UNIVERSITY OF LEICESTER

Department of Economics

Afriat's Theorem and Samuelson's `Eternal Darkness'

Matthew Polisson, University of Leicester
Ludovic Renou, University of Essex

Working Paper No. 16/09

Afriat's Theorem and Samuelson's 'Eternal Darkness'

Matthew Polisson and Ludovic Renou*

April 28, 2016

Abstract: Suppose that we have access to a finite set of expenditure data drawn from an individual consumer, i.e., how much of each good has been purchased and at what prices. Afriat (1967) was the first to establish necessary and sufficient conditions on such a data set for rationalizability by utility maximization. In this note, we provide a new and simple proof of Afriat's Theorem, the explicit steps of which help to more deeply understand the driving force behind one of the more curious features of the result itself, namely that a concave rationalization is without loss of generality in a classical finite data setting. Our proof stresses the importance of the non-uniqueness of a utility representation along with the finiteness of the data set in ensuring the existence of a concave utility function that rationalizes the data.

Keywords: Afriat's Theorem, concavity, revealed preference, utility maximization

JEL classification numbers: C60, D11

Suppose that we have access to a finite set of expenditure data drawn from an individual consumer, i.e., how much of each good has been purchased and at what prices. When is such a consumer's behavior consistent with the maximization of a stable preference over consumption goods? Afriat (1967) was the first to establish necessary and sufficient conditions on a finite set of price and demand observations in order to provide a definitive answer to this question. In this note, we provide a new and simple proof of Afriat's Theorem, the explicit steps of which help to more deeply understand the driving force behind one of the more curious features of the result itself, namely that a *concave* rationalization is without loss of generality in a classical finite data setting.

Formally, let $\mathcal{D} = (p^t, x^t)_{t \in T}$ be a finite data set, where $x^t = (x_1^t, x_2^t, \ldots, x_\ell^t) \geq 0$ denotes the consumption of ℓ goods purchased at prices $p^t = (p_1^t, p_2^t, \ldots, p_\ell^t) \gg 0$. The data set

* Email addresses: matthew.polisson@le.ac.uk; lrenou@essex.ac.uk. We are very grateful to Chris Chambers, John Quah, and two anonymous referees for helpful comments.

$\mathcal{D} = (p^t, x^t)_{t\in T}$ is said to be *rationalizable* if there exists a preference relation $\succsim^*$ on $\mathbb{R}^\ell_+$, which is complete, transitive, and locally non-satiated, such that, at every observation $t \in T$, $x^t \succsim^* x$ for any $x \in B^t := \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leq p^t \cdot x^t\}$. In words, a data set is rationalizable if we are unable to reject the hypothesis that a consumer has consistently chosen a preferred option from the set of feasible alternatives. It is worth stressing that we require the consumer's preference to be stable, i.e., unchanging across observations.

Theorem 1 (Afriat's Theorem)**.** *The following statements are equivalent:*

(1) The data set $\mathcal{D} = (p^t, x^t)_{t\in T}$ is rationalizable.

(2) Given the data set $\mathcal{D} = (p^t, x^t)_{t\in T}$, there exists $(v^t, \lambda^t)_{t\in T}$, with $(v^t, \lambda^t) \in \mathbb{R} \times \mathbb{R}_{++}$ for all $t \in T$, such that

$$v^{t'} \leq v^t + \lambda^t p^t \cdot (x^{t'} - x^t) \text{ for any } (t, t') \in T \times T. \qquad \text{(Afriat inequalities)}$$

(3) There exists a continuous, strictly increasing, and concave utility function $v : \mathbb{R}^\ell_+ \to \mathbb{R}$, such that, at every observation $t \in T$, $x^t \in \arg\max_{x\in B^t} v(x)$.

For seminal statements and proofs of Afriat's Theorem, see Afriat (1967), Diewert (1973), and Varian (1982); for the relationship between our approach and the broader literature, we refer the reader to the discussion which immediately follows the proof. Before formally proving Theorem 1, we emphasize a number of distinctive features of our approach. Firstly, and most importantly, our proof (in its construction) stresses the importance of the *non-uniqueness* of a utility representation along with the *finiteness* of the data set in ensuring the existence of a *concave* utility function that rationalizes the data. Secondly, like Varian's (1982) algorithmic proof, ours is entirely constructive; however, unlike Varian (1982), we first construct the numbers $(v^t)_t$ (interpreting v^t as the utility of consuming the bundle x^t) and then the numbers $(\lambda^t)_t$ (interpreting λ^t as the shadow value of the expenditure $p^t \cdot x^t$). Lastly, most renditions of Afriat's Theorem typically incorporate a further equivalent statement, namely that the data set obeys an intuitive no-cycling condition known as the *generalized axiom of revealed preference* (GARP);[1] we omit this statement of the theorem as our proof does *not* appeal to GARP.

[1] A typical route to proving the theorem is to first show that rationalizability implies GARP, subsequently that GARP implies the Afriat inequalities, and finally that the Afriat inequalities imply rationalizability.

We first prove that (1) $\implies$ (2), i.e., the necessity of the Afriat inequalities for rationalizability by a complete, transitive, and locally non-satiated preference relation, in three distinct lemmas. Throughout, let $\mathcal{X} := \{x^t : t \in T\}$ denote the finite set of observed consumption bundles. By convention, the infimum (resp., supremum) of the empty set is $+\infty$ (resp., $-\infty$), $r/0 = +\infty$ for any $r > 0$, and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is the extended real line.

Lemma 1. *If the data set $\mathcal{D} = (p^t, x^t)_{t \in T}$ is rationalizable, then there exists a complete and transitive preference relation $\succcurlyeq \subseteq \mathcal{X} \times \mathcal{X}$, such that*

$$x^{t'} \succcurlyeq x^t \implies p^t \cdot (x^{t'} - x^t) \geq 0 \text{ for any } (t, t') \in T \times T, \tag{i}$$

$$x^{t'} \succ x^t \implies p^t \cdot (x^{t'} - x^t) > 0 \text{ for any } (t, t') \in T \times T. \tag{ii}$$

Condition (i) states that if $x^{t'}$ is *weakly preferred* to x^t, then $x^{t'}$ must cost *at least as much* as x^t when x^t is purchased; condition (ii) states that if $x^{t'}$ is *strictly preferred* to x^t, then $x^{t'}$ must cost *strictly more* than x^t when x^t is purchased.[2]

Proof of Lemma 1. Assume the data set $\mathcal{D} = (p^t, x^t)_{t \in T}$ is rationalizable by the complete, transitive, and locally non-satiated preference relation $\succcurlyeq^*$. (a) Suppose that there exists some (t, t') such that $x^{t'} \succcurlyeq^* x^t$ and $p^t \cdot (x^{t'} - x^t) < 0$. By the completeness and local non-satiation of $\succcurlyeq^*$, there exists some $\varepsilon > 0$ and some x in the open ball of radius ε around $x^{t'}$ such that $p^t \cdot x \leq p^t \cdot x^t$ and $x \succ^* x^{t'}$. By the transitivity of $\succcurlyeq^*$, $x \succ^* x^{t'}$ and $x^{t'} \succcurlyeq^* x^t$ imply that $x \succ^* x^t$, contradicting the optimality of x^t. (b) Suppose that there exists some (t, t') such that $x^{t'} \succ^* x^t$ and $p^t \cdot (x^{t'} - x^t) \leq 0$. This immediately contradicts the optimality of x^t. It follows from (a) and (b) that (i) and (ii) hold, with $\succcurlyeq$ the restriction of $\succcurlyeq^*$ to $\mathcal{X} \times \mathcal{X}$. $\square$

We now apply Lemma 1 in order to construct a *specific* utility representation of $\succcurlyeq$. A set of real numbers $(v^t)_{t \in T}$ is said to *represent* the preference relation $\succcurlyeq$ if $v^t \geq v^{t'} \iff x^t \succcurlyeq x^{t'}$ for any $(t, t') \in T \times T$.

[2] It is not uncommon to equivalently represent conditions (i) and (ii) using their contrapositives, i.e., (i') $p^t \cdot (x^{t'} - x^t) < 0 \implies x^t \succ x^{t'}$ for any $(t, t') \in T \times T$, and (ii') $p^t \cdot (x^{t'} - x^t) \leq 0 \implies x^t \succcurlyeq x^{t'}$ for any $(t, t') \in T \times T$. Condition (ii') states that if $x^{t'}$ is *affordable* when x^t is purchased, then x^t must be *weakly preferred* to $x^{t'}$; condition (i') states that if $x^{t'}$ costs *strictly less* that x^t when x^t is purchased, then x^t must be *strictly preferred* to $x^{t'}$. Conditions (i) and (ii) are similar to the conditions in Varian (1984) for cost minimization.

Lemma 2. *Given the data set $\mathcal{D} = (p^t, x^t)_{t\in T}$, if there exists a complete and transitive preference relation $\succcurlyeq \subseteq \mathcal{X}\times\mathcal{X}$, such that (i) and (ii) hold, then there exists a representation $(v^t)_{t\in T}$ of $\succcurlyeq$, such that*

$$\frac{v^{t'} - v^t}{v^t - v^{t''}} \leq \frac{p^t\cdot(x^{t'} - x^t)}{p^t\cdot(x^t - x^{t''})}, \tag{iii}$$

for all triplets $(t', t, t'') \in T\times T\times T$ satisfying $x^{t'} \succ x^t \succ x^{t''}$ and $p^t\cdot(x^t - x^{t''}) \geq 0$.

To understand condition (*iii*), consider *any* function v that rationalizes the data, and suppose that for some (t', t, t'') satisfying $v(x^{t'}) > v(x^t) > v(x^{t''})$ and $p^t\cdot(x^t - x^{t''}) \geq 0$,

$$\frac{v(x^{t'}) - v(x^t)}{v(x^t) - v(x^{t''})} > \frac{p^t\cdot(x^{t'} - x^t)}{p^t\cdot(x^t - x^{t''})},$$

i.e., condition (*iii*) is violated (with $v(x^t) = v^t$, $v(x^{t'}) = v^{t'}$, and $v(x^{t''}) = v^{t''}$). This statement is then equivalent to

$$\frac{p^t\cdot(x^t - x^{t''})}{p^t\cdot(x^{t'} - x^t) + p^t\cdot(x^t - x^{t''})}v(x^{t'}) + \frac{p^t\cdot(x^{t'} - x^t)}{p^t\cdot(x^{t'} - x^t) + p^t\cdot(x^t - x^{t''})}v(x^{t''}) > v(x^t),$$

i.e., a convex combination of $v(x^{t'})$ and $v(x^{t''})$ is strictly greater than $v(x^t)$ (recall that $p^t\cdot(x^{t'} - x^t) > 0$ since $v(x^{t'}) > v(x^t)$; see Lemma 1). Moreover, notice that the bundle

$$y^t := \frac{p^t\cdot(x^t - x^{t''})}{p^t\cdot(x^{t'} - x^t) + p^t\cdot(x^t - x^{t''})}x^{t'} + \frac{p^t\cdot(x^{t'} - x^t)}{p^t\cdot(x^{t'} - x^t) + p^t\cdot(x^t - x^{t''})}x^{t''}$$

exhausts the budget at prices p^t, i.e., $p^t\cdot y^t = p^t\cdot x^t$. Since v rationalizes the data set, it must be that $v(x^t) \geq v(y^t)$. Combining this inequality with the above inequality, it follows that v *cannot* be concave. Therefore, condition (*iii*) is essential in ensuring the existence of a concave rationalization. It is worth stressing that the existence of a representation satisfying condition (*iii*) rests heavily on non-uniqueness properties. If the representation was unique up to an affine transformation, condition (*iii*) would not necessarily hold. Indeed, if v rationalizes the data but fails condition (*iii*), then any affine transformation $\alpha v+\beta$, with $(\alpha,\beta)\in\mathbb{R}_{++}\times\mathbb{R}$, would also violate condition (*iii*) since the ratios remain unchanged, i.e.,

$$\frac{v(x^{t'}) - v(x^t)}{v(x^t) - v(x^{t''})} = \frac{(\alpha v(x^{t'}) + \beta) - (\alpha v(x^t) + \beta)}{(\alpha v(x^t) + \beta) - (\alpha v(x^{t''}) + \beta)},$$

for any $(x^{t'}, x^t, x^{t''})$. For instance, in the domain of risk, the expected utility representation is unique (up to an affine transformation),[3] and concavity of the Bernoulli function is not

[3] By uniqueness up to an affine transformation, we mean uniqueness within the class of expected utility representations.

without loss of generality, as demonstrated by Polisson, Quah, and Renou (2015). (See also the examples under uncertainty in Bayer *et al.* (2013) and Echenique and Saito (2015).)

Proof of Lemma 2. Since the preference relation $\succcurlyeq$ is complete and transitive, there exists a partition $\{T_0, T_1, \ldots, T_n\}$ of T such that $x^t \succ x^{t'}$ for all $(t, t') \in T_i \times T_j$ with $i > j$, and $x^t \sim x^{t'}$ for all $(t, t') \in T_i \times T_i$, i.e., we can partition the set of observations into equivalence classes, ordered from the worst to the best. By the finiteness of $\mathcal{X}$, a representation $(v^t)_{t \in T}$ of $\succcurlyeq$ exists. Since there is nothing to prove if $T \backslash (T_0 \cup T_n) = \emptyset$, assume that $T \backslash (T_0 \cup T_n) \neq \emptyset$.

For all $t \in T$, define the sets $U^t := \{t' \in T : x^{t'} \succ x^t\}$ and $L^t := \{t'' \in T : x^t \succ x^{t''} \text{ and } p^t \cdot (x^t - x^{t''}) \geq 0\}$. For every $t \in T \backslash (T_0 \cup T_n)$, let

$$m^t = \inf_{(t', t'') \in U^t \times L^t} \frac{p^t \cdot (x^{t'} - x^t)}{p^t \cdot (x^t - x^{t''})}.$$

By construction, $m^t > 0$ for all $t \in T \backslash (T_0 \cup T_n)$, so that $\inf_{t \in T \backslash (T_0 \cup T_n)} m^t > 0$. Choose any $m^* \in (0, \inf_{t \in T \backslash (T_0 \cup T_n)} m^t] \backslash \{+\infty\}$, where $(0, \inf_{t \in T \backslash (T_0 \cup T_n)} m^t]$ can be viewed as a half-open interval on the extended real line $\overline{\mathbb{R}}$.

Let $v^t = 0$ for all $t \in T_0$ and $v^t = \sum_{j=1}^{i} (m^*/(m^*+1))^j$ for all $t \in T_i$, $i = 1, 2, \ldots, n$. Clearly, $(v^t)_{t \in T}$ is a representation of $\succcurlyeq$. We now argue that $(v^t)_{t \in T}$ is also a representation of $\succcurlyeq$ which satisfies condition (iii). Choose any triplet $(t', t, t'') \in T \times T \times T$ satisfying $x^{t'} \succ x^t \succ x^{t''}$ and $p^t \cdot (x^t - x^{t''}) \geq 0$. Assume that $t' \in T_{i'}$, $t \in T_i$, and $t'' \in T_{i''}$, with $i' > i > i'' \geq 1$. (If $i'' = 0$, a similar argument applies.) We have that

$$\begin{aligned}
\frac{v^{t'} - v^t}{v^t - v^{t''}} &= \frac{\sum_{j=1}^{i'} \left(\frac{m^*}{m^*+1}\right)^j - \sum_{j=1}^{i} \left(\frac{m^*}{m^*+1}\right)^j}{\sum_{j=1}^{i} \left(\frac{m^*}{m^*+1}\right)^j - \sum_{j=1}^{i''} \left(\frac{m^*}{m^*+1}\right)^j} = \frac{\sum_{j=i+1}^{i'} \left(\frac{m^*}{m^*+1}\right)^j}{\sum_{j=i''+1}^{i} \left(\frac{m^*}{m^*+1}\right)^j} \\
&\leq \frac{\sum_{j=i+1}^{i'} \left(\frac{m^*}{m^*+1}\right)^j}{\left(\frac{m^*}{m^*+1}\right)^i} = \sum_{j=i+1}^{i'} \left(\frac{m^*}{m^*+1}\right)^{j-i} \\
&\leq \lim_{i' \to \infty} \sum_{j=i+1}^{i'} \left(\frac{m^*}{m^*+1}\right)^{j-i} = m^*,
\end{aligned}$$

where the inequalities follow from the fact that we are summing over positive terms and $\sum_{j=i''+1}^{i} \left(\frac{m^*}{m^*+1}\right)^j \geq \left(\frac{m^*}{m^*+1}\right)^i > 0$. Finally, by construction, we have that

$$m^* \leq m^t \leq \frac{p^t \cdot (x^{t'} - x^t)}{p^t \cdot (x^t - x^{t''})},$$

which completes the argument. $\square$

We now apply Lemmas 1 and 2 in order to construct *multipliers*, thereby completing the proof that the Afriat inequalities are a necessary condition for rationalizability by a complete, transitive, and locally non-satiated preference relation.

Lemma 3. *Given the data set $\mathcal{D} = (p^t, x^t)_{t\in T}$, if there exists a representation $(v^t)_{t\in T}$, satisfying (iii), of the complete and transitive preference relation $\succcurlyeq \subseteq \mathcal{X}\times\mathcal{X}$, satisfying (i) and (ii), then there exists $(\lambda^t)_{t\in T}$, with $\lambda^t \in \mathbb{R}_{++}$ for all $t \in T$, such that*

$$v^{t'} \leq v^t + \lambda^t p^t \cdot (x^{t'} - x^t) \;\; \textit{for all} \;\; (t,t') \in T \times T. \qquad \text{(Afriat inequalities)}$$

Proof of Lemma 3. For every $t \in T$, consider the interval

$$I^t := \left[\sup_{t'\in U^t} \frac{v^{t'} - v^t}{p^t \cdot (x^{t'} - x^t)}, \; \inf_{t''\in L^t} \frac{v^t - v^{t''}}{p^t \cdot (x^t - x^{t''})}\right] \subseteq \overline{\mathbb{R}}.$$

By the definition of L^t, we have that

$$\inf_{t''\in L^t} \frac{v^t - v^{t''}}{p^t \cdot (x^t - x^{t''})} > 0,$$

for all $t \in T$. From Lemma 2, we have that

$$\inf_{t''\in L^t} \frac{v^t - v^{t''}}{p^t \cdot (x^t - x^{t''})} \geq \sup_{t'\in U^t} \frac{v^{t'} - v^t}{p^t \cdot (x^{t'} - x^t)},$$

for all $t \in T$ such that $U^t \times L^t \neq \emptyset$. It then follows that $I^t \cap \mathbb{R}_{++} \neq \emptyset$ for all $t \in T$. Choose any $\lambda^t \in I^t \cap \mathbb{R}_{++}$ for all $t \in T$.

We now verify that $(v^t, \lambda^t)_{t\in T}$ as constructed does indeed satisfy the Afriat inequalities. For any $t \in T$, there are three cases to consider. Firstly, choose any $t'' \in T$ such that $v^t > v^{t''}$. If $p^t \cdot (x^{t''} - x^t) < 0$, then

$$v^t + \lambda^t p^t \cdot (x^{t''} - x^t) \geq v^t + \frac{v^t - v^{t''}}{p^t \cdot (x^t - x^{t''})} p^t \cdot (x^{t''} - x^t) \geq v^{t''},$$

since $0 < \lambda^t \leq (v^t - v^{t''})/p^t \cdot (x^t - x^{t''})$. Similarly, if $p^t \cdot (x^{t''} - x^t) \geq 0$, then

$$v^t + \lambda^t p^t \cdot (x^{t''} - x^t) \geq v^{t''},$$

since $\lambda^t > 0$. Secondly, choose any $t' \in T$ such that $v^{t'} > v^t$. We have that

$$v^t + \lambda^t p^t \cdot (x^{t'} - x^t) \geq v^t + \frac{v^{t'} - v^t}{p^t \cdot (x^{t'} - x^t)} p^t \cdot (x^{t'} - x^t) \geq v^{t'},$$

since $\lambda^t \geq (v^{t'} - v^t)/p^t \cdot (x^{t'} - x^t) > 0$ and $p^t \cdot (x^{t'} - x^t) > 0$. Thirdly, choose any t^* such that $v^t = v^{t^*}$ (hence, $x^t \sim x^{t^*}$). Lemma 1 guarantees that $\lambda^t p^t \cdot (x^{t^*} - x^t) \geq 0$, and therefore $v^t + \lambda^t p^t \cdot (x^{t^*} - x^t) \geq v^{t^*}$ since $\lambda^t > 0$. This completes the proof. $\square$

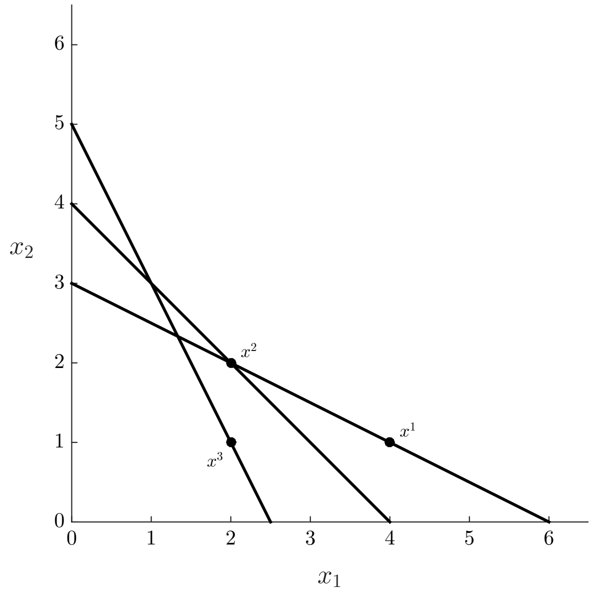

Figure 1: Example

In the progression from Lemmas 1 to 3, we have proven that the Afriat inequalities are a necessary condition for rationalizability by a complete, transitive, and locally non-satiated preference relation. To further illustrate the mechanics of our construction, consider the following numerical example.[4] Suppose that a consumer is observed to have chosen the bundle $x^1 = (4, 1)$ at the prices $p^1 = (1, 2)$, $x^2 = (2, 2)$ at $p^2 = (2, 2)$, and $x^3 = (2, 1)$ at $p^3 = (2, 1)$. This scenario is depicted in Figure 1.

Suppose that a complete, transitive, and locally non-satiated preference relation $\succcurlyeq^*$ rationalizes the data, with $x^1 \succ^* x^2 \succ^* x^3$. Since $p^2 \cdot (x^1 - x^2) = 2 > 0$, $p^3 \cdot (x^1 - x^3) = 4 > 0$, and $p^3 \cdot (x^2 - x^3) = 1 > 0$, Lemma 1 is satisfied, with $\succcurlyeq$ the restriction of $\succcurlyeq^*$ to $\mathcal{X} \times \mathcal{X}$. Turning to Lemma 2, we have $T_0 = \{3\}$, $T_1 = \{2\}$, and $T_2 = \{1\}$; $U^1 = \emptyset$, $U^2 = \{1\}$, and $U^3 = \{1, 2\}$; $L^1 = \{2, 3\}$, $L^2 = \{3\}$, and $L^3 = \emptyset$. It follows that $m^2 = p^2 \cdot (x^1 - x^2)/p^2 \cdot (x^2 - x^3) = 1$. Letting $m^* = m^2 = 1$, the numbers $v^3 = 0$, $v^2 = 1/2$, and $v^1 = 3/4$ satisfy Lemma 2.[5] Turning to Lemma 3, we have $I^1 = (-\infty, 3/8]$, $I^2 = [1/8, 1/4]$, and $I^3 = [1/2, +\infty)$. Choosing $\lambda^1 = 1/8$, $\lambda^2 = 1/4$, and $\lambda^3 = 1/2$, the set $(v^t, \lambda^t)_{t=1,2,3}$ satisfies the Afriat inequalities.

As a final step, we prove that (2) $\Longrightarrow$ (3), i.e., the sufficiency of the Afriat inequalities

[4] We are grateful to an anonymous referee for providing this example.

[5] Suppose the consumer had chosen $\tilde{x}^3 = (1, 3)$ instead of $x^3 = (2, 1)$. In this case, $p^2 \cdot (x^2 - x^3) = 0$, and therefore $m^2 = +\infty$. Following Lemma 2, we are free to choose any $m^* \in (0, +\infty)$, e.g., $m^* = 1$.

for rationalizability by a continuous, strictly increasing, and concave utility function.

Lemma 4. *If there exists $(v^t, \lambda^t)_{t\in T}$, with each $(v^t, \lambda^t) \in \mathbb{R}\times\mathbb{R}_{++}$ for all $t \in T$, that satisfies the Afriat inequalities, then there exists a continuous, strictly increasing, and concave utility function $v : \mathbb{R}^\ell_+ \to \mathbb{R}$, such that $x^t \in \arg\max_{x\in B^t} v(x)$ for all $t \in T$.*

Proof of Lemma 4. This proof is not new, but we provide it for completeness.[6] Let the utility function $v : \mathbb{R}^\ell_+ \to \mathbb{R}$ be defined by $v(x) = \inf\{v^t + \lambda^t p^t \cdot (x - x^t) : t \in T\}$ for any $x \in \mathbb{R}^\ell_+$. Notice that the piecewise linear utility function v is continuous, strictly increasing, and concave. By the definition of v, for all $t \in T$, $v(x^t) = v^{t^*} + \lambda^{t^*} p^{t^*} \cdot (x^t - x^{t^*}) \leq v^t + \lambda^t p^t \cdot (x^t - x^t) = v^t$ for some $t^* \in T$, and if this inequality were to hold strictly, then the Afriat inequalities would be violated. Therefore, $v(x^t) = v^t$ for all $t \in T$. Now, at any $t \in T$, choose some $x \in B^t = \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leq p^t \cdot x^t\}$. Again by the definition of v, for this $t \in T$ and $x \in B^t$, $v(x) \leq v^t + \lambda^t p^t \cdot (x - x^t)$. Since $x \in B^t$, $p^t \cdot (x - x^t) \leq 0$, and therefore $v(x) \leq v^t$. Finally, using that $v(x^t) = v^t$ for all $t \in T$, we have $v(x) \leq v^t = v(x^t)$. $\square$

Since rationalization by a continuous, strictly increasing, and concave utility function clearly implies rationalization by a complete, transitive, and locally non-satiated preference relation, we have shown an equivalence between (1), (2), and (3), and the proof of Afriat's Theorem is therefore complete.

Some background and concluding remarks are helpful to situate the main insights of our approach. Afriat's Theorem was initially stated and proven by Afriat (1967) using a combinatorial inductive approach. Several years later, the result was simplified, qualified, and extended by Diewert (1973), who related the theorem more directly to a specific linear programming problem, and by Varian (1982), who more explicitly linked GARP to the *cyclical consistency* property of Afriat (1967), and who generally promoted the broad approach as a nonparametric alternative to demand analysis. More recently, Fostel, Scarf, and Todd (2004) provided two new proofs of Afriat's Theorem, one inductive and another exploiting the dual structure inherent in linear programming problems.

Several extensions and adaptations to the Afriat (1967) approach have proven insightful and useful over the years. Matzkin (1991), Chavas and Cox (1993), Forges and Minelli (2009),

[6] It first appears in Afriat (1967), then Diewert (1973) and Varian (1982), among many others.

and Cherchye, Demuynck, and De Rock (2014) provided complete characterizations under more general (not necessarily linear) constraints. Teo and Vohra (2003) showed that rationalizability is related to the identification of cycles in graphs, and Geanakoplos (2013) linked rationalizability to the existence of equilibria in zero-sum games. Reny (2015) circumvented the Afriat inequalities entirely and directly proved that GARP is necessary and sufficient for rationalizability (in both finite and infinite data), somewhat in the spirit of Rochet (1987) and Brown and Calsamiglia (2007). Lastly, Fujishige and Yang (2012), Polisson and Quah (2013), Cosaert and Demuynck (2014), and Forges and Iehlé (2014) allowed for discreteness and indivisibilities, which is a natural consideration in many empirical applications.

Our approach delivers further clarity on a particular issue, namely that a concave rationalization is without loss of generality in a finite data setting. Firstly, let us say that a preference relation on $\mathcal{X}$ is consistent with the data if it satisfies conditions (i) and (ii). Lemma 2 then states that *any* preference relation on $\mathcal{X}$, consistent with the data, admits a specific representation, i.e., a representation that satisfies condition (iii). In turn, Lemma 3 states that *this* specific representation admits multipliers, which satisfy the Afriat inequalities, and which therefore extends to a concave representation on the entire consumption space. This observation complements a recent result of Quah (2014), who stated that any preference relation on $\mathcal{X}$, consistent with the data, extends to a concave rationalization on the entire consumption space; in this note, we have explicitly constructed a specific representation on $\mathcal{X}$, which extends to a concave representation on the entire consumption space.

One advantage of proving the necessity of the Afriat inequalities directly (and without appealing to GARP) is to highlight the role of *non-uniqueness* (up to affine transformations), and therefore the main insight of the approach can be found in Lemma 2. As Diewert (1973) concluded, "it is perhaps somewhat surprising that the [utility function] constructed from a finite body of price and quantity data ... is continuous, increasing and concave when the decision-maker's 'true' [preference] only satisfies the much weaker regularity conditions ... thus the data will never be able to reveal backward bending indifference curves or non-convex indifference sets." Diewert (1973) goes on to quote Samuelson (1950), who made the earlier observation that "any point where the indifference curves are convex rather than concave cannot be observed in a competitive market," and that "such points are shrouded in eternal darkness." What this note has highlighted is that the lack of uniqueness of a utility representation in a restricted finite data setting is largely responsible for such an equivalence.

When budgets are nonlinear, however, a lack of uniqueness (say, up to affine transformations) of the representation does not necessarily guarantee the equivalence. Indeed, we also require the ratios of utilities $(v^{t'} - v^t)/(v^t - v^{t''})$ to be bounded from above by ratios of *linear* functionals. To see this, let us briefly revisit the work of Forges and Minelli (2009). In Forges and Minelli (2009), a data set is a finite collection $(x^t, B^t)_{t\in T}$, where $B^t := \{x \in \mathbb{R}^\ell_+ : g^t(x) \leq 0\}$ is a generalized budget set (for all $t \in T$, $g^t : \mathbb{R}^\ell_+ \to \mathbb{R}$ is a continuous and increasing function satisfying $g^t(x^t) = 0$). Assume that the data set $(x^t, B^t)_{t\in T}$ is rationalizable. It is straightforward to verify that Lemmas 1 to 3 generalize to this richer environment. Firstly, the generalized version of Lemma 1 states the existence of a complete and transitive preference relation $\succcurlyeq \subseteq \mathcal{X} \times \mathcal{X}$, such that $x^{t'} \succcurlyeq x^t$ implies $g^t(x^{t'}) \geq 0$ and $x^{t'} \succ x^t$ implies $g^t(x^{t'}) > 0$ for all $(t, t') \in T \times T$. Secondly, the generalized version of Lemma 2 states the existence of a representation $(v^t)_{t\in T}$, such that

$$\frac{v^{t'} - v^t}{v^t - v^{t''}} \leq \frac{g^t(x^{t'})}{-g^t(x^{t''})},$$

for all triplets $(t', t, t'') \in T \times T \times T$ satisfying $x^{t'} \succ x^t \succ x^{t''}$ and $g^t(x^{t''}) \leq 0$. Thirdly, the generalized version of Lemma 3 states the existence of a set of positive multipliers $(\lambda^t)_{t\in T}$, such that for all $(t, t') \in T \times T$,

$$v^{t'} \leq v^t + \lambda^t g^t(x^{t'}).$$

However, this does *not* guarantee the existence of a concave rationalization. To guarantee a concave rationalization, we need to find a finite collection of (normalized) hyperplanes $(\mathcal{H}^t)_{t\in T}$, such that (a) $B^t \subseteq \{x \in \mathbb{R}^\ell_+ : \inf_{h^t\in\mathcal{H}^t} h^t \cdot (x - x^t) \leq 0\}$ for all $t \in T$, (b) for any $(t, t') \in T \times T$, $x^{t'} \succ x^t$ implies that $\inf_{h^t\in\mathcal{H}^t} h^t \cdot (x^{t'} - x^t) > 0$, and (c)

$$\frac{v^{t'} - v^t}{v^t - v^{t''}} \leq \frac{\inf_{h^t\in\mathcal{H}^t} h^t \cdot (x^{t'} - x^t)}{\inf_{h^t\in\mathcal{H}^t} h^t \cdot (x^t - x^{t''})},$$

for all triplets $(t', t, t'') \in T \times T \times T$ satisfying $x^{t'} \succ x^t \succ x^{t''}$ and $g^t(x^{t''}) \leq 0$ (hence, $\inf_{h^t\in\mathcal{H}^t} h^t \cdot (x^t - x^{t''}) \geq 0$). Notice that when the budget sets are classically linear, we have that $\mathcal{H}^t = \{p^t\}$ for all $t \in T$. We refer the reader to Theorem 3 in Cherchye, Demuynck, and De Rock (2014) for details.

Finally, the approach developed in this paper may be particularly profitable outside of a classical demand setting, i.e., when combinatorial GARP-like conditions do not straightforwardly exist, or when stronger assumptions on the preference relation or utility function no longer hold without loss of generality. An important message is that as soon as further structure on the preference ordering is introduced, e.g., some form of separability, then any utility representation may be unique up to a more restrictive transformation than monotonic transformation, which then precludes the flexible construction of a *concave* utility function that rationalizes the data.

References

AFRIAT, S. N. (1967): "The Construction of Utility Functions from Expenditure Data," *International Economic Review*, 8(1), 67–77.

BAYER, R.-C., S. BOSE, M. POLISSON, and L. RENOU (2013): "Ambiguity Revealed," *IFS Working Papers*, W13/05.

BROWN, D. J., and C. CALSAMIGLIA (2007): "The Nonparametric Approach to Applied Welfare Analysis," *Economic Theory*, 31(1), 183–188.

CHAVAS, J.-P., and T. L. COX (1993): "On Generalized Revealed Preference Analysis," *Quarterly Journal of Economics*, 108(2), 493–506.

COSAERT, S., and T. DEMUYNCK (2015): "Revealed Preference Theory for Finite Choice Sets," *Economic Theory*, 59(1), 169-200.

CHERCHYE, L., T. DEMUYNCK, and B. DE ROCK (2014): "Revealed Preference Analysis for Convex Rationalizations on Nonlinear Budget Sets," *Journal of Economic Theory*, 152 (C), 224–236.

DIEWERT, W. E. (1973): "Afriat and Revealed Preference Theory," *Review of Economic Studies*, 40(3), 419–425.

ECHENIQUE, F., AND K. SAITO (2015): "Supplement to 'Savage in the Market'," *Econometrica Supplemental Material*, 83, http://dx.doi.org/10.3982/ECTA12273.

FOSTEL, A., H. E. SCARF, and M. J. TODD (2004): "Two New Proofs of Afriat's Theorem," *Economic Theory*, 24(1), 211–219.

FORGES, F., and E. MINELLI (2009): "Afriat's Theorem for General Budget Sets," *Journal of Economic Theory*, 144(1), 135–145.

FORGES, F., and V. IEHLÉ (2014): "Afriat's Theorem for Indivisible Goods," *Journal of Mathematical Economics*, 54(C), 1–6.

FUJISHIGE, S., and Z. F. YANG (2012): "On Revealed Preference and Indivisibilities," *Modern Economy*, 3(6), 752–758.

GEANAKOPLOS, J. D. (2013): “Afriat from MaxMin,” *Economic Theory*, 54(3), 443–448.

MATZKIN, R. L. (1991): “Axioms of Revealed Preference for Nonlinear Choice Sets,” *Econometrica*, 59(6), 1779–1786.

POLISSON, M., and J. K.-H. QUAH (2013): “Revealed Preference in a Discrete Consumption Space,” *American Economic Journal: Microeconomics*, 5(1), 28–34.

POLISSON, M., J. K.-H. QUAH, and L. RENOU (2015): “Revealed Preferences over Risk and Uncertainty,” *IFS Working Papers*, W15/25.

QUAH, J. K.-H. (2014): “A Test for Weakly Separable Preferences,” *Department of Economics, University of Oxford, Discussion Paper Series*, 708.

RENY, P. J. (2015): “A Characterization of Rationalizable Consumer Behavior,” *Econometrica*, 83(1), 175–192.

ROCHET, J.-C. (1987): “A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context,” *Journal of Mathematical Economics*, 16(2), 191–200.

SAMUELSON, P. A. (1950): “The Problem of Integrability in Utility Theory,” *Economica*, 17(68), 355–385.

TEO, C. P., and R. V. VOHRA (2003): “Afriat’s Theorem and Negative Cycles,” *Center for Mathematical Studies in Economics and Management Science, Kellogg School of Management, Northwestern University*, 1377.

VARIAN, H. R. (1982): “The Nonparametric Approach to Demand Analysis,” *Econometrica*, 50(4), 945–973.

VARIAN, H. R. (1984): “The Nonparametric Approach to Production Analysis,” *Econometrica*, 52(3), 579–598.