $\pi \in S(r)$ and $g \in G^r$, and as $S(\delta)^{\pi} = S(\delta'')$ for some $\delta'' \in \Delta(n, r)$, we just need to show that $S(\delta)^g \cap S(\delta') \in \mathcal{W}$ for all $\delta, \delta' \in \Delta(n, r)$ and $g \in G^r$. Now as $S(\delta)^g \cap S(\delta') = (S(\delta)^g \cap S([1, r])) \cap S(\delta')$, we just need to show that $(S(\delta)^g \cap S([1, r])) = S(\delta'')$ for some $\delta'' \in \Delta(n, r)$, then we will have $S(\delta)^g \cap S(\delta') = S(\delta'') \cap S(\delta') = S(\delta'' \wedge \delta')$.

Now any element in $S(\delta)$ can be written as a matrix $P(\sigma)$ where

$$P(\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

for some suitable $\sigma \in \mathrm{Sym}(r)$.(Of course this σ is actually in a particular Young subgroup $\mathrm{Sym}(\lambda)$ of $\mathrm{Sym}(r)$ with $\mathrm{Sym}(\lambda) \cong S(\delta)$.) Also, any element $g \in G^r$ is a matrix of the form $g = \mathrm{diag}(g_1, g_2, \ldots, g_r)$, with each $g_i \in G$. Therefore any element in $S(\delta)^g$ is a matrix

$$(gP(\sigma)g^{-1})_{i,j} = \begin{cases} g_i g_j^{-1} & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

so that $P(\sigma) \in S(\delta) \cap S([1, r])$ if and only if $g_i = g_j$ whenever $\sigma(j) = i$.

Now each dissection $\delta \in \Delta(n, r)$ can also be thought of as an equivalence relation $\sim_\delta$ on $\underline{r}$, so that $i \sim_\delta j$ if and only if i and j are both in the same part δ_s of δ for some $s \in \underline{n}$. We then define a new equivalence relation by letting $i \sim_{\delta,g} j$ if and only if $i \sim_\delta j$ and $g_i = g_j$. This equivalence relation on r leads to a dissection of r which we will denote by $\delta^g \in \Delta(n,r)$. Hence we also get a Young subgroup $S(\delta^g)$ of $G \wr \mathrm{Sym}(r)$. Then

$P(\sigma) \in S(\delta)^g \cap S([1,r])$

$\Longleftrightarrow$ $g_i = g_j$ whenever $\sigma(j) = i$

$\Longleftrightarrow$ $g_i = g_j$ whenever $i \sim_\delta j$

$\Longleftrightarrow$ $i \sim_{\delta,g} j$

$\Longleftrightarrow$ $P(\sigma) \in S(\delta^g)$. Therefore $S(\delta)^g \cap S([1,r]) = S(\delta^g)$ so that $S(\delta)^g \cap S(\delta') = (S(\delta)^g \cap S([1,r])) \cap S(\delta') = S(\delta^g) \cap S(\delta') = S(\delta^g \wedge \delta')$ and of course $\delta^g \wedge \delta' \in \Delta(n,r)$ so that $\mathcal{W}$ is closed under intersection. □

We can now show the following.

LEMMA 6.4. *If p and $|G|$ are coprime, then $\mathcal{W}$ is a p-Mackey system for $G \wr \mathrm{Sym}(r)$.*

PROOF. (M1) Put $\lambda = 1^r$. Then $S(\lambda) = \{1\}$ and so $\{1\} \in \mathcal{W}$.
(M2)(i) By definition $\mathcal{W}$ is closed under conjugation in $G \wr \mathrm{Sym}(r)$.
(M2)(ii) $\mathcal{W}$ is closed under intersection by the preceeding lemma.
(M3) Since p does not divide $|G|$, any $P \in \mathrm{Syl}_p(G \wr \mathrm{Sym}(r))$ must be a subgroup of $S(r) \leqslant G \wr \mathrm{Sym}(r)$. Then putting $\lambda = (r)$ we get that $S(\lambda) = S(r)$, so that $S(r) \in \mathcal{W}$ and P is a subgroup of a member of $\mathcal{W}$. □

We now have a Mackey system so we can apply Grabmeier's count.

6.3. The Count

We now modify the definition of p-power partitions for this particular chapter. For a prime p we say a composition $\lambda \in \Lambda_2(n,r)$ is a

p-power partition if $\lambda = (1^{a_0}p^{a_1}(p^2)^{a_2}\dots)$. We will denote the set of these by $\beth_p(n,r)$. From previous adventures with Young subgroups we can deduce the following.

LEMMA 6.5. *The set of $\mathcal{W}$-vertices, denoted $\mathcal{W}_0$, is given by*

$$\mathcal{W}_0 = \{S(\lambda)^\omega \mid \lambda \in \beth_p(n,r), \text{ and } \omega \in G \wr \mathrm{Sym}(r)\}.$$

Hence for $\mathcal{W}_0'$, the set of $\mathcal{W}$-vertices up to conjugacy, we can take

$$\mathcal{W}_0' = \{S(\lambda) \mid \lambda \in \beth_p(n,r)\}.$$

The next step in Grabmeier's count is for us to work out $\frac{N_{G\wr \mathrm{Sym}(r)}(P)}{P}$ for each vertex $P \in \mathcal{W}_0'$.

LEMMA 6.6. *Let $\lambda = (1^{a_0}p^{a_1}(p^2)^{a_2}\dots) \in \beth_p(n,r)$. Then*

$$\frac{N_{G\wr \mathrm{Sym}(r)}(S(\lambda))}{S(\lambda)} \text{ is isomorphic to}$$

$$(G \wr \mathrm{Sym}(a_0)) \times (G \wr \mathrm{Sym}(a_1)) \times (G \wr \mathrm{Sym}(a_2)) \times \dots$$

We call this group $N(\lambda)_G$.

PROOF. Let $\lambda \in \beth_p(n,r)$, and let $H = S(\lambda)$. For each $i \in \underline{n}$ let $a_i = \sum_{l=1}^{i-1} \lambda_i$ and define A_i^j to be the subset of $\Omega \times \Gamma$ given by

$$A_i^j = \{(a_i+1)^j, (a_i+2)^j, \dots, (a_i+\lambda_i)^j\},$$

so that if λ is in $\Lambda(n,r)$ then the set

$$\{A_i^j \mid i \in \underline{n} \text{ and } j \in \underline{m}\}$$

forms a complete set of H-orbits of $\Omega \times \Gamma$.

For $b \geqslant 0$, set $B_b = \{A_i^1, \dots, A_1^m \text{ such that } |A_i^1| = p^b\}$, so that $|B_b| = ma_b$. Let $N(H)= N_{G\wr \mathrm{Sym}(r)}(H)$. Now $N(H)$ permutes the H-orbits of $\Omega \times \Gamma$ of the same size, and we have a wreath product action on each B_b such that each $G \wr \mathrm{Sym}(B_b)$ is isomorphic to $G \wr \mathrm{Sym}(a_b)$. We can know define a homomorphism

$$f : N(H) \to (G \wr \mathrm{Sym}(B_0) \times G \wr \mathrm{Sym}(B_1) \times \dots)$$
$$\text{via } f : \omega(A_i^j) \mapsto \omega(A_i^j)$$

which has $\ker f = H$. We now need to show that f is onto.

Now say that for each $l \in \underline{m}$ the H-orbits $A_j^l = \{x_1^l, x_2^l, \dots, x_{p^i}^l\}$ and $A_k^l = \{z_1^l, z_2^l, \dots, z_{p^i}^l\}$ are in the set B_i. Now define $s \in G \wr \mathrm{Sym}(r)$ via $s(x_a^l) = z_a^l$ and $s(z_a^l) = x_a^l$ for each $a \in [1, p^i]$ and $l \in \underline{m}$ and let s fix all other elements of $\Omega \times \Gamma$. Let $\omega \in H$ then

$$(s^{-1}\omega s)A_j^l = (s^{-1}\omega)A_k^l = s^{-1}A_k^l = sA_k^l = A_j^l$$

for each $l \in \underline{m}$. Therefore $s^{-1}\omega s \in H$ and so $s \in N(H)$.

We must now deal with the base groups. Recall from earlier parts of this chapter that $G = \langle g_1, \dots, g_t \rangle$, and for each $d \in \underline{r}$ we get corresponding elements $\gamma_{i,d} = \prod_{j=1}^{l}(d^{g_{i,j}(1)}d^{g_{i,j}(2)} \dots d^{g_{i,j}(k(j))}) \in \mathrm{Sym}(\Omega \times \Gamma)$, and with the property that the set $\{\gamma_{i,d} \mid i \in [1,t] \text{ and } d \in [1,r]\}$ generates the base group $G^r \leqslant G \wr \mathrm{Sym}(r)$.

Now say that $A_j^1 \in B_i$ so that for each $l \in \underline{m}$ we have that A_j^l is also in B_i. Let $a_e = \sum_{u=1}^{e-1} \lambda_u$ and define for each λ_e in λ an element $\pi_e \in G \wr \mathrm{Sym}(r)$ via

$$\pi_e = \gamma_{i,(a_e+1)}\gamma_{i,(a_e+2)} \cdots \gamma_{i,(a_e+\lambda_e)}.$$

If we again let $\omega \in H$ then for each $l \in [1, m]$ we have

$$(\pi_e^{-1}\omega\pi_e)A_{\lambda_e}^l = \pi_e^{-1}\omega A_{\lambda_e}^{l'} = \pi_e^{-1}A_{\lambda_e}^{l'} = A_{\lambda_e}^l$$

for some $l' \in [1, m]$. So we have that $\pi_e^{-1}\omega\pi_e \in H$ and $\pi_e \in N(H)$.

Now all standard generators of $G \wr \mathrm{Sym}(B_i)$ can be constructed as an $f(\pi_e)$, or as an $f(s)$, and therefore f is onto.

So, by the homomorphism theorem, we have that

$$\frac{N(H)}{H} \cong G \wr \mathrm{Sym}(B_0) \times G \wr \mathrm{Sym}(B_1) \times G \wr \mathrm{Sym}(B_2) \times \dots$$

which is isomorphic to

$$N(\lambda)_G = G \wr \mathrm{Sym}(a_0) \times G \wr \mathrm{Sym}(a_1) \times G \wr \mathrm{Sym}(a_2) \times \dots$$

and we are done. $\square$

Therefore, by Grabmeier's count, when $p \nmid |G|$ we have

$$\mathrm{TSM}(\mathcal{W}) = \textstyle\sum_{\lambda \in \Pi_p(n,r)} \#_p(N(\lambda)_G)$$

where as usual $\#_p$ denotes the number of p'-classes of G. Recall that when $p \nmid |G|$, and there exist precisely m non-isomorphic simple kG-modules, then $\#_p(G \wr \mathrm{Sym}(r)) = |\Lambda^+_m(n,r)_{col}|$. Therefore $\#_p(N(\lambda)_G) =$ the number of m-tuples $(\psi_1, \dots, \psi_m)$ where for $j \in \underline{m}$ we have that $\psi_j = (\psi_j(0), \psi_j(1), \dots)$ are sequences of column p-regular partitions such that $\sum_{j=1}^m \psi_j(i) = a_i$ for each $i \geqslant 0$. Therefore Grabmeier's count now tells us that

$$\begin{aligned}\mathrm{TSM}(\mathcal{W}) &= \textstyle\sum_{\lambda \in \Pi_p(n,r)} \#_p(N(\lambda)_G)\\ &= \textstyle\sum_{\lambda \in \Pi_p(n,r)} \text{number of } m\text{-tuples } (\psi_1, \dots, \psi_m) \text{ as above}\end{aligned}$$

so that in fact $\mathrm{TSM}(\mathcal{W})$ is the number of $(m+1)$-tuples $(\lambda, \psi_1, \dots, \psi_m)$, where $\lambda \in \sqsupset_p(n,r)$ and $\psi_1, \dots, \psi_m$ are as above. Call the set of such $(m+1)$-tuples ß. Now define a map

$$\kappa : \text{ß} \to \Lambda^+_m(n,r) \text{ via } (\lambda, \psi_1, \dots, \psi_m) \mapsto (\alpha_1, \dots, \alpha_m)$$

where $\alpha_j = \sum_{i \geqslant 0} p^i \psi_j(i)$ for each $j \in \underline{m}$.

Lemma 6.7. *κ is a bijection.*

Proof. Due to uniqueness of p-adic expansions, we know that κ is both well defined and injective. Now say $(\alpha_1, \alpha_2, \dots, \alpha_m) \in \Lambda^+_m(n,r)$. Again, by uniqueness, for each $j \in \underline{m}$ we can write $\alpha_j = \sum_{i \geqslant 0} \xi_j(i)$, where the $\xi_j(i)$ are uniquely determined. Now $r = \sum_{j=1}^m |\alpha_j(i)| = \sum_{i \geqslant 0} p^i (\sum_{j=1}^m |\xi_j(i)|)$, so by putting $b_i = \sum_{j=1}^m |\xi_j(i)|$ for all $i \geqslant 0$ we retrieve an $(m+1)$-tuple $(\lambda', \xi_1, \dots, \xi_m)$ where $\lambda' = (\emptyset; \dots; \emptyset; 1^{b_0} p^{b_1} \dots)$

and for $j \in \underline{m}$ each $\xi_j = (\xi_j(0), \xi_j(1), \dots)$ is a sequence of column p-regular partitions such that $\sum_{j=1}^{m} |\xi_j(i)| = b_i$.

Therefore $(\lambda', \xi_1, \dots, \xi_m) \in$ ß with $\kappa(\lambda', \xi_1, \dots, \xi_m) = (\alpha_1, \dots, \alpha_m)$ and κ is surjective also.

$\square$

We get the following theorem as a corollary.

THEOREM 6.8. *When p does not divide $|G|$ we have that* $\mathrm{TSM}(\mathcal{W}) = |\#_0(G \wr \mathrm{Sym}(r))|$ *and the p-Mackey system $\mathcal{W}$ for the group $G \wr \mathrm{Sym}(r)$ is balanced.*

CHAPTER 7

Mackey Systems and Partition Algebras

In this chapter we consider a new Mackey system and determine when it is balanced. These methods will then help to us consider the irreducible representations of the partition algebras.

An unlikely Mackey system

We let $\mathcal{P}$ be the following system of subgroups of $\mathrm{Sym}(n)$:

$$\mathcal{P} = \{\mathrm{Sym}(j)^{\omega} \mid j \in \{1,2,\ldots,n\}\,,\, \omega \in \mathrm{Sym}(n)\}.$$

Note that $\mathcal{P}$ can also be written as

$$\mathcal{P} = \{\mathrm{Sym}(X) \mid X \subseteq \{1,2,\ldots,n\}\}.$$

At first glance this looks too simple to be a Mackey system. However, we have the following.

LEMMA 7.1. *Let p be a prime. Then $\mathcal{P}$ is a p-Mackey system for* $\mathrm{Sym}(n)$.

PROOF. (M1) $\mathrm{Sym}(\emptyset) = \{1\}$ is in $\mathcal{P}$.
(M2)(i) $\mathcal{P}$ is closed under conjugation by definition.
(M2)(ii) Recall that each member of $\mathcal{P}$ looks like $\mathrm{Sym}(X)$ for some $X \subseteq \{1,2,\ldots,n\}$. Now say $X, Y \subseteq \{1,2,\ldots,n\}$ then

$$\mathrm{Sym}(X) \cap \mathrm{Sym}(Y) = \mathrm{Sym}(X \cap Y)$$

and since $X \cap Y \subseteq \{1,2,\ldots,r\}$ we have that

$$\mathrm{Sym}(X) \cap \mathrm{Sym}(Y) \in \mathcal{P}$$

and $\mathcal{P}$ is closed under intersection.

(M3) $\mathrm{Sym}(n) \in \mathcal{P}$ and therefore this is trivially satisfied. $\square$

Now for the vertices, which we recall are

$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid p \text{ divides } |P : B| \text{ for all } B \in \mathcal{P} \text{ with } B < P\}.$$

LEMMA 7.2. *We have*

$\mathcal{P}_0 = \{\{1\}\} \cup \{\mathrm{Sym}(X) \mid X \subseteq \{1, 2, \ldots, n\}$ *and* p *divides* $|X|\}$.

PROOF. $\{1\}$ is trivially a member of $\mathcal{P}_0$.

Now say $X \subseteq \{1, 2, \ldots, n\}$ with $|X| = px$ for some positive integer x, and put $P = \mathrm{Sym}(X)$. Then every proper subgroup of P in $\mathcal{P}$ is a subgroup of some $H = \mathrm{Sym}(Y)$, where $Y \subset X$ with $|Y| = px - 1$, or is H itself. Since $|P : H| = px$ we must have p divides $|P : K|$ for all $K \in \mathcal{P}$ with $K < P$. Therefore each $\mathrm{Sym}(X)$ with $X \subseteq \{1, 2, \ldots, n\}$ and $|X| = px$ is a member of $\mathcal{P}_0$.

Conversely, if a member of $\mathcal{P}$ is not of the above form, then it must look like $G = \mathrm{Sym}(X)$, where $X \subseteq \{1, 2, \ldots, n\}$ with $|X| = px + y$ where x is a non-negative integer and $1 \leqslant y \leqslant p - 1$. Each G has subgroup $H = \mathrm{Sym}(Y)$ where $Y \subset X$ and $|Y| = px + y - 1$. Now $|G : H| = px + y$, and since $(p, y) = 1$ we get that $|G : H|$ and p are also coprime. Therefore each of the above subgroups G has a proper subgroup, like H, of p' index and we are done. $\square$

So for $\mathcal{P}'_0$, the set of vertices up to conjugacy, we can take the set $\{\mathrm{Sym}(ip) \mid 0 \leqslant i \leqslant \lfloor n/p \rfloor\}$, bearing in mind that $\mathrm{Sym}(0) = \{1\}$. Now each of these is a Young subgroup and from [**27**, 4.6.3] we know the normalizer in $\mathrm{Sym}(n)$ of each of its Young subgroups. From this we get the following.

LEMMA 7.3. *For* $0 \leqslant i \leqslant \lfloor n/p \rfloor$ *we have*

$$\frac{N_{\mathrm{Sym}(n)}(\mathrm{Sym}(ip))}{\mathrm{Sym}(ip)} \cong \mathrm{Sym}(n-ip).$$

Let $\Lambda^+(r)$ denote the set of all partitions of r, so that when $r \leqslant n$ we have that $\Lambda^+(n,r) = \Lambda^+(r)$. Then for $s \geqslant 0$, let $\Lambda^+(s)_p$ denote the set of partitions of s which correspond to p-regular conjugacy classes of $\mathrm{Sym}(s)$ i.e. if $\lambda \in \Lambda^+(s)_p$ then $(\lambda_i, p) = 1$ for all $1 \leqslant i \leqslant s$. Call these class p-regular partitions. Of course when $s \leqslant n$ we have $|\Lambda^+(s)_p| = |\Lambda^+(n,s)_{col}|$, so that applying Grabmeier's count we have that

$$\mathrm{TSM}(\mathcal{P}) = \textstyle\sum_{i\geqslant 0} \#_p(\mathrm{Sym}(n-ip)) = \sum_{i\geqslant 0} |\Lambda^+(n-ip)_p|.$$

The corollory at the end of the following discussion will enable us to decide precisely when the Mackey system $\mathcal{P}$ is balanced.

Let $A = \bigsqcup_{i\geqslant 0} \Lambda^+(i) \times \Lambda^+(n-ip)_p$, and define

$$f : A \to \Lambda^+(n) \text{ via } f((\mu,\nu)) = p\mu \cup \nu$$

where for $\mu = (\mu_1, \mu_2, \ldots, \mu_i) \in \Lambda^+(i)$ and our prime p we have that $p\mu = (p\mu_1, p\mu_2, \ldots, p\mu_i)$, and for $\tau(1) \in \Lambda^+(a)$ and $\tau(2) \in \Lambda^+(n-a)$ we have that $\tau(1) \cup \tau(2)$ is the member of $\Lambda^+(n)$ given by arranging the parts $\tau(1)_1, \tau(1)_2, \ldots, \tau(1)_a, \tau(2)_1, \tau(2)_2, \ldots, \tau(2)_{n-a}$ in descending order.

LEMMA 7.4. *f is a bijection.*

PROOF. Say $f((\mu,\nu)) = f((\mu',\nu')) \in \Lambda^+(n)$. Then the parts of these two partitions which are divisible by p must be the same, and hence on dividing these parts by p we get that $\mu = \mu'$. The parts left over must be the same in each partition, and all must be coprime to p, which gives that both ν and ν' are p-regular and moreover that $\nu = \nu'$. Therefore $f((\mu,\nu)) = f((\mu',\nu')) \implies (\mu,\nu) = (\mu',\nu')$ and f is injective.

Now say that $\lambda \in \Lambda^+(n)$. We can arrange all parts of λ which are coprime to p in descending order to give a class p-regular partition $\tau_p \in \Lambda^+(n-a)_p$, say. The leftover parts will all be divisible by p, and so on dividing these parts by p we get a partition $\tau \in \Lambda^+(a/p)$. Putting $a = ip$, which it must do for some $i \geqslant 0$, we have a pair of partitions $(\tau, \tau_p) \in \Lambda^+(i) \times \Lambda^+(n-ip)_p$ with $f((\tau, \tau_p)) = \lambda$, which tells us that f is surjective. $\square$

COROLLARY 7.5. $|\Lambda^+(n)| = \sum_{i \geqslant 0} |\Lambda^+(i)||\Lambda^+(n-ip)_p|$, $\forall n \geqslant 0$.

We can now prove the following.

PROPOSITION 7.6. *$\mathcal{P}$ is a balanced p-Mackey system for* Sym(n) *if and only if $p > n/2$ i.e.* $\mathrm{TSM}(\mathcal{P}) = |\Lambda^+(n)| \iff p > n/2$.

PROOF. If $i > \lfloor n/p \rfloor$ then $|\Lambda^+(n-ip)_p| = 0$. This gives us that

$$\mathrm{TSM}(\mathcal{S}) = \textstyle\sum_{i \geqslant 0} |\Lambda^+(n-ip)_p| = \sum_{i=0}^{\lfloor n/p \rfloor} |\Lambda^+(n-ip)_p|$$

and from the above corollary we know that

$$|\Lambda^+(n)| = \textstyle\sum_{i \geqslant 0} |\Lambda^+(i)||\Lambda^+(n-ip)_p| = \sum_{i=0}^{\lfloor n/p \rfloor} |\Lambda^+(i)||\Lambda^+(n-ip)_p|.$$

Therefore, for $|\Lambda^+(n)|$ and $\mathrm{TSM}(\mathcal{P})$ to be equal, we would require that $|\Lambda^+(i)| = 1$ for each $0 \leqslant i \leqslant \lfloor n/p \rfloor$, but since $|\Lambda^+(i)| > 1$ whenever $i > 1$ we must have that $0 \leqslant i \leqslant 1$. Therefore we have equality if and only if $\lfloor n/p \rfloor \leqslant 1$ i.e. if and only if $p > n/2$. $\square$

7. 1. $\mathcal{P}$ in action

We have now applied Grabmeier's count to the p-Mackey system $\mathcal{P}$ of Sym(n). We can now show that in certain cases the point stabilisers of basis elements of a Sym(n)-action on a familiar space, called $E_n^{\otimes r}$, are nearly the same as our Mackey system $\mathcal{P}$. Then for particular values of p, we can use our count of the last section to count the number of trivial source modules of $E_n^{\otimes r}$ as a $\mathrm{Sym}(n)$-module, and hence calculate the number of irreducible $\mathrm{End}_{\mathrm{Sym}(n)}(E_n^{\otimes r})$-modules to boot.

Let $r, n \geqslant 0$, and also let E_n be an n-dimensional k-vector space with basis $\{e_1, e_2, \ldots, e_n\}$. Then r^{th} tensor power of E_n, denoted $E_n^{\otimes r}$, has k-basis $\{e_\mathbf{i} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} | \mathbf{i} = (i_1, i_2, \ldots, i_r) \in I(n,r)\}$. Of course, we usually denote this module by $E^{\otimes r}$, but now n is of increased importance. As in chapter 1 this is a right $\mathrm{Sym}(r)$-module, the action being via place permutation, and the resulting endomorphism algebra $\mathrm{End}_{k\,\mathrm{Sym}(r)}(E_n^{\otimes r})$ is the classical Schur algebra.

But we can also make $\mathrm{Sym}(n)$ act on the left of $I(n,r)$. The action of $\pi \in \mathrm{Sym}(n)$ on an element $\mathbf{i} = (i_1, i_2, \ldots, i_r) \in I(n,r)$ is given by

$$\pi\mathbf{i} = \pi(i_1, i_2, \ldots, i_r) = (\pi(i_1), \pi(i_2), \ldots, \pi(i_r)).$$

We can now transfer this action to $E_n^{\otimes r}$ via $\pi e_\mathbf{i} = e_{\pi\mathbf{i}}$. Extending this action linearly makes $E_n^{\otimes r}$ into a left $k\,\mathrm{Sym}(n)$-module, and if $n \geqslant 2r$, then $\mathrm{End}_{k\,\mathrm{Sym}(n)}(E_n^{\otimes r})$ is isomorphic to the partition algebra $P_r(n)$ as in chapter 1, or [**26**, 5.5].

We now look at the point stabilizers of standard basis elements of the $\mathrm{Sym}(n)$-module $E_n^{\otimes r}$. Say $n \geqslant r+2$ then as every basis element $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$ of $E_n^{\otimes r}$ has at most r of the vectors $e_1, e_2, \ldots, e_n$ in it , there must be at least two of these vectors which do not figure in each basis element respectively. Therefore, the smallest possible point stabilizer in these cases is $\mathrm{Sym}(2)$, or a conjugate of this, and so the identity subgroup does not appear as a point stabilizer. Therefore when $n \geqslant r+2$ the point stabilizers of the $\mathrm{Sym}(n)$-module $E_n^{\otimes r}$ do not form a Mackey system. We therefore restrict our attention to the case $n \leqslant r+1$.

For $1 \leqslant s \leqslant n$, let $\mathcal{P}(s) = \{\mathrm{Sym}(X) \mid X \subseteq \{1,2,\ldots,n\}, |X| \leqslant s\}$, so that $\mathcal{P}(n) = \mathcal{P}$ itself. Then it is easy to see that for $n \leqslant r+1$, the set of point stabilizers of basis elements of the $\mathrm{Sym}(n)$-module $E_n^{\otimes r}$ is precisely $\mathcal{P}(n-1)$. We then have the following.

LEMMA 7.7. *If p does not divide n then $\mathcal{P}(n-1)$ is a p-Mackey system for* $\mathrm{Sym}(n)$.

PROOF. We know already that $\mathcal{P}(n-1)$ is closed under conjugation and intersection, and that the identity subgroup $\{1\}$ is a member of $\mathcal{P}(n-1)$. Now since p does not divide n any $S \in \mathrm{Syl}_p(\mathrm{Sym}(n))$ must be a subgroup of some subgroup of $\mathrm{Sym}(n)$ which is conjugate to $\mathrm{Sym}(n-1)$. Since all such subgroups are in $\mathcal{P}(n-1)$ we can conclude that $S \in \mathcal{P}(n-1)$ and $\mathcal{P}(n-1)$ is a p-Mackey system. $\square$

Now for the vertices.

LEMMA 7.8. *If p does not divide n then $\mathcal{P}(n-1)'_0 = \mathcal{P}(n)'_0$*

PROOF. If p does not divide n then $|\mathrm{Sym}(n) : \mathrm{Sym}(n-1)|$ is also not divisible by p, and $\mathrm{Sym}(n) \notin \mathcal{P}(n)_0$. Therefore $\mathcal{P}(n)_0 \subseteq \mathcal{P}(n-1)_0$ but also we trivially have $\mathcal{P}(n-1)_0 \subseteq \mathcal{P}(n)_0$. $\square$

So using the above lemma and the count from the last section we get the following.

COROLLARY 7.9. *If $p \nmid n$ then* $\mathrm{TSM}(\mathcal{P}(n-1)) = \mathrm{TSM}(\mathcal{P}(n))$.

Now if M is a direct summand of $E_n^{\otimes r}$, and if m is a basis vector of M with $\mathrm{stab}_{\mathrm{Sym}(n)}(m) = \mathrm{Sym}(i)$ for some $1 \leqslant i \leqslant n$ then our discussion in chapter 4 we know that $M \cong k \uparrow_{\mathrm{Sym}(i)}^{\mathrm{Sym}(n)}$. As all summands of $E_n^{\otimes r}$ must be indecomposable components of such an M, we get the following.

PROPOSITION 7.10. *When char $k = p$ does not divide n the* $\mathrm{Sym}(n)$*-module* $E_n^{\otimes r}$ *has* $\sum_{i\geqslant 0}|\Lambda^+(n-ip)_p|$ *isomorphism types of indecomposable summands.*

Now we can apply Fitting's Theorem.

COROLLARY 7.11. *When char $k = p$ does not divide n, the k-algebra* $\mathrm{End}_{k\,\mathrm{Sym}(n)}(E_n^{\otimes r})$ *has* $\sum_{i\geqslant 0}|\Lambda^+(n-ip)_p|$ *irreducibles. There are* $|\Lambda^+(n)|$ *of these if and only if* $p > n/2$.

Remark

As outlined in chapter 1, if $n \leqslant r+1$ and $p = 0$, or indeed if $p > n$, then $\mathcal{S}(n-1)$ is a p-Mackey system for $\mathrm{Sym}(n)$, as in this case the Sylow p-subgroup is the identity subgroup, and we already know that the other conditions are satisfied. In this case $\{1\}$ is the only $\mathcal{P}(n-1)$-vertex, and all conjucacy classes are p-regular, so $\mathrm{TSM}(\mathcal{P}(n-1)) = |\Lambda^+(n)|$ and $\mathrm{End}_{k\,\mathrm{Sym}(n)}(E_n^{\otimes r})$ has $|\Lambda^+(n)|$ irreducibles.

Also, in [**20**], Halverson looks at $P = \mathrm{End}_{\mathbb{C}\,\mathrm{Sym}(n)}(E_n^{\otimes r})$, the partition algebra over $\mathbb{C}$, and shows using Bratteli diagrams that the set $\widehat{P_r(n)} = \{\lambda \vdash n \mid |\lambda^*| \leqslant r\}$ where $\lambda^* = (\lambda_2, \lambda_3, \lambda_4, \dots) \in \Lambda^+(n-\lambda_1)$ indexes the irreducible $\mathbb{C}\,\mathrm{Sym}(n)$-modules in $E_n^{\otimes r}$ for all r and n. This means that he has shown that P has $|\widehat{P_r(n)}|$ irreducibles, by theorem 1.2 (Fitting's Theorem). However all is not lost as when $n \leqslant r+1$ we have that $\widehat{P_r(n)} = \Lambda^+(n)$, and the two counts agree, as they should.

7.2. Irreducible modules of Partition Algebras

As we have seen, when $n > r+1$ we have no Mackey system to play with. This is fine however, as in this case we can use other methods to count the number of irreducibles for $\mathrm{End}_{k\,\mathrm{Sym}(n)}(E_n^{\otimes r})$, and also to give a formula for the dimension of these modules in terms of certain modules for the general linear group. When $n \geqslant 2r$ this gives us the number of and dimensions of the irreducible modules for the partition algebra in characteristic p. As usual we will use Fitting's Theorem, but this time we make use of both results it gives us.

As in chapter 1, for $\lambda \in \Lambda(n,r)$ it is standard (see [8], for example) to define induced modules M^λ, or $M(\lambda)$, to be $M^\lambda = k \uparrow_{\mathsf{Sym}(\lambda)}^{\mathsf{Sym}(r)}$. Also, for $n, r \geqslant 1$ let $\mathrm{Hook}(r) = \{(n-j, 1^j) \in \Lambda^+(n) | 1 \leqslant j \leqslant r\}$. We now see how $E_n^{\otimes r}$ decomposes into a direct sum of permutation modules M^α, for $\alpha \in \mathrm{Hook}(r)$. Now for positive integers u and v we let $St(u,v)$ denote the Stirling number i.e. the number of ways of partitioning a set of size u into exactly v parts.

LEMMA 7.12. *If* $n, r \geqslant 1$ *then* $n^r = \sum_{j=1}^r St(r,j)n!/(n-j)!$

PROOF. Let $I(n,r)_j = \{f \in I(n,r) \mid f \text{ takes exactly } j \text{ values}\}$. Then $n^r = |I(n,r)| = |\bigsqcup_{j=1}^r I(n,r)_j| = \sum_{j=1}^r |I(n,r)_j|$. Now the number of ways of partitioning r into exactly j parts is the Stirling number $St(r,j)$, and the number of ways of choosing j values from 1 to n for these to take is $\binom{n}{j}$, and there are $j!$ ways of fitting these into the j parts. Therefore

$$|I(n,r)_j| = St(r,j)\binom{n}{j}j! = St(r,j)n!/(n-j)!$$

so that

$$n^r = \sum_{j=1}^r |I(n,r)_j| = \sum_{j=1}^r St(r,j)n!/(n-j)!$$

and we are done. □

Now let $\alpha = (n-j, 1^j) \in \mathrm{Hook}(r)$, and let $f(\alpha) \in I(n,r)$ such that $f(\alpha)$ takes exactly j different values. Then as every standard basis element e_i in the cyclic module $k\,\mathrm{Sym}(n)e_{f(\alpha)}$ has point stabilizer which is conjugate to $\mathrm{Sym}(n-j)$, it is easy to see that

$$M^{\alpha} \cong k\,\mathrm{Sym}(n)e_{f(\alpha)}, \text{ as left } k\,\mathrm{Sym}(n)\text{-modules.}$$

Note that the choice of $f(\alpha)$ is not unique, nor does it lead to a unique module. In fact, choosing different values of $f(\alpha)$ leads to $St(r,j)$ different but isomorphic modules, as you would expect from the above identity involving n^r. Then using the same argument as in the lemma above we have the following.

COROLLARY 7.13. *For $n, r \geqslant 1$ we have that*
$$E_n^{\otimes r} = \bigoplus_{j=1}^{r}(k\,\mathrm{Sym}(n)e_{f(\alpha)})^{\oplus St(r,j)} \cong \bigoplus_{j=1}^{r} M(n-j,1^j)^{\oplus St(r,j)}$$

To apply Fitting's Theorem we just need to know how many isomorphism types of indecomposable summands there are in $E_n^{\otimes r}$. So our above decomposition tells us that we can throw away the multiplicities and just count the number of isomorphism types of indecomposable summands of the left $k\,\mathrm{Sym}(n)$-module $\bigoplus_{j=1}^{r} M(n-j,1^j)$. Now it is well known (see for example [8], for the q-case) that the indecomposable summands of the permutation modules for the symmetric group are the so called Young modules, which are again labelled by partitions of n, and so we will denote them by $\mathrm{Young}(\lambda)$, for $\lambda \in \Lambda^+(n)$. So we need to work out how the modules M^{α}, where $\alpha \in \mathrm{Hook}(r)$, decompose into Young modules. Luckily, in [7], Donkin gives us a recipe for doing just this.

LEMMA 7.14. *Let $\alpha, \lambda \in \Lambda^+(n)$ and let $L(\lambda)^{\alpha}$ denote the α weight space of the simple GL_n-module $L(\lambda)$. Then*
$$(M^{\alpha}|\,\mathrm{Young}(\lambda)) = \dim L(\lambda)^{\alpha}.$$
In particular let $\lambda = \sum_{i=0} p^i\lambda(i)$ be the unique p-adic expansion of λ, so that each $\lambda(i)$ is column p-regular. Then $(M^{\alpha}|\,\mathrm{Young}(\lambda)) \neq 0$ if there exists an expansion $\alpha = \sum_{i=0} p^i\alpha(i)$ where each $\alpha(i)$ is a composition, and $\overline{\alpha(i)} \leqslant \lambda(i)$ for all $i \geqslant 0$.

We can now prove the following.

LEMMA 7.15. *Let p be a prime which does not divide n. Let $\lambda \in \Lambda^+(n)$, and let $\lambda = \sum_{i=0} p^i \lambda(i)$ be the unique p-adic expansion of λ. Then there exists an $\alpha = \sum_{i=0} p^i \alpha(i) \in \mathrm{Hook}(r)$ with each $\alpha(i)$ a composition with $\overline{\alpha(i)} \leqslant \lambda(i)$ for all $i \geqslant 0$ if and only if $\lambda = \lambda(0) + px$ where $x \geqslant 0$, $\lambda(0)$ is column p-regular and $\lambda_1 \geqslant n - r$.*

PROOF. Say $\lambda = \lambda(0) + px$ where $x \geqslant 0$, $\lambda(0)$ is column p-regular and $\lambda_1 \geqslant n - r$, and also say that $\lambda_2 \neq 0$. Then if $\lambda = \lambda(0) + px$ put $\lambda' = (\lambda(0)_1, 1^{|\lambda(0)| - \lambda(0)_1}) + px$. Then $\lambda' \in \mathrm{Hook}(r)$ with $\lambda'(i) \leqslant \lambda(i)$ for all $i \geqslant 0$, and we are done. Now say $\lambda_2 = 0$ so that $\lambda = z + px$ where $1 \leqslant z \leqslant p - 1$. Now $\lambda' = (n-1, 1) = (z-1, 1) + px$ so $\lambda' = (n-1, 1) \in \mathrm{Hook}(r)$ and $\lambda'(i) \leqslant \lambda(i)$ for all $i \geqslant 0$. Note this works even when $z = 1$.

Conversely say λ is not of this form. Then either:

(i) $\lambda(0)$ is not column p-regular;

(ii) $\lambda_1 < n - r$; or

(iii) $\lambda(i)_2 \neq 0$ for some $i \geqslant 1$.

We consider each case in turn and show each cannot occur.

(i) If $\lambda = \sum_{i=0} p^i \lambda(i)$ is a p-adic expansion, we must have that all $\lambda(i), i \geqslant 0$ are column p-regular. Therefore in particular $\lambda(0)$ must be column p-regular.

(ii) If $\overline{\alpha(i)} \leqslant \lambda(i)$ for all $i \geqslant 0$ we must have that $\alpha \leqslant \lambda$. Since the "smallest" $\alpha \in \mathrm{Hook}(r)$ has $\alpha_1 = n - r$ we must have that $\lambda_1 \geqslant n - r$.

(iii) All possible expansions of $\alpha \in \mathrm{Hook}(r)$ must have $\alpha(i)_j = 0$ for $i \geqslant 1, j \geqslant 2$, therefore for $i \geqslant 1$ each $\alpha(i)$ must be a single part partition. Since we require $\overline{\alpha(i)} \leqslant \lambda(i)$ for all $i \geqslant 0$ we must have that $|\overline{\alpha(i)}| = |\lambda(i)|$ for all $i \geqslant 0$. Therefore $\lambda(i) = \alpha(i)$ for all $i \geqslant 0$

i.e. $\lambda(i)_2 = 0$ as the only other choices with $|\alpha(i)| = |\lambda(i)|$ would give $\lambda(i) < \alpha(i)$. □

Write $\Lambda^+(n; r, p)$ for the subset of $\Lambda^+(n)$ consisting of partitions λ where $\lambda(0) + px$ for some $x \geqslant 0$, $\lambda(0)$ a column p-regular partition and $\lambda_1 \geqslant n - r$. The above lemma tells us that

$$(M^\alpha | \operatorname{Young}(\lambda)) \neq 0 \text{ for some } \alpha \in \operatorname{Hook}(r) \text{ and } \lambda \in \Lambda^+(n)$$
$$\text{if and only if } \lambda \in \Lambda^+(n; r, p),$$

and so $E_n^{\otimes r}$ has $|\Lambda^+(n; r, p)|$ types of indecomposable summand. Again we apply Fitting's Theorem.

PROPOSITION 7.16. *When p, the characteristic of k, does not divide n, our algebra* $\operatorname{End}_{k\,\mathrm{Sym}(n)}(E_n^{\otimes r})$ *has* $|\Lambda^+(n; r, p)|$ *non-isomorphic irreducible modules. In particular, when* $n \geqslant 2r$ *the partition algebra* $P_r(n)$ *has* $|\Lambda^+(n; r, p)|$ *non-isomorphic irreducible modules.*

Dimensions

We have now shown that there exist irreducible modules $U(\lambda)$ for $\operatorname{End}_{k\,\mathrm{Sym}(n)}(E_n^{\otimes r})$ such that $\{U(\lambda) | \lambda \in \Lambda^+(n; r, p)\}$ is full set of such objects. For each $\lambda \in \Lambda^+(n; r, p)$, Fitting's Theorem tells us that the multiplicity of Young(λ) in the decomposition of $E_n^{\otimes r}$ into Young modules is exactly $\dim U(\lambda)$. We can now show the following.

PROPOSITION 7.17. *Let* $L(\lambda)^\alpha$ *be the* α*-weight space of the irreducible* GL_n*-module* $L(\lambda)$*. Then for* $\lambda \in \Lambda^+(n; r, p)$ *we get*

$$\dim U(\lambda) = \textstyle\sum_{\alpha \in \mathrm{Hook}(r)} St(r, j) \dim L(\lambda)^\alpha$$

where each $\alpha = (n - j, 1^j)$ *for some* $1 \leqslant j \leqslant r$.

PROOF. Fitting tells us that $\dim U(\lambda) = (E_n^{\otimes r} | \operatorname{Young}(\lambda))$, but we know that $E_n^{\otimes r} = \bigoplus_{j=1}^{r} M(n - j, 1^j)^{\oplus St(r,j)} = \bigoplus_{\alpha \in \mathrm{Hook}(r)} M(\alpha)^{\oplus St(r,j)}$, so using the fact that $(M^\alpha|\,\mathrm{Young}(\lambda)) = \dim L(\lambda)^\alpha$ we arrive at the above formula. $\square$

COROLLARY 7.18. *When $n \geqslant 2r$, and p does not divide n, the irreducible modules for the partition algebra, $P_r(n)$, have dimensions as given by the formula above.*

Remarks (a) If $n \leqslant r+1$ then $\lambda_1 \geqslant n-r$ is no restriction at all. So the size of $\Lambda^+(n;r,p)$ is given by adding up the number of column p-regular partitions of $n-ip$, for each $i \geqslant 0$. So in this case $|\Lambda^+(n;r,p)| = \sum_{i\geqslant 0} |\Lambda^+(n-ip)_p|$ and this agrees with our count using the Mackey system.

(b) Say $p > n/2$, and recall that $\widehat{P_r(n)} = \{\lambda \vdash n | \lambda_1 \geqslant n-r\}$. Let $\lambda \in \widehat{P_r(n)}$. Then if λ is column p-regular, λ is also in $\Lambda^+(n;r,p)$. If λ is not column p-regular, in this case we must have that $\lambda = \lambda(0)+p$, with $\lambda(0)$ being column p-regular. Therefore $\lambda \in \Lambda^+(n;r,p)$ and so when $p > n/2$, we have that $|\Lambda^+(n;r,p)| = |\widehat{P_r(n)}|$. Conversely if $p \leqslant n/2$ then the partition $(n-p,p) \in \widehat{P_r(n)} \setminus \Lambda^+(n;r,p)$ and therefore when $p \leqslant n/2$ we have that $|\Lambda^+(n;r,p)| < |\widehat{P_r(n)}|$, as certainly we have that $\Lambda^+(n;r,p) \subseteq \widehat{P_r(n)}$. This is the analogue of determining when the "Mackey system" is "balanced" for this type of count.

(c) In [**26**, 5.5] Martin and Woodcock show that the irreducible modules for $P_r(n)$ are labelled by the set $\bigsqcup_{i=0}^r \Lambda^+(n,i)_{col}$, so that there are $\sum_{i=0}^r |\Lambda^+(n,i)_{col}|$ of them. Now say that $n \geqslant 2r$, so that certainly $\Lambda^+(n,i)_{col} = \Lambda^+(n,i-1)_{col}$ for all $0 \leqslant i \leqslant r$. Then we define

$$f : \bigsqcup_{i=0}^r \Lambda^+(n-1,i)_{col} \to \Lambda^+(n;r,p)$$
$$f(\lambda_1,\lambda_2,\ldots,\lambda_{n-1}) = (n-i,\lambda_1,\lambda_2,\ldots,\lambda_{n-1}).$$

Then f is a bijection, so that $\sum_{i=0}^r |\Lambda^+(n,i)_{col}| = |\Lambda^+(n;r,p)|$, and the two counts agree, as they should.

CHAPTER 8

Conjectures

In this section we conjecture how some of our results may, or may not, extend to complex reflection groups and Ariki-Koike algebras. Firstly, we define these and make some comments about them and related objects. Our notation for complex reflection groups is based on [**30**, Section 9].

Let $m, r \geqslant 1$. Then the group $\mathrm{G}(m,1,r)$, sometimes called a generalized symmetric group, is given by generators $s_0, s_1, \dots, s_{r-1}$ and relations

$$s_0^m = 1,$$
$$s_i s_j = s_j s_i \text{ if } |i-j| > 1, \text{ and if } j=0 \text{ and } 1<i<r,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } 1 \leqslant i \leqslant r-2$$
$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0.$$

It is well known that $\mathrm{G}(m,1,r)$ is isomorphic to the wreath product $\frac{\mathbf{Z}}{m\mathbf{Z}} \wr \mathrm{Sym}(r)$, so that $\mathrm{G}(1,1,r)$ is $\mathrm{Sym}(r)$, and $\mathrm{G}(2,1,r)$ is our beloved hyperoctahedral group. We also wish to know about some subgroups of $\mathrm{G}(m,1,r)$. Let $t = s_0 s_1 s_0 s_1$, and then we define $\mathrm{G}(m,m,r) = \langle t, s_1, s_2, \dots, s_{r-1}\rangle$. Then $\mathrm{G}(m,m,r)$ is a normal subgroup of $\mathrm{G}(m,1,r)$ of index m, and is isomorphic to $(\frac{\mathbf{Z}}{m\mathbf{Z}})^{r-1} \rtimes \mathrm{Sym}(r)$. Also note that $\mathrm{G}(2,2,r)$ is $\mathrm{D}(r)$, the Weyl group of type D. Let $a|m$ so that $m = ba$ for some $b \in \underline{m}$, and put $u = s_0^a$, so that u has order b. Then we define $\mathrm{G}(m,a,r) = \langle t, u, s_1, s_2, \dots, s_{r-1}\rangle$. Then $\mathrm{G}(m,a,r)$ is isomorphic to $((\frac{\mathbf{Z}}{m\mathbf{Z}})^{r-1} \times \frac{\mathbf{Z}}{b\mathbf{Z}}) \rtimes \mathrm{Sym}(r)$, and $\mathrm{G}(m,a,r)$ is a normal subgroup of $\mathrm{G}(m,1,r)$ of index a. Of course $\mathrm{G}(m,1,r)$ and $\mathrm{G}(m,m,r)$ are special cases of $\mathrm{G}(m,a,r)$.

The groups $\mathrm{G}(m,a,r)$, where $m,a,r \geqslant 1$ with $a|m$, give all the infinite families of complex reflection groups. There are also 34 sporadic complex reflection groups, but we do not consider them here. See [**33**] for the full classification.

We can quantize the groups $\mathrm{G}(m,1,r)$ to give us the Ariki-Koike (or cyclotomic Hecke) algebras, which are given as follows. Let k be a field and let $q, Q_1, Q_2, \ldots, Q_r$ be members of k, with q invertible. Then the Ariki-Koike algebra $\mathfrak{H}(m,r)$ has generators $T_{s_0}, T_{s_1}, \ldots, T_{s_{r-1}}$ and relations

$$
\begin{aligned}
(T_{s_i} - q)(T_{s_i} + 1) &= 0 \text{ for } 1 \leqslant i \leqslant r-1, \\
(T_{s_0} - Q_1)(T_{s_0} - Q_2)\cdots(T_{s_0} - Q_r) &= 0 \\
T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} \text{ if } |i-j| > 1, \text{ and if } j=0 \text{ and } 1<i<r, \\
T_{s_i}T_{s_{i+1}}T_{s_i} &= T_{s_{i+1}}T_{s_i}T_{s_{i+1}} \text{ for } 1 \leqslant i \leqslant r-2 \\
T_{s_0}T_{s_1}T_{s_0}T_{s_1} &= T_{s_1}T_{s_0}T_{s_1}T_{s_0}.
\end{aligned}
$$

If $w = s_{a(1)}s_{a(2)}\ldots s_{a(b)}$ is a reduced expression for $w \in \langle s_1, \ldots, s_{r-1}\rangle$ then we write T_w for $T_{s_{a(1)}}T_{s_{a(2)}}\cdots T_{s_{a(b)}}$. Note that when $m=1$ and $Q_1 = q$ we have that $\mathfrak{H}(1,r) = \mathrm{Hec}(r)$, the Hecke algebra of type A, and if $m = 2, Q_1 = Q$ and $Q_2 = -1$ we have that $\mathfrak{H}(2,r) = \mathcal{H}(r)$, the type B Hecke algebra. Also note that if ζ is an m^{th} root of 1 of order m, then putting $q = 1$ and $Q_i = \zeta^i$ for $1 \leqslant i \leqslant r$ gives us that $\mathfrak{H}(m,r) = k\mathrm{G}(m,1,r)$, the group algebra of $\mathrm{G}(m,1,r)$.

As in chapter 1, we let $\lambda = (\lambda(1);\lambda(2);\ldots;\lambda(m))$ be an m-tuple of compositions of integers in $\{0,1,\ldots,r\}$ such that each $\lambda(i)$ has at most n parts, and such that $\sum_{i=1}^{m}|\lambda(i)| = r$. Then we call λ an m-composition of r, and we denote the set of these by $\Lambda_m(n,r)$. If each

$\lambda(i)$ is a partition then we call λ an m-partition and denote the set of these by $\Lambda_m^+(n,r)$.

In [6], Dipper, James and Mathas assign to each $\lambda \in \Lambda_m(n,r)$ an $\mathfrak{H}(m,r)$-module M^λ as follows. Let $\lambda \in \Lambda_m(n,r)$ and put $a_0 = 0$ and $a_i = a_{i-1} + |\lambda(i)|$ for $i \geqslant 0$. Then the sequence $\mathbf{a} = (a_0, a_1, \ldots, a_m)$ is called the cumulative norm sequence, or c.n.s, of λ. For $1 \leqslant i \leqslant r$ we also let $L_i = q^{1-i} T_{s_{i-1}} \cdots T_{s_1} T_{s_0} T_{s_1} \cdots T_{s_{i-1}}$. Let $x_\lambda = \sum_u T_u$, where the sum is over all $u \in \mathrm{Sym}(\lambda(1)) \times \cdots \times \mathrm{Sym}(\lambda(m))$, and also define

$$\pi_\lambda = \prod_{l=2}^{m} \prod_{i=1}^{a_{l-1}} (L_i - Q_l)$$

and finally we put $m_\lambda = \pi_\lambda x_\lambda$. Then our module M^λ is given by

$$M^\lambda = m_\lambda \mathfrak{H}(m,r).$$

Then Du and Rui call the algebra

$$\mathcal{S}_q^m(n,r) = \mathrm{End}_{\mathfrak{H}(m,r)}\left(\bigoplus\nolimits_{\lambda \in \Lambda_m(n,r)} M^\lambda\right)$$

the q-Schur^m algebra, as in [10], and Dipper, James and Mathas define the cyclotomic q-Schur algebra similarly. Note that when we are back in the group case, so that $q = 1$ and $Q_i = \zeta^i$, we write $\mathcal{S}^m(n,r)$ for the Schur^m-algebra $\mathrm{End}_{k\mathrm{G}(m,1,r)}\left(\bigoplus_{\lambda \in \Lambda_m(n,r)} M^\lambda\right)$.

8. 1. A tensor space for $\mathrm{G}(m,1,r)$

In the hope that we can see $\bigoplus_{\lambda \in \Lambda_m(n,r)} M^\lambda$ as a genuine tensor space, we now generalize the construction of the $k\,\mathrm{Hyp}(r)$-module $Y^{\otimes r}$ to the complex reflection group $\mathrm{G}(m,1,r)$. Let k be a field of arbitrary characteristic containing m distinct m^{th} roots of unity. Let X_m be the k-vector space with basis

$$\{\, x_j^{a,b} \mid 1 \leqslant j \leqslant n \ , \ 1 \leqslant a \leqslant b \leqslant m \}$$

where $r \leqslant n$. Let $X_m^{\otimes r}$ be its r^{th} tensor power, so a typical basis element in $X_m^{\otimes r}$ looks like

$$x_{j_1}^{a_1,b_1} \otimes x_{j_2}^{a_2,b_2} \otimes \cdots \otimes x_{j_r}^{a_r,b_r}$$

Then $X_m^{\otimes r}$ is $(n\sum_{i=1}^m i)^r$-dimensional and we have the following.

LEMMA 8.1. *G$(m,1,r)$ acts on the right of $X_m^{\otimes r}$. The action is as follows:*

For $i>0$, s_i swaps the i^{th} and $(i+1)^{th}$ places in

$$x_{j_1}^{a_1,b_1}\otimes x_{j_2}^{a_2,b_2}\otimes\cdots\otimes x_{j_r}^{a_r,b_r}$$

and, for $y\in X_m^{\otimes r-1}$,

$$(x_j^{a,b}\otimes y)s_0=\begin{cases}(x_j^{a+1,b}\otimes y) & \text{if } a<b\\ (e_1(\ a)x_j^{a,a}-e_2(a)x_j^{a-1,a}+\dots & \\ \cdots+(-1)^{a+1}e_a(a)x_j^{1,a})\otimes y & \text{if } a=b\end{cases}$$

where for $t\in\{1,\dots,a\}$, $e_t(a)$ denotes the t^{th} elementary symmetric function in the variables $\{1=\zeta^0,\zeta,\zeta^2,\dots,\zeta^{a-1}\}$, with ζ a primitive m^{th} root of unity in k, of order m.

PROOF. We need to show that the defining relations for G$(m,1,r)$ hold when applied to our tensor space $X_m^{\otimes r}$. We already know that any of these relations not involving s_0 hold, by [**18**, 2.6], for example. Also note that

$$x.s_0s_i=x.s_is_0 \text{ for } i>1, \text{ and } x\in X_m^{\otimes r}$$

as s_0 only acts on the first coordinate of $X_m^{\otimes r}$, whilst the elements s_i only act on the other $r-1$ coordinates, when $i>1$. Therefore we just need to show that

(i)s_0^m and

(ii)$(s_0s_1)^4$

act trivially on our tensor space. Here we go.

(i) Since the action of s_0 only sees the first coordinate of $X_m^{\otimes r}$, we can just consider the case $r=1$ here, and the general case will follow from that. This means that we need to show

$$x_j^{a,b}.s_0^m = x_j^{a,b} \text{ for all } j \in \underline{n}, \text{ and } 1 \leqslant a \leqslant b \leqslant m.$$

Let $1 \leqslant b \leqslant m$. As the action of s_0 can't affect the j's, and thinking of a linear combination of elements $x_j^{1,b}, x_j^{2,b}, \ldots, x_j^{b,b}$ as a row vector (or b-tuple) then the action of s_0 on such elements can be represented by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & (-1)^{b+1}e_b(b) \\ 1 & 0 & 0 & \ldots & 0 & (-1)^b e_{b-1}(b) \\ 0 & 1 & 0 & \ldots & 0 & (-1)^{b+1}e_{b-2}(b) \\ 0 & 0 & 1 & \ldots & 0 & (-1)^b e_{b-3}(b) \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & e_1(b) \end{pmatrix}.$$

Showing that $A^m = I_b$ will prove this case. Now

$$\det(A - \lambda I_b) = \begin{vmatrix} -\lambda & 0 & 0 & \ldots & 0 & (-1)^{b+1}e_b(b) \\ 1 & -\lambda & 0 & \ldots & 0 & (-1)^b e_{b-1}(b) \\ 0 & 1 & -\lambda & \ldots & 0 & (-1)^{b+1}e_{b-2}(b) \\ 0 & 0 & 1 & \ldots & 0 & (-1)^b e_{b-3}(b) \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & e_1(b) - \lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 0 & \ldots & 0 & (-1)^b e_{b-1}(b) \\ 1 & -\lambda & \ldots & 0 & (-1)^{b+1}e_{b-2}(b) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & e_1(b) - \lambda \end{vmatrix}$$

$$+e_b(b)\begin{vmatrix} 1 & -\lambda & 0 & \dots & 0 \\ 0 & 1 & -\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

$$= (-\lambda)^2\begin{vmatrix} -\lambda & 0 & \dots & 0 & (-1)^{b+1}e_{b-2}(b) \\ 1 & -\lambda & \dots & 0 & (-1)^b e_{b-3}(b) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & e_1(b)-\lambda \end{vmatrix}$$

$$+(-\lambda)e_{b-1}(b)\begin{vmatrix} 1 & -\lambda & 0 & \dots & 0 \\ 0 & 1 & -\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} + e_b(b)$$

$$= (-\lambda)^3\begin{vmatrix} -\lambda & 0 & \dots & 0 & (-1)^b e_{b-3}(b) \\ 1 & -\lambda & \dots & 0 & (-1)^{b+1}e_{b-4}(b) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & e_1(b)-\lambda \end{vmatrix}$$

$$+(-\lambda)^2 e_{b-2}(b)\begin{vmatrix} 1 & -\lambda & 0 & \dots & 0 \\ 0 & 1 & -\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} + (\lambda)e_{b-1}(b) + e_b(b),$$

bearing in mind that the determinant of an upper triangular matrix with each entry on the main diagonal being 1 is equal to 1. Continuing in this manner gives

$$\det(A-\lambda I_b)=\textstyle\sum_{i=0}^{b}(-\lambda)^{b-i}e_i(b),$$

where we take $e_0(b)=1$, which can be rewritten as

$$\det(A-\lambda I_b)=(-1)^b(\lambda-1)(\lambda-\zeta)(\lambda-\zeta^2)\cdots(\lambda-\zeta^{b-1}).$$

Therefore A has eigenvalues $1,\zeta,\zeta^2,\dots,\zeta^{b-1}$, and as ζ is an m^{th} root of unity of order m, and $b\leqslant m$, these eigenvalues are all distinct. So A is a $b\times b$ matrix with b distinct eigenvalues, and therefore A can be diagonalized i.e. we can find a non-singular matrix P such that
$A=P^{-1}DP$, where $D=$diag$(1,\zeta,\zeta^2,\dots,\zeta^{b-1})$.
Then $A^m=(P^{-1}DP)^m=P^{-1}D^mP$
$=P^{-1}\ \mathrm{diag}(1^m,\zeta^m,(\zeta^2)^m,\dots,(\zeta^{b-1})^m)P$
$=P^{-1}\ \mathrm{diag}(1,1,1,\dots,1)P$ (since $\zeta^m=1$)
$=P^{-1}I_bP=I_b$,
so that $A^m=I_b$, so this relation holds.

(ii) We must now check that $s_0s_1s_0s_1=s_1s_0s_1s_0$. It suffices to check this for the case $r=2$, as s_0 and s_1 only see the first two coordinates of the tensor space. In fact, we can also drop the suffices i and j from elements like $x_i^{a,b}\otimes x_j^{c,d}$, as the action of G$(m,1,2)$ cannot alter them.

We must check this holds on all elements $x_i^{a,b}\otimes x_j^{c,d}\in X_m^{\otimes r}$. There are four cases to consider.

(1) $a<b, c<d$. We have
$(x^{a,b}\otimes x^{c,d})s_0s_1s_0s_1=(x^{a+1,b}\otimes x^{c,d})s_1s_0s_1=(x^{c,d}\otimes x^{a+1,b})s_0s_1$
$=(x^{c+1,d}\otimes x^{a+1,b})s_1=x^{a+1,b}\otimes x^{c+1,d}$,
and
$(x^{a,b}\otimes x^{c,d})s_1s_0s_1s_0=(x^{c,d}\otimes x^{a,b})s_0s_1s_0=(x^{c+1,d}\otimes x^{a,b})s_1s_0$

$= (x^{a,b} \otimes x^{c+1,d})s_0 = x^{a+1,b} \otimes x^{c+1,d},$

so the two expressions are equal.

(2) $a = b$, $c < d$. We have

$(x^{b,b} \otimes x^{c,d})s_0s_1s_0s_1$

$= ((e_1(b)x^{b,b} - e_2(b)x^{b-1,b} + \cdots + (-1)^{b+1}e_b(b)x^{1,b}) \otimes x^{c,d})s_1s_0s_1$

$= (x^{c,d} \otimes (e_1(b)x^{b,b} - e_2(b)x^{b-1,b} + \cdots + (-1)^{b+1}e_b(b)x^{1,b}))s_0s_1$

$= (x^{c+1,d} \otimes (e_1(b)x^{b,b} - e_2(b)x^{b-1,b} + \cdots + (-1)^{b+1}e_b(b)x^{1,b}))s_1$

$= ((e_1(b)x^{b,b} - e_2(b)x^{b-1,b} + \cdots + (-1)^{b+1}e_b(b)x^{1,b}) \otimes x^{c+1,d})$

and

$(x^{b,b} \otimes x^{c,d})s_1s_0s_1s_0 = (x^{c,d} \otimes x^{b,b})s_0s_1s_0$

$= (x^{c+1,d} \otimes x^{b,b})s_1s_0 = (x^{b,b} \otimes x^{d,d})s_0$

$= ((e_1(b)x^{b,b} - e_2(b)x^{b-1,b} + \cdots + (-1)^{b+1}e_b(b)x^{1,b}) \otimes x^{c+1,d}),$

so the two expressions are equal.

(3) $a < b$, $c = d$. We have

$(x^{a,b} \otimes x^{d,d})s_0s_1s_0s_1 = (x^{a+1,b} \otimes x^{d,d})s_1s_0s_1 = (x^{d,d} \otimes x^{a+1,b})s_0s_1$

$= ((e_1(d)x^{d,d} - e_2(d)x^{d-1,d} + \cdots + (-1)^{d+1}e_d(d)x^{1,d}) \otimes x^{a+1,b})s_1$

$= x^{a+1,b} \otimes (e_1(d)x^{d,d} - e_2(d)x^{d-1,d} + \cdots + (-1)^{d+1}e_d(d)x^{1,d})$

and

$(x^{a,b} \otimes x^{d,d})s_1s_0s_1s_0 = (x^{d,d} \otimes x^{a,b})s_0s_1s_0$

$= ((e_1(d)x^{d,d} - e_2(d)x^{d-1,d} + \cdots + (-1)^{d+1}e_d(d)x^{1,d}) \otimes x^{a,b})s_1s_0$

$= (x^{a,b} \otimes (e_1(d)x^{d,d} - e_2(d)x^{d-1,d} + \cdots + (-1)^{d+1}e_d(d)x^{1,d}))s_0$

$= x^{a+1,b} \otimes (e_1(d)x^{d,d} - e_2(d)x^{d-1,d} + \cdots + (-1)^{d+1}e_d(d)x^{1,d}),$

so the two expressions are equal.

(4) $a < b$, $c < d$. We have

$(x^{b,b} \otimes x^{d,d})s_0s_1s_0s_1$

$= ((e_1(b)x^{b,b} - \cdots + (-1)^{b+1}e_b(b)x^{1,b}) \otimes x^{d,d})s_1s_0s_1$

$= (x^{d,d} \otimes (e_1(b)x^{b,b} - \cdots + (-1)^{b+1}e_b(b)x^{1,b}))s_0s_1$

$= (e_1(b)x^{b,b} - \cdots + (-1)^{b+1}e_b(b)x^{1,b}) \otimes (e_1(d)x^{d,d} - \cdots + (-1)^{d+1}e_d(d)x^{1,d})$

and

$(x^{b,b} \otimes x^{d,d})s_1s_0s_1s_0 = (x^{d,d} \otimes x^{b,b})s_0s_1s_0$

$((e_1(d)x^{d,d} - \cdots + (-1)^{d+1}e_d(d)x^{1,d}) \otimes x^{b,b})s_1s_0$

$(x^{b,b} \otimes (e_1(d)x^{d,d} - \cdots + (-1)^{d+1}e_d(d)x^{1,d}))s_0$

$= (e_1(b)x^{b,b} - \cdots + (-1)^{b+1}e_b(b)x^{1,b}) \otimes e_1(d)x^{d,d} - \cdots + (-1)^{d+1}e_d(d)x^{1,d})$

so the two expressions are equal, and the proof is complete. □

Extending this action linearly makes $X_m^{\otimes r}$ into a right $k\mathrm{G}(m,1,r)$-module. We now come to our first conjecture.

CONJECTURE 8.2. *As right $k\mathrm{G}(m,1,r)$-modules*

$$X_m^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda_m(n,r)} M^\lambda.$$

This would also tell us that

$$\mathcal{S}^m(n,r) \cong \mathrm{End}_{k\mathrm{G}(m,1,r)}(X_m^{\otimes r}).$$

This would extend all our results of chapter 3 to the generalized symmetric groups $\mathrm{G}(m,1,r)$. However all attempts to quantize this action to give us a genuine tensor space for the Ariki-Koike algebras $\mathfrak{H}(m,r)$ were fruitless.

8.2. The Mackey system of Young subgroups for $\mathrm{G}(m,a,r)$

As $\mathrm{G}(m,1,r)$ is isomorphic to the wreath product $\frac{\mathbf{Z}}{m\mathbf{Z}} \wr \mathrm{Sym}(r)$, we have from chapter 6 that the set of Young subgroups

$$\mathcal{Y} = \{\mathrm{S}(\lambda)^g \mid g \in \mathrm{G}(m,1,r), \lambda \in \Lambda(n,r)\}$$

of $\mathrm{G}(m,1,r)$ is a Mackey system for $\mathrm{G}(m,1,r)$, and moreover if $p \nmid m$ then we know that $\mathcal{Y}$ is a balanced p-Mackey system for $\mathrm{G}(m,1,r)$.

Recall from chapter 1 that Grabmeier shows that if H is a subgroup of a finite group G, and if $\mathcal{M}$ is a p-Mackey system for G, then

$$\mathcal{M}\downarrow_H = \{A \cap H | A \in \mathcal{M}\}$$

is a p-Mackey system for H. So we have that

$$\mathcal{Y}\downarrow_{G(m,a,r)} = \{S(\lambda)^g \cap G(m,a,r) \mid S(\lambda)^g \in \mathcal{Y}\}$$

is a p-Mackey system for $G(m,a,r)$ when $p \nmid m$. We make the following claim.

CONJECTURE 8.3. *If $p \nmid m$ then $\mathcal{Y}\downarrow_{G(m,a,r)}$ is a balanced p-Mackey system for $G(m,a,r)$.*

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