

# Exact Friedmann Solutions in Higher-Order Gravity Theories

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## Abstract

We find the general behaviour of homogeneous and isotropic cosmological models in some fourth-order theories of gravity. Explicit, exact, general solutions are given for both empty universes and those filled with a perfect fluid. For the vacuum case, solutions are found with closed, open and flat geometries, whilst the perfect fluid solutions are all spatially flat. Both early and late-time limits are studied, and attractor behaviour towards simple power-law expansion is identified. Multiple solutions to the same theories, with the same matter content and topology are found. It is shown that these solutions exhibit great variety in their evolution.

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# 1 Introduction

Generalisations of the usual Einstein-Hilbert action of general relativity (GR) have been extensively studied in the literature. One frequently considered modification is to replace the Ricci scalar,  $R$ , by some analytic function,  $f(R)$  (see e.g [1, 2, 3, 4]). Theories of this type are referred to as fourth-order, as the field equations generated from them are generically fourth-order in derivatives of the metric. Motivations for such studies are found from various different sources. One often cited reason is that early attempts to create a perturbatively renormalizable quantum field theory of gravity found success by adding extra terms, quadratic in the Ricci curvature, to the action [5]. More recently, the effective actions of some string theories have been shown to include higher-order curvature terms [6, 7]. Studies of  $f(R)$  theories have also been performed in cosmological settings, often in attempts to better understand the late-time accelerating expansion of the universe [8, 9], cosmological inflation [10, 11, 12] or the nature of an initial singularity [13, 14, 15].

Difficulty can arise, in studies of fourth-order theories, as the field equations involved are considerably more complex than their counterparts in GR. This extra complexity brings new and interesting behaviour, such as violations of Birkhoff's theorem [16] or violations of the no hair theorems of de Sitter space, [12]. However, this extra complexity also makes it even more difficult to find exact solutions of the field equations. Whilst some exact solutions have been found in the spherically symmetric [16, 17] and cosmological [17, 18, 19, 20, 21, 25] situations, these are all particular solutions. To date, there have been no general solutions published in the literature, for any set of non-maximal symmetries. Exact solutions, and particularly exact general solutions, are of great importance for understanding a theory. In this paper we will present homogeneous and isotropic exact general solutions, obtained through direct integration of the field equations, to some fourth-order theories of gravity. We expect these solutions to be of use for helping to understand fourth-order theories of gravity, and the evolution of the universes they govern.

Investigations of fourth-order theories usually follow one of two approaches: Either they take the full-theory and look for approximate solutions, or they approximate the theory and look for exact solutions. We will take the later approach and consider theories that have a Lagrangian proportional to  $R^n$ . Such theories are scale invariant, and reduce to GR in the limit  $n \rightarrow 1$ . These theories may be considered as the limit of a more general Lagrangian that has a power of  $R$  dominating in some particular regime, or as a simple deviation from the standard theory in their own right. We find that for homogeneous and isotropic cosmologies the field equations of these theories can often be integrated directly. For spatially flat vacuum cosmologies the general solution can be found for any  $n$ , for spatially curved vacuum cosmologies we find the general solutions when  $n = 3/2$ , and for spatially flat perfect fluid cosmologies (with equation of state  $p = (\gamma - 1)\rho$ ) we find the general solutions when  $n = 3\gamma/(3\gamma - 1)$  and when  $n = (10 - 3\gamma)/(2(7 - 3\gamma))$ . These dependences are shown graphically in figure 1, below.

Investigations of homogeneous and isotropic cosmologies in  $R^n$  theories have been performed before in [17, 21]. In these papers the authors find power-law exact, particular solutions and perform a dynamical systems analysis of the phase space of more general solutions. These studies show that the general solutions are often attracted to simple power-law solutions, at

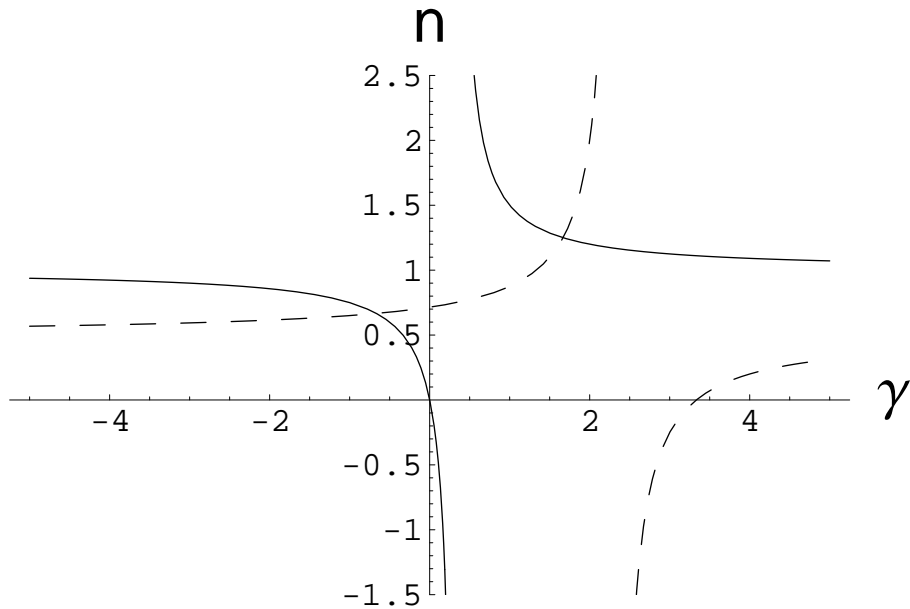


Figure 1: A plot of  $n = 3\gamma/(3\gamma - 1)$  (solid line) and  $n = (10 - 3\gamma)/(2(7 - 3\gamma))$  (dashed line). For these values of  $n$  the field equations for homogeneous and isotropic perfect fluid cosmologies can be integrated directly.

both late and early times. We will confirm this behaviour here, and add to the previous work by finding explicit expressions for the general evolution of these universes.

We will now comment on the extent to which the solutions found in this paper can be considered candidates for the description of our universe. It has been shown in [16] and [17] that gravitational theories derived from a Lagrangian of the form  $R^n$  are constrained by observations in our local universe to have  $n$  very close to 1. It has also been suggested (see e.g. [22], [23]) that it may not be possible for general  $f(R)$  theories of gravity to transition between eras dominated by different fluids in the same way that occurs in the usual general relativistic description. These results suggest that our neighbourhood of space-time should be well described by a theory that is very close to the Einstein-Hilbert one. However, with these constraints in mind, there are still good reasons to be interested in deviations of the  $R^n$  form. Firstly, even if  $n$  is very tightly constrained by local observations, any small deviation from  $n = 1$  may produce significant deviations at early times in the universe's history. This will be shown explicitly below. Secondly, theories of the type  $R + \alpha R^n$  are often considered in the literature. If  $n > 1$  then we may expect the effective gravitational theory in the early universe to be well described by a Lagrangian of the form  $R^n$ . In such cases a more appropriate description of the evolution of the early universe would be given by an  $R^n$  theory, even though observations from our local universe show strong agreement with an effective theory of the Einstein-Hilbert type.

This paper will proceed as follows. In section 2 we will give the field equations for the theory, and show how they can be simplified by recasting them in terms of new variables and transforming time coordinates. Simple power-law exact, particular solutions to these equations are given. In section 3 we show how the field equations for vacuum cosmologies can be decoupled by a further transformation of variables. These equations are then integrated directly, and the

solutions given explicitly in terms of the metric. Section 4 follows a similar prescription, this time solving the field equations in the presence of a perfect fluid. In section 5 we perform a brief analysis of the vacuum solutions, and in section 6 we analyse the perfect fluid solutions. Section 7 provides a closing discussion. For readers interested in the solutions, but not the derivations, the solutions in sections 3 and 4 are boxed to make them easily identifiable.

## 2 Field equations

We consider here a gravitational theory derived from the Lagrangian density

$$\mathcal{L}_G = \frac{1}{\chi} \sqrt{-g} R^{1+\delta}, \quad (1)$$

where  $\delta$  is a real number and  $\chi$  is a constant. The limit  $\delta \rightarrow 0$  gives the Einstein-Hilbert Lagrangian of GR, and we are interested in isotropic and homogeneous cosmological solutions with  $\delta \neq 0$ . Theories of this kind are dynamically equivalent to scalar-tensor theories (see e.g. [24]).

We denote the matter action as  $S_m$  and ignore the boundary term. Extremizing

$$S = \int \mathcal{L}_G d^4x + S_m,$$

with respect to the metric  $g_{ab}$ , then gives [19]

$$\begin{aligned} \delta(1 - \delta^2) R^\delta \frac{R_{,a} R_{,b}}{R^2} - \delta(1 + \delta) R^\delta \frac{R_{;ab}}{R} + (1 + \delta) R^\delta R_{ab} - \frac{1}{2} g_{ab} R R^\delta \\ - g_{ab} \delta(1 - \delta^2) R^\delta \frac{R_{,c} R_{,c}}{R^2} + \delta(1 + \delta) g_{ab} R^\delta \frac{\square R}{R} = \frac{\chi}{2} T_{ab}, \end{aligned} \quad (2)$$

where  $T_{ab}$  is the energy-momentum tensor of the matter fields, and is defined in terms of  $S_m$  and  $g_{ab}$  in the usual way. We take the quantity  $R^\delta$  to be the positive real root of  $|R|$ .

We are concerned with idealised homogeneous and isotropic space-times described by the Friedmann-Robertson-Walker metric with spatial curvature parameter  $\kappa$ :

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{(1 - \kappa r^2)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \quad (3)$$

Substituting this metric ansatz into the field equations (2), and assuming the universe to be filled with a perfect fluid of pressure  $p$  and density  $\rho$ , gives the generalised version of the Friedmann equations

$$(1 - \delta) R^{1+\delta} + 3\delta(1 + \delta) R^\delta \left( \frac{\ddot{R}}{R} + 3 \frac{\dot{a}}{a} \frac{\dot{R}}{R} \right) - 3\delta(1 - \delta^2) R^\delta \frac{\dot{R}^2}{R^2} = \frac{\chi}{2} (\rho - 3p) \quad (4)$$

$$-3 \frac{\ddot{a}}{a} (1 + \delta) R^\delta + \frac{R^{1+\delta}}{2} + 3\delta(1 + \delta) \frac{\dot{a}}{a} \frac{\dot{R}}{R} R^\delta = \frac{\chi}{2} \rho \quad (5)$$

where, as usual,

$$R = 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + 6\frac{\kappa}{a^2}. \quad (6)$$

It can be seen that in the limit  $\delta \rightarrow 0$  these equations reduce to the standard Friedmann equations of GR. A study of the vacuum solutions to these equations has been made by Schmidt, see the review [20], and a study of the perfect-fluid evolution has been made by Carloni et al [21]. In [17] Clifton and Barrow investigated solutions when  $\kappa = 0$ , and showed that attractor power-law solutions exist. Various conclusions are also immediate from the general analysis of  $f(R)$  Lagrangians made in [3] by specialising them to the case  $f = R^{1+\delta}$ . These previous studies have focussed on the existence of particular exact solutions, or qualitative investigations of the phase space of general solutions. In what follows we shall be interested in determining the general evolution of  $a$  in terms of explicit, exact solutions. This should lead to fresh insights into the evolution of universes described by these theories.

Assuming a perfect-fluid equation of state of the form  $p = (\gamma - 1)\rho$ , the usual conservation equation gives  $\rho = \rho_0 a^{-3\gamma}$ . Substituting this into equations (4) and (5), with  $\kappa = 0$ , gives the particular power-law exact Friedmann solution, for  $\gamma \neq 0$ ,

$$a(t) = t^{\frac{2(1+\delta)}{3\gamma}}. \quad (7)$$

Alternatively, if  $\gamma = 0$ , there exists the de Sitter solution  $a(t) = e^{mt}$ . These power-law solutions reduce to the usual general relativistic solutions in the limit  $\delta \rightarrow 0$ . Furthermore, the power-law solutions

$$a(t) = t^{\frac{\delta(1+2\delta)}{(1-\delta)}} \quad (8)$$

and

$$a(t) = t^{\frac{1}{2}} \quad (9)$$

also exist. These solutions are independent of  $\gamma$ , and therefore of the matter content of the universe. The  $\delta = 0$  limit of (8) clearly corresponds to Minkowski space. These three particular solutions were both shown in [17] to be attractors of the spatially flat general solution, under certain conditions. In what follows we will be able to show this attractor explicitly in terms of exact general solutions.

The equations (4), (5) and (6) can be cast in a simpler form by defining the new variables

$$\phi \equiv \sqrt{3} \ln R^\delta \quad \text{and} \quad \bar{a} \equiv e^{\phi/2\sqrt{3}} a.$$

Transforming to the time coordinate

$$d\eta \equiv e^{\phi/2\sqrt{3}} dt$$

and setting  $\chi = 1$  we then have two second-order evolution equations, for the two variables  $\bar{a}$  and  $\phi$ ,

$$6\frac{\ddot{\bar{a}}}{\bar{a}} = -\dot{\phi}^2 + V_0 e^{-\lambda} + \frac{(2-3\gamma)\rho_0}{2\bar{a}^{3\gamma}} e^{\sqrt{3}(\gamma-4/3)\phi/2} \quad (10)$$

$$\ddot{\phi} + 3\frac{\dot{\bar{a}}}{\bar{a}}\dot{\phi} = \lambda V_0 e^{-\lambda\phi} - \frac{\sqrt{3}(\gamma-4/3)\rho_0}{2\bar{a}^{3\gamma}} e^{\sqrt{3}(\gamma-4/3)\phi/2} \quad (11)$$

and the constraint equation

$$6\frac{\dot{\bar{a}}^2}{\bar{a}^2} = \frac{\dot{\phi}^2}{2} + V_0 e^{-\lambda\phi} - \frac{\kappa}{\bar{a}^2} + \frac{\rho_0}{\bar{a}^{3\gamma}} e^{\sqrt{3}(\gamma-4/3)\phi/2} \quad (12)$$

where

$$V_0 = \frac{\delta}{(1+\delta)} \text{sign}(R) \quad \text{and} \quad \lambda = \frac{(\delta-1)}{\sqrt{3}\delta}.$$

Over-dots here denote differentiation with respect to  $\eta$ . In rescaling time in this way we must remember that the more physically significant cosmological time coordinate  $t$  may diverge at finite values of the new time coordinate. In such cases we will consider  $t$  as the more physically meaningful time, and allow the new coordinate to take values only over an appropriate range. This set of equations is similar to those obtained in GR, when considering homogeneous and isotropic cosmologies with a scalar field in an exponential potential, and a perfect fluid. This similarity is due to the conformal equivalence between  $f(R)$  theories of gravity and GR (see e.g. [25, 26, 27]). One should notice, however, that the terms corresponding to the perfect fluid have a more complicated form than may have otherwise been obtained in GR (except for the special case of black-body radiation,  $\gamma = 4/3$ , which is conformally invariant). The corresponding set of equations in GR has been solved in vacuum by Russo [28], and in the presence of a perfect fluid by Dehnen, Gavrillov and Melnikov [29].

### 3 Vacuum solutions

#### 3.1 Spatially flat solutions

In the absence of any matter fields the equations (10) and (11) are identical to homogeneous and isotropic general relativistic cosmologies, with a scalar field in an exponential potential. The spatially flat general solutions to these equations have already been found by Russo [28]. Here we will briefly reiterate the method of solving these equations, and use the results to find the general solution to spatially flat vacuum cosmologies in  $R^n$  gravity.

Firstly, making the transformation of variables  $\{a, \phi\} \rightarrow \{u, v\}$  by the definitions

$$\bar{a}^3 \equiv e^{v+u} \quad \text{and} \quad \phi \equiv \sqrt{\frac{4}{3}}(v-u)$$

and defining the new time coordinate

$$d\eta \equiv e^{\frac{\lambda}{2}\phi} d\tau$$

allows the evolution equations (10) and (11) to be written as

$$\ddot{v} + \left(1 - \frac{\lambda}{\sqrt{3}}\right) \dot{v}^2 - \frac{3V_0}{8} \left(1 + \frac{\lambda}{\sqrt{3}}\right) = 0 \quad (13)$$

$$\ddot{u} + \left(1 + \frac{\lambda}{\sqrt{3}}\right) \dot{u}^2 - \frac{3V_0}{8} \left(1 - \frac{\lambda}{\sqrt{3}}\right) = 0 \quad (14)$$

where over-dots now denote differentiation with respect to  $\tau$ . In terms of these new variables, the constraint equation (12) reads

$$\dot{u}\dot{v} = \frac{3V_0}{8}. \quad (15)$$

We will first solve for two special cases, and then give the solutions for more general cases.

### 3.1.1 $\delta = -1/2$

The special case  $\lambda = \sqrt{3}$  (corresponding to  $\delta = -1/2$ ) gives the solutions to equations (13) and (14), under the constraint (15), as

$$u = c_1 + \frac{1}{2} \ln(\tau + c_2) \quad \text{and} \quad v = c_3 + \frac{3V_0}{8}(\tau + c_2)^2$$

where the  $c_i$  are constants. Transforming these results back, and absorbing constants into redefinitions of the coordinates, gives the metric

$$ds^2 = -\frac{e^{\pm\tau^2}}{\tau^{2/3}} d\tau^2 + \tau^{2/3} d\mathbf{x}^2 \quad (16)$$

where  $d\mathbf{x}^2$  is the line-element of flat three dimensional Euclidean space. The  $\pm$  sign here is due to the dependence of  $V_0$  on the sign of the Ricci scalar.

### 3.1.2 $\delta = 1/4$

The case  $\lambda = -\sqrt{3}$  (corresponding to  $\delta = 1/4$ ) is also special and gives the solutions

$$u = c_4 + \frac{3V_0}{8}(\tau + c_5)^2 \quad \text{and} \quad v = c_6 + \frac{1}{2} \ln(\tau + c_5)$$

i.e. the same as the solutions in the previous case, under the exchange of  $u$  and  $v$ . Now, transforming variables back to the originals and absorbing constants into coordinate freedoms, we have

$$ds^2 = -e^{\pm 2\tau^2} \tau^{2/3} d\tau^2 + e^{\pm\tau^2} d\mathbf{x}^2. \quad (17)$$

Here the two  $\pm$  signs must be chosen together.



### 3.1.3 $\delta \neq -1/2$ or $1/4$

Under the conditions  $V_0(\lambda^2 - 3) > 0$  and  $\delta \neq -1/2$  or  $1/4$ , the solutions to equations (13), (14) and (15) are

$$\begin{aligned} u &= c_7 + \frac{\sqrt{3}}{\sqrt{3} + \lambda} \ln \left[ \cos \left\{ \sqrt{\frac{3V_0}{8} \left( \frac{\lambda^2}{3} - 1 \right)} (\tau + c_8) \right\} \right] \\ v &= c_9 + \frac{\sqrt{3}}{\sqrt{3} - \lambda} \ln \left[ \sin \left\{ \sqrt{\frac{3V_0}{8} \left( \frac{\lambda^2}{3} - 1 \right)} (\tau + c_8) \right\} \right]. \end{aligned}$$

Transforming back to the original variables, and again absorbing constants, gives

$$\boxed{ds^2 = -\frac{\cos^{\frac{2}{4\delta-1}} \tau}{\sin^{\frac{2}{2\delta+1}} \tau} d\tau^2 + \cos^{\frac{4\delta}{4\delta-1}} \tau d\mathbf{x}^2.} \quad (18)$$

In the limit  $\delta \rightarrow 0$  this solution can be seen to approach Minkowski space.

Alternatively, when  $V_0(\lambda^2 - 3) < 0$  the solutions are

$$\begin{aligned} u &= c_{10} + \frac{\sqrt{3}}{\sqrt{3} + \lambda} \ln \left[ e^{\sqrt{\frac{3V_0}{8} \left( 1 - \frac{\lambda^2}{3} \right)} \tau} - c_{11} e^{-\sqrt{\frac{3V_0}{8} \left( 1 - \frac{\lambda^2}{3} \right)} \tau} \right] \\ v &= c_{12} + \frac{\sqrt{3}}{\sqrt{3} - \lambda} \ln \left[ e^{\sqrt{\frac{3V_0}{8} \left( 1 - \frac{\lambda^2}{3} \right)} \tau} + c_{11} e^{-\sqrt{\frac{3V_0}{8} \left( 1 - \frac{\lambda^2}{3} \right)} \tau} \right] \end{aligned}$$

which in terms of the original quantities corresponds to the metric

$$\boxed{ds^2 = -\frac{(e^\tau - c_{11}e^{-\tau})^{\frac{2}{4\delta-1}}}{(e^\tau + c_{11}e^{-\tau})^{\frac{2}{2\delta+1}}} d\tau^2 + (e^\tau - c_{11}e^{-\tau})^{\frac{4\delta}{4\delta-1}} d\mathbf{x}^2.} \quad (19)$$

Here we have absorbed the constants  $c_{10}$  and  $c_{12}$  into coordinate redefinitions whilst retaining  $c_{11}$ . It should be noticed that whilst the magnitude of  $c_{11}$  could have been absorbed, it has been left as the special case  $c_{11} = 0$  corresponds to the attractor solution. Again, this solution approaches Minkowski space as  $\delta \rightarrow 0$ .

The conditions imposed upon  $V_0$  and  $\lambda$  here do not imply any restriction on  $\delta$ . This is not immediately obvious as the sign of  $V_0$  depends on the sign of  $R$ , and hence on the solution. However, calculating the Ricci scalar for solutions (18) and (19), and substituting this into the definition of  $V_0$ , shows that both of these solutions exist for any given  $\delta$  ( $\neq -1/2$  or  $1/4$ ).

## 3.2 Spatially curved solutions

### 3.2.1 $\delta = 1/2$

We will now present the general solution for a spatially curved vacuum universe, when  $\delta = 1/2$ . When  $\rho_0 = 0$  the change of variables

$$\bar{a} \equiv (uv)^{1/4} \quad \text{and} \quad \phi \equiv \sqrt{\frac{3}{4}} \ln \left( \frac{u}{v} \right),$$

and the new time coordinate  $d\tau \equiv \bar{a}d\eta$ , allow the field equations (10), (11) and (12) to be recast in the simple form

$$\begin{aligned} \ddot{u} &= 0 \\ \ddot{v} &= \frac{2}{3}V_0 \\ \dot{u}\dot{v} &= \frac{2}{3}(V_0u - \kappa). \end{aligned}$$

Solving these equations yields

$$u = c_{13}(\tau - c_{14}) \quad \text{and} \quad v = \frac{1}{3}(\tau - c_{14})^2 V_0 - \frac{2}{3} \frac{\kappa}{c_{13}} (\tau - c_{15})$$

which on transformation back to the original variables gives

$$ds^2 = -\frac{d\tau^2}{\tau} + \left\{ c_{16} - \kappa\tau + \frac{V_0}{2}\tau^2 \right\} \left( \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right)$$

where  $c_{16} = \kappa c_{13}(c_{15} - c_{14})$  and all other constants have been absorbed into coordinate redefinitions. In this case we can transform time coordinates back to proper time  $t$ , giving

$$ds^2 = -dt^2 + \{c_{17} - \kappa t^2 \pm t^4\} \left( \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right). \quad (20)$$

As well as this solution there exists a second with

$$u = \frac{\kappa}{V_0} \quad \text{and} \quad v = c_{18} + c_{19}\tau + \frac{V_0}{3}\tau^2.$$

Transforming this solution back gives Milne space.

## 4 Perfect fluid solutions

Perfect fluid dominated solutions to equations (10), (11) and (12) will now be presented. These solutions are for spatially flat ( $\kappa = 0$ ) cosmologies.

## 4.1 $\delta = 1/(3\gamma - 1)$

For the case  $\delta = 1/(3\gamma - 1)$  we can integrate the field equations exactly, to find the general solution. Notable exceptions are given by  $\gamma = 0$  and  $\gamma = 1/3$ . For  $\gamma = 0$ , often associated with vacuum energy, the corresponding value of  $\delta$  is  $-1$ , which gives a gravitational theory derived from a Lagrangian of the form  $R^0 = \text{constant}$ . Clearly this is of little interest. For  $\gamma \rightarrow 1/3$  the value of  $\delta \rightarrow \infty$ , and we have a theory that is not clearly defined. Having excluded these cases we continue by changing the time coordinate to

$$d\tau \equiv d\eta \bar{a}^3 e^{-\lambda\phi},$$

and considering the variables  $u$  and  $v$  defined by the transformations

$$\bar{a} \equiv u^{\frac{1}{2(3+\sqrt{3}\lambda)}} v^{\frac{1}{2(3-\sqrt{3}\lambda)}} \quad \text{and} \quad \phi \equiv \frac{\ln v}{(\sqrt{3} - \lambda)} - \frac{\ln u}{(\sqrt{3} + \lambda)}.$$

These definitions force us to exclude  $\lambda = \pm\sqrt{3}$  (corresponding to  $\gamma = -1/3$  and  $\gamma = 5/3$ ). When  $\lambda = 2/\sqrt{3} - \sqrt{3}\gamma$  (corresponding to  $\delta = 1/(3\gamma - 1)$ ) we can then write the field equations (10), (11) and (12) as

$$\begin{aligned} \ddot{u} &= -\frac{1}{2}\gamma(5 - 3\gamma)\rho_0 v^{-\frac{1+6\gamma}{1+3\gamma}} \\ \ddot{v} &= 0 \\ \dot{u}\dot{v} &= \frac{1}{6}(5 - 3\gamma)(1 + 3\gamma) \left( V_0 + \rho_0 v^{-\frac{3\gamma}{1+3\gamma}} \right). \end{aligned} \quad (21)$$

Solutions to these equations are

$$\begin{aligned} u &= \frac{1}{6}(5 - 3\gamma)(1 + 3\gamma)^2 \rho_0 (\tau - c_{20})^2 (c_{21}(\tau - c_{20}))^{-\frac{1+6\gamma}{1+3\gamma}} + \frac{(5 - 3\gamma)(1 + 3\gamma)V_0}{6c_{21}} (\tau - c_{22}) \\ v &= c_{21}(\tau - c_{20}). \end{aligned}$$

Transforming back to the original variables, and absorbing constants into coordinate redefinitions, gives the metric

$$\boxed{ds^2 = -\tau^{-\frac{6\gamma}{(1+3\gamma)}} \left( (1 + 3\gamma)\rho_0 \tau^{\frac{1}{(1+3\gamma)}} + V_0(\tau + c_{23}) \right)^{-\frac{6(1-\gamma)}{5-3\gamma}} d\tau^2 + \left( (1 + 3\gamma)\rho_0 \tau^{\frac{1}{(1+3\gamma)}} + V_0(\tau + c_{23}) \right)^{\frac{2}{5-3\gamma}} d\mathbf{x}^2} \quad (22)$$

where  $c_{23} = c_{21}(c_{20} - c_{22})$ . This solution is valid for all  $\gamma \neq -1/3, 0, 1/3$  or  $5/3$ .

A second solution to the equations (21) also exists, when  $V_0 < 0$ . This solution is given by

$$\begin{aligned} u &= c_{24} + c_{25}\tau - \frac{1}{4}\gamma(5 - 3\gamma)\rho_0 \left( -\frac{V_0}{\rho_0} \right)^{\frac{1+6\gamma}{3\gamma}} \tau^2 \\ v &= \left( -\frac{V_0}{\rho_0} \right)^{-\frac{1+3\gamma}{3\gamma}}. \end{aligned}$$

Which corresponds to the metric

$$\begin{aligned}
 ds^2 = & - \left( (c_{24} + c_{25}\tau) - \frac{1}{4}\gamma(5 - 3\gamma)\rho_0\tau^2 \right)^{-\frac{6(1-\gamma)}{5-3\gamma}} d\tau^2 \\
 & + \left( (c_{24} + c_{25}\tau) - \frac{1}{4}\gamma(5 - 3\gamma)\rho_0\tau^2 \right)^{\frac{2}{5-3\gamma}} d\mathbf{x}^2.
 \end{aligned} \tag{23}$$

Again, this solution is valid for all  $\gamma \neq -1/3, 0, 1/3$  or  $5/3$ .

## 4.2 $\delta = -(4 - 3\gamma)/(2(7 - 3\gamma))$

For the theories  $\delta = -(4 - 3\gamma)/(2(7 - 3\gamma))$  we can also integrate the field equations directly, to obtain general solutions for spatially flat cosmologies. We should note that the particular cases  $\gamma = 4/3$ ,  $\gamma = 7/3$  and  $\gamma = 10/3$  are not usefully solved for here. Unfortunately, these exceptional cases include the physically interesting case of  $\gamma = 4/3$ , a fluid of black-body radiation. Here  $\gamma = 4/3$  corresponds to  $\delta = 0$ , which is GR. The case  $\gamma \rightarrow 7/3$  corresponds to the limit  $\delta \rightarrow \infty$ , and  $\gamma = 10/3$  corresponds to  $\delta = -1$ . Neither of these values of  $\delta$  are of physical interest.

Making the transformation to the variables  $u$  and  $v$  via the definitions

$$a \equiv e^{\left(\frac{\lambda}{\sqrt{3}}u+v\right)} \quad \text{and} \quad \phi \equiv \sqrt{12} \left( \frac{\lambda}{\sqrt{3}}v + u \right)$$

and redefining time as  $d\eta \equiv a^3 d\tau$  allows the field equations (10), (11) and (12) to be written as

$$\begin{aligned}
 \ddot{u} &= \frac{(4 - 3\gamma)}{12} \rho_0 e^{\frac{4(5-3\gamma)}{(4-3\gamma)}u} \\
 \ddot{v} &= \frac{1}{2} V_0 e^{-\frac{24(5-3\gamma)}{(4-3\gamma)^2}v} \\
 \dot{u}^2 - \dot{v}^2 &= \frac{(4 - 3\gamma)^2}{24(5 - 3\gamma)} \rho_0 e^{\frac{4(5-3\gamma)}{(4-3\gamma)}u} + \frac{(4 - 3\gamma)^2}{24(5 - 3\gamma)} V_0 e^{-\frac{24(5-3\gamma)}{(4-3\gamma)^2}v}
 \end{aligned}$$

where we have taken

$$\frac{\lambda}{\sqrt{3}} = -\frac{3(2 - \gamma)}{(4 - 3\gamma)},$$

which corresponds to  $\delta = -(4 - 3\gamma)/(2(7 - 3\gamma))$ . The case  $\gamma = 5/3$  does not give a sensible limit, and so we will exclude it from consideration. When  $\gamma \neq 5/3$  solutions to these equations are given by

$$\begin{aligned}
 u &= -\frac{(4 - 3\gamma)}{4(5 - 3\gamma)} \ln \left[ \frac{(5 - 3\gamma)\rho_0}{6c_{26}^2} \sin^2(c_{26}(\tau - c_{27})) \right] \\
 v &= \frac{(4 - 3\gamma)^2}{24(5 - 3\gamma)} \ln \left[ -\frac{(5 - 3\gamma)V_0}{6c_{26}^2} \sin^2 \left( \frac{6c_{26}}{(4 - 3\gamma)}(\tau - c_{28}) \right) \right].
 \end{aligned}$$

The two sin functions here can be transformed to cos functions by suitably redefining the constants  $c_{27}$  and  $c_{28}$ . These sin or cos functions can then be transformed to sinh or cosh function by the transformation  $c_{26} \rightarrow ic_{26}$ . It is, of course, important that solutions remain real for at least some range of  $\tau$ . This restricts which functions should be taken as physically interesting (assuming  $\rho_0 > 0$ ), so that the form of the corresponding metric depends upon the sign of  $5 - 3\gamma$ . We will treat the different cases separately below.

#### 4.2.1 $5 - 3\gamma > 0$

The condition  $5 - 3\gamma > 0$  includes the important cases of pressureless dust and vacuum energy density, and leads to the metric

$$ds^2 = -b_i^2(\tau)d\tau^2 + a_i^2(\tau)d\mathbf{x}^2 \quad (24)$$

where  $i = 1, 2$  or  $3$ . We then have

$$a_1 = \sin^{-\frac{1}{(5-3\gamma)}} \{\tau - c_{29}\} \sin^{-\frac{(4-3\gamma)}{6(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\}$$

$$b_1 = \sin^{-\frac{(7-3\gamma)}{(5-3\gamma)}} \{\tau - c_{29}\} \sin^{\frac{(1-\gamma)(4-3\gamma)}{2(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\},$$

where constants have been absorbed into coordinate redefinitions, and  $c_{29} = c_{26}(c_{27} - c_{28})$ . These sin functions can be transformed to cos functions through redefinitions of the origin of the time coordinate  $\tau$ , and the constant  $c_{29}$ . A second solution also exists with

$$a_2 = \sinh^{-\frac{1}{(5-3\gamma)}} \{\tau - c_{29}\} \sinh^{-\frac{(4-3\gamma)}{6(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\}$$

$$b_2 = \sinh^{-\frac{(7-3\gamma)}{(5-3\gamma)}} \{\tau - c_{29}\} \sinh^{\frac{(1-\gamma)(4-3\gamma)}{2(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\},$$

and a third with

$$a_3 = \sinh^{-\frac{1}{(5-3\gamma)}} \{\tau - c_{30}\} \cosh^{-\frac{(4-3\gamma)}{6(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\}$$

$$b_3 = \sinh^{-\frac{(7-3\gamma)}{(5-3\gamma)}} \{\tau - c_{30}\} \cosh^{\frac{(1-\gamma)(4-3\gamma)}{2(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\}.$$

These solutions are valid for all  $\gamma < 5/3$  and  $\neq 4/3$ .

#### 4.2.2 $5 - 3\gamma < 0$

The condition  $5 - 3\gamma < 0$  contains the important case of a scalar field,  $\gamma = 2$ , and gives the metric

$$ds^2 = -b_j^2(\tau)d\tau^2 + a_j^2(\tau)d\mathbf{x}^2 \quad (25)$$

where  $j = 4$  or  $5$ . Here

$$a_4 = \cosh^{-\frac{1}{(5-3\gamma)}} \{\tau - c_{31}\} \sinh^{-\frac{(4-3\gamma)}{6(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\}$$

$$b_4 = \cosh^{-\frac{(7-3\gamma)}{(5-3\gamma)}} \{\tau - c_{31}\} \sinh^{\frac{(1-\gamma)(4-3\gamma)}{2(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\}$$

and a second solution is

$$a_5 = \cosh^{-\frac{1}{(5-3\gamma)}} \{\tau - c_{32}\} \cosh^{-\frac{(4-3\gamma)}{6(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\}$$

$$b_5 = \cosh^{-\frac{(7-3\gamma)}{(5-3\gamma)}} \{\tau - c_{32}\} \cosh^{\frac{(1-\gamma)(4-3\gamma)}{2(5-3\gamma)}} \left\{ \frac{6}{(4-3\gamma)} \tau \right\}.$$

These solutions are valid for all  $\gamma > 5/3$  and  $\neq 7/3$  or  $10/3$ .

## 5 Analysis of vacuum cosmologies

In this section we will perform an analysis of the solutions found in section 3. The special cases of  $\delta = -1/2$  and  $\delta = 1/4$  will be investigated in the appendix. These solutions contain a number of constants, as a result of integrating the field equations. The physical significance of these constants varies: In some cases they can be absorbed into a rescaling of coordinates, in others they cannot and must be specified by initial conditions<sup>2</sup>. If the latter is the case, then these constants are physically meaningful quantities and their value is important for the evolution of the space-time. We will describe the effect of taking different values for these constants in the analysis below.

### 5.1 Spatially flat cosmologies

We begin with the spatially flat solution (19). In the case  $c_{11} = 0$  the solution (19) can be written as

$$ds^2 = -e^{-\frac{4(\delta-1)}{(4\delta-1)(2\delta+1)}\tau} d\tau^2 + e^{\frac{4\delta}{(4\delta-1)}\tau} d\mathbf{x}^2$$

$$= -dt^2 + t^{\frac{2\delta(1+2\delta)}{(1-\delta)}} d\mathbf{x}^2,$$

which is simply the power-law solution (8). It can be seen directly from (19) that for all  $c_{11}$  this simple solution is the attractor as  $\tau \rightarrow \infty$ .

When  $c_{11} = 0$  the power-law solution above describes the evolution of the universe all the way back to the initial singularity, but for  $c_{11} \neq 0$  more general behaviour occurs as the singularity is approached. The form of this early-time behaviour depends upon the sign of  $c_{11}$  (recall that the magnitude of  $c_{11}$  can be absorbed into coordinate redefinitions).

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<sup>2</sup>This is not a new phenomenon, and can be shown to be the case in, for example, the general Friedmann solutions of Brans-Dicke theory [30].

For  $c_{11} > 0$  a power series expansion of the metric gives

$$\begin{aligned} ds^2 &\simeq -\tau^{\frac{2}{(4\delta-1)}} d\tau^2 + \tau^{\frac{4\delta}{(4\delta-1)}} d\mathbf{x}^2 \\ &= -dt^2 + td\mathbf{x}^2 \end{aligned}$$

when  $\tau - \frac{1}{2} \log(c_{11}) \ll 1$ . This limit is the same as the power-law exact solution (9).

For  $c_{11} < 0$  a similar expansion about  $\frac{1}{2} \log(-c_{11})$  gives

$$\begin{aligned} ds^2 &\simeq -\tau^{\frac{-2}{(1+2\delta)}} d\tau^2 + \left(1 + \frac{\delta}{(4\delta-1)} \tau^2\right)^2 d\mathbf{x}^2 \\ &= -dt^2 + \left(1 + \frac{\delta}{(4\delta-1)} t^{\frac{(1+2\delta)}{\delta}}\right)^2 d\mathbf{x}^2. \end{aligned}$$

This solution is clearly quite different from the power-law limits that have been found so far. For  $\delta > 1/4$  or  $< 0$  it corresponds to a non-zero minimum of expansion, or bounce. For  $0 < \delta < 1/4$  it is a maximum of expansion. It should be noted that when  $-1/2 < \delta < 0$  the power of  $t$  in the scale factor diverges as  $t \rightarrow 0$ . In this case there is no staticity, and the scale-factor diverges as  $t \rightarrow 0$ . Bounces occur here without any violation of energy conditions, as there are no matter fields present to commit such violations. The bounce is entirely due to the vacuum dynamics of the theory.

We have now shown both the early and late-time behaviour of the solution (19). At late-times all solutions approach the power-law exact particular solution (8). At early-times the evolution of the universe is qualitatively different depending upon the value of the constant  $c_{11}$ . For  $c_{11} = 0$  the solution (8) is valid all the way back to the initial singularity. For  $c_{11} > 0$  the general solution has an early period of expansion of the form of (9), in the vicinity of the singularity. The expansion then evolves into the late-time attractor. When  $c_{11} < 0$  there is a non-zero minimum of expansion (or maximum if  $0 < \delta < 1/4$ , or divergence if  $-1/2 < \delta < 0$ ). The universe then evolves from this static past towards the late-time attractor. Some representative evolutions of the scale-factor, in terms of the proper time coordinate  $t$ , are shown in figure 2.

We will now investigate the solution (18). Firstly, expanding around the point  $\tau = 0$  we get the approximation

$$\begin{aligned} ds^2 &\simeq -\tau^{-\frac{1}{(1+2\delta)}} d\tau^2 + \left(1 - \frac{4\delta}{(4\delta-1)} \tau^2\right)^2 d\mathbf{x}^2 \\ &= -dt^2 + \left(1 - \frac{4\delta}{(4\delta-1)} t^{\frac{(1+2\delta)}{\delta}}\right)^2 d\mathbf{x}^2, \end{aligned}$$

when  $\tau \ll 1$ . This is remarkably similar to the form of the previous metric, (19), in the vicinity of its minimum of expansion. A noticeable difference is the sign before the second term in the brackets. This change in sign shows that when the previous solution was a minimum ( $\delta > 1/4$  or  $< -1/2$ ) this solution is a maximum of expansion. Correspondingly, when the previous solution was a maximum ( $0 < \delta < 1/4$ ), this solution is a minimum. Again, there is divergence when  $-1/2 < \delta < 0$ .

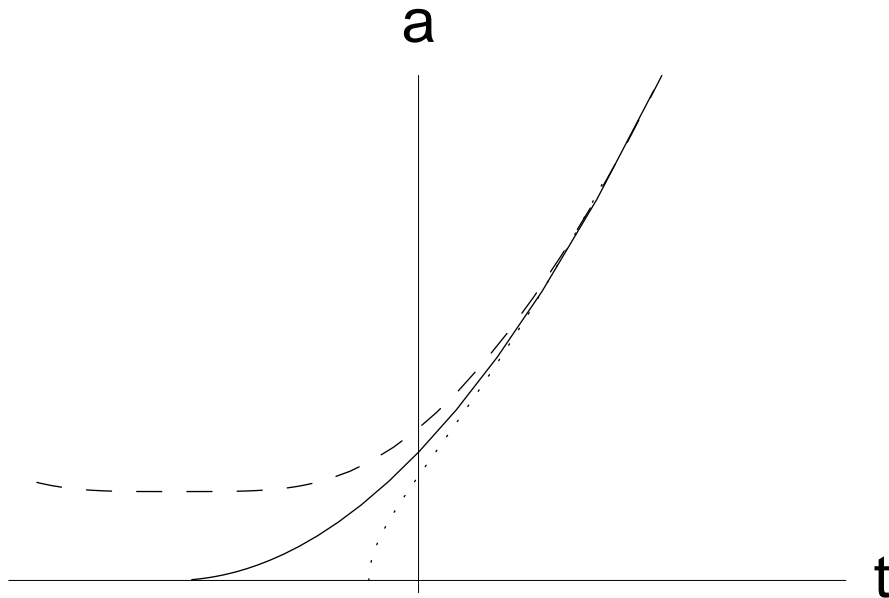


Figure 2: The evolution of the scale factor of solution (19) in terms of proper time,  $t$ . The value of  $\delta$  chosen here is  $1/2$ . The solid line corresponds to  $c_{11} = 0$ , or the power-law exact solution (8). The dotted line corresponds to  $c_{11} > 0$  and shows an early period of  $t^{1/2}$  expansion, that is attracted towards (8) at later times. The dashed line corresponds to  $c_{11} < 0$  and shows a non-zero minimum of expansion, and subsequent evolution towards (8).

We will now investigate the form of (18) around  $\tau = \pm\pi/2$ . Performing a power series expansion about either of these points gives

$$\begin{aligned} ds^2 &\simeq -\tau^{\frac{2}{(4\delta-1)}} d\tau^2 - \tau^{\frac{4\delta}{(4\delta-1)}} d\mathbf{x}^2 \\ &= -dt^2 + t d\mathbf{x}^2, \end{aligned}$$

when  $\tau \mp \pi/2 \ll 1$ . In the vicinity of both of these points the evolution of (19) is therefore of the form of the power-law exact solution (9).

We have shown that the evolution of the solution (18) depends upon the sign of  $\delta$ . When  $\delta > 1/4$  or  $< -1/2$  the solution, in terms of proper time  $t$ , has a maximum of expansion with its early and late-time evolution going as  $t^{1/2}$ . When  $0 < \delta < 1/4$  there is a minimum of expansion, and late and early-time expansion again goes as  $t^{1/2}$ . For  $-1/2 < \delta < 0$  the solution diverges in the region which is static for all other values of  $\delta$ . The form of some representative solutions are shown in figure 3. The behaviour of these solutions are markedly different from those found by Russo [28]. Although a late-time power-law attractor can still be seen to exist in one of the solutions, very little else is comparable.

## 5.2 Spatially curved cosmologies

For spatially curved vacuum universes we found the general solutions (20) and Milne space, when  $\delta = 1/2$ . The existence of Milne space as a solution should not be surprising, as any Ricci



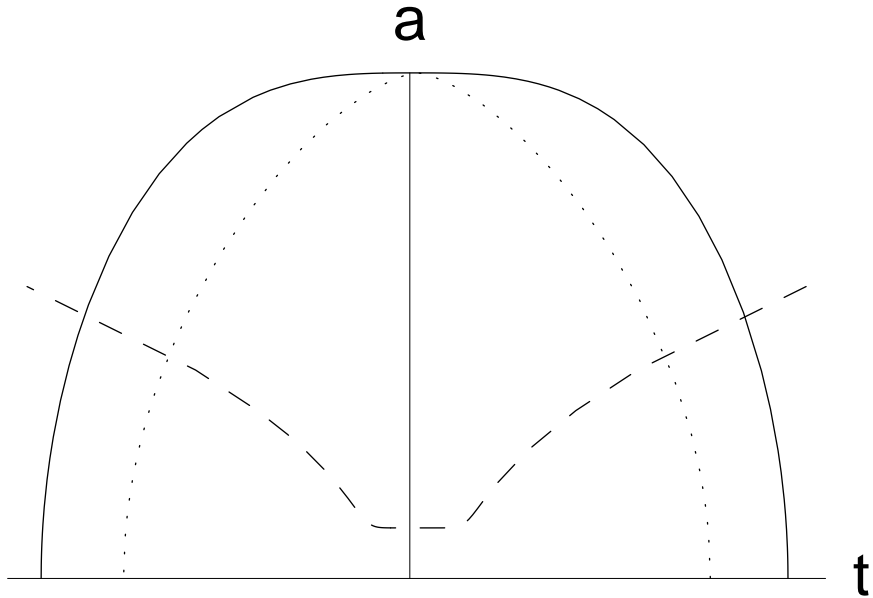


Figure 3: The evolution of the scale factor of solution (18) in terms of proper time,  $t$ . The solid line corresponds to  $\delta = 1$ , the dashed line to  $\delta = 1/8$  and the dotted line to  $\delta = -2$ .

flat solution of GR in a vacuum is also a solution of the field equations (2) in vacuum (at least when  $\delta > 0$ ). We will now investigate the form of the solution (20).

When  $\kappa = 0$  this solution reduces to

$$ds^2 = -dt^2 + t^4 d\mathbf{x}^2,$$

which corresponds to the power-law particular solution (8), when  $\delta = 1/2$ . When  $\kappa \neq 0$  we see that this solution exhibits new behaviour, dependent on the value of  $\kappa$  and which branch of the  $\pm$  sign is chosen.

When  $\kappa > 0$  and  $c_{17} > 0$ , then in the vicinity of  $t = 0$  this solution exhibits a maximum of expansion. Subsequent evolution of the scale-factor, as  $t \rightarrow \pm\infty$ , depends upon the chosen branch. Taking the positive branch we see that the universe can start to expand again, as  $|t|$  increases. The existence of this expansion phase depends upon the magnitude of  $c_{17}$ . If  $c_{17} \leq \kappa^2/4$  then the effect of the second term in the scale-factor is too great and the universe collapses to a singularity, before the  $t^4$  term can become dominant. When  $c_{17} > \kappa^2/4$  the  $t^4$  term dominates the late-time evolution and the power-law solution (8) is approached, as  $|t| \rightarrow \infty$ . When the negative branch is taken then the maximum of expansion at  $t = 0$  always leads to collapse to a singularity, as  $t$  either increases or decreases.

When  $\kappa < 0$  and  $c_{17} = 0$  then there exists an initial singularity, with the evolution of the scale factor in its immediate vicinity expanding proportionally to  $t$ . The late-time evolution of such solutions will be attracted to either the power-law solution (8), if the positive branch is chosen, or to eventual collapse to singularity, if the negative branch is chosen. The case  $c_{17} > 0$  does not have a singularity at  $t = 0$ , but instead has a minimum of expansion, or bounce. The evolution away from this minimum proceeds as in the  $c_{17} = 0$  case. The  $c_{17} < 0$  case again features an initial singularity and displays approximately the same evolution as when  $c_{17} = 0$

(a notable deviation is that the period of  $a \propto t$  expansion near the singularity is absent if  $c_{17}$  is sufficiently negative).

We have shown that when  $\kappa \neq 0$  there exists a minimum of expansion when  $c_{17} > 0$ , and an initial singularity when  $c_{17} \leq 0$ . Initial evolution away from an early minimum proceeds as  $a \propto t$ , either expanding or collapsing, depending on the sign of  $\kappa$ . If there is an initial singularity, then evolution away from it proceeds as  $a \propto t$ , unless  $c_{17}$  is sufficiently negative (in which case the  $t^4$  term dominates right from the beginning). The late-time evolution of these solutions is dominated by the  $t^4$  term in (20). The negative branch of this term causes collapse to a singularity, and the positive branch leads to asymptotic expansion and approach towards the power-law solution (8) (unless the term proportional to  $\kappa$  causes collapse to singularity before the  $t^4$  becomes dominant). The form of some representative solutions are shown in figure 4.

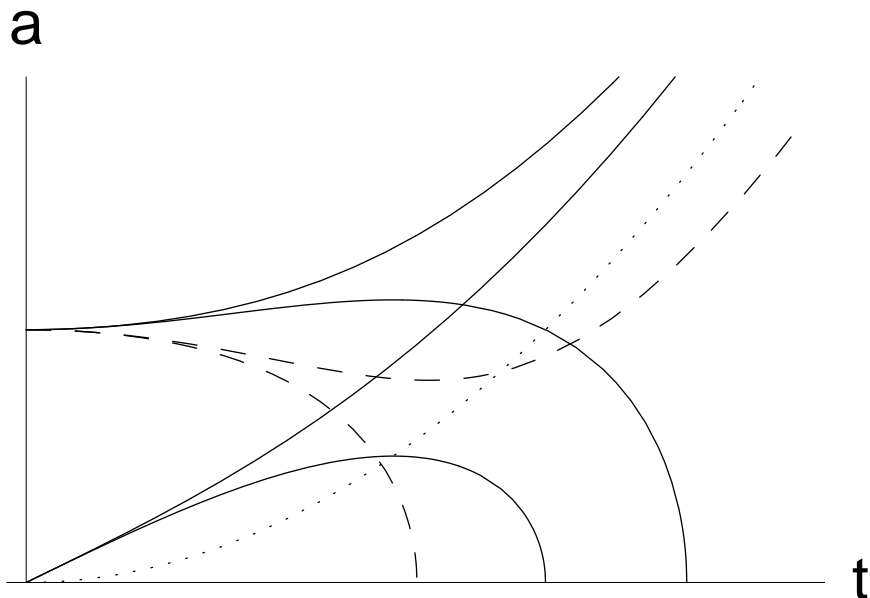


Figure 4: The evolution of the scale factor of solution (20) in terms of proper time,  $t$ . The solid lines correspond to universes with  $\kappa > 0$ , dashed lines to those with  $\kappa < 0$  and the dotted line to  $\kappa = 0$ . Those solutions starting from the origin have  $c_{17} = 0$ , and those starting from a non-zero minimum have  $c_{17} > 0$ . All solutions collapsing to singularity correspond to the negative branch of  $t^4$ , and all solutions expanding at late-times to the positive branch. See the main body of text for an explanation of these different behaviours.

## 6 Analysis of perfect fluid cosmologies

We will now analyse the evolution of universes described by the perfect fluid solutions (22), (23), (24) and (25). Again, these solutions contain a number of constants, some of which can be absorbed into coordinate redefinitions and some of which cannot. We will explain the significance of such constants below.

## 6.1 $\delta = 1/(3\gamma - 1)$

When  $\delta = 1/(3\gamma - 1)$  we found from direct integration of the field equations the general solutions (22) and (23). We will first consider (22).

The form of the scale factor in this solution depends upon which of the two terms in the brackets is dominant. When the first term dominates the metric is approximately given by

$$\begin{aligned} ds^2 &\simeq -\tau^{-\frac{6(1+4\gamma-3\gamma^2)}{(5-3\gamma)(1+3\gamma)}} d\tau^2 + \tau^{\frac{2}{(5-3\gamma)(1+3\gamma)}} d\mathbf{x}^2 \\ &= -dt^2 + t d\mathbf{x}^2, \end{aligned}$$

which corresponds to the power-law particular solution (9). When the second term in the brackets dominates we have

$$\begin{aligned} ds^2 &\simeq -\tau^{-\frac{6(1+7\gamma-6\gamma^2)}{(5-3\gamma)(1+3\gamma)}} + \tau^{\frac{2}{(5-3\gamma)(1+3\gamma)}} d\mathbf{x}^2 \\ &= -dt^2 + t^{\frac{2(1+3\gamma)}{(2-3\gamma)(1-3\gamma)}} d\mathbf{x}^2, \end{aligned}$$

which corresponds to the vacuum dominated, power-law particular solution (8). It now remains to investigate the conditions under which each of these approximations dominates, at early and late times.

When  $c_{23} = 0$ , and  $\gamma > 0$ , then the first term always dominates as  $\tau \rightarrow 0$ , and the universe evolves like  $a \sim t^{\frac{1}{2}}$ . As  $\tau \rightarrow \infty$  the second term dominates, and the evolution of the universe is attracted towards the form of (8).

When  $c_{23} \neq 0$ , and  $\gamma > 0$ , then this has the effect of offsetting the origin of the second term relative to the first. The second term will then be the dominant one for most of the range of  $\tau$ , with a contribution expected from the first term in the vicinity of  $\tau = -c_{23}$ .

Phantom fluids, with  $\gamma < 0$ , have the same behaviour as outlined above, but with the dominance of the two terms reversed in each case.

We will now consider the solution (23). The evolution of this space-time again depends on which of the two terms in the brackets is dominant. If the first term dominates then the metric look like

$$\begin{aligned} ds^2 &\simeq -\tau^{-\frac{6(1-\gamma)}{(5-3\gamma)}} d\tau^2 + \tau^{\frac{2}{(5-3\gamma)}} d\mathbf{x}^2 \\ &= -dt^2 + t d\mathbf{x}^2, \end{aligned}$$

which is again the power-law evolution (9). We will call this vacuum domination, as it corresponds to evolution being dominated by a term independent of  $\rho_0$ . If the second term in brackets dominates then we now have

$$\begin{aligned} ds^2 &\simeq -\tau^{-\frac{12(1-\gamma)}{(5-3\gamma)}} d\tau^2 + \tau^{\frac{4}{(5-3\gamma)}} d\mathbf{x}^2 \\ &= -dt^2 + t^{\frac{4}{(3\gamma-1)}} d\mathbf{x}^2, \end{aligned}$$

which corresponds to the matter dominated power-law expansion (7). We must now investigate the conditions under which each of these approximations dominates. This will depend upon the value of the constants  $c_{24}$  and  $c_{25}$ .

When  $c_{24} = c_{25} = 0$  we see that the matter dominated power-law expansion (7) holds all the way back to the initial singularity. If  $c_{24} = 0$  and  $c_{25} \neq 0$ , then there is an initial period of vacuum dominated expansion of the form  $a \propto t^{\frac{1}{2}}$  and a late-time evolution towards the matter dominated power-law solution (7). If  $c_{24} \neq 0$  then this has the effect of offsetting the origin of first term in the brackets, relative to the second. This can result in a non-zero minimum of expansion (depending on the signs of  $c_{24}$  and  $c_{25}$ ), as in this case the first term can both dominate and be non-zero at  $\tau = 0$ . As  $|\tau| \rightarrow \infty$  the matter dominated power-law solution (7) is then approached.

## 6.2 $\delta = -(4 - 3\gamma)/(2(7 - 3\gamma))$

We will now investigate the solutions (24) and (25), obtained by integrating the field equations in the presence of a perfect fluid when  $\delta = -(4 - 3\gamma)/(2(7 - 3\gamma))$ .

### 6.2.1 $5 - 3\gamma > 0$

We will first consider the solution  $\{a_1, b_1\}$ . Expanding this solution around the points  $\tau = n\pi$  and  $\tau = c_{29} + n\pi$ , where  $n$  is an integer, gives the evolution of the scale factor in their vicinity. To first order, an expansion about  $\tau = n\pi$  results in

$$\begin{aligned} ds^2 &\simeq -\tau^{\frac{(4-3\gamma)(1-\gamma)}{(5-3\gamma)}} d\tau^2 + \tau^{-\frac{(4-3\gamma)}{(5-3\gamma)}} d\mathbf{x}^2 \\ &= -dt^2 + t^{\frac{2(4-3\gamma)}{3(7-3\gamma)(\gamma-2)}} d\mathbf{x}^2, \end{aligned}$$

which corresponds to the vacuum dominated power-law expansion, (8). A similar expansion about  $\tau = c_{29} + n\pi$  gives

$$\begin{aligned} ds^2 &\simeq -(\tau - c_{29})^{-\frac{2(7-3\gamma)}{(5-3\gamma)}} d\tau^2 + (\tau - c_{29})^{-\frac{2}{(5-3\gamma)}} d\mathbf{x}^2 \\ &= -dt^2 + t d\mathbf{x}^2, \end{aligned}$$

which is the power-law evolution described by (9). Whether or not these points correspond to a curvature singularity depends upon the value of  $\gamma$ , and can be read off from the Ricci scalar

$$R = \frac{6(10 - 3\gamma)}{(5 - 3\gamma)(4 - 3\gamma)} \sin^{\frac{2(7-3\gamma)}{(5-3\gamma)}} \{\tau - c_{29}\} \sin^{-\frac{(2-\gamma)(7-3\gamma)}{(5-3\gamma)}} \left\{ \frac{6}{(4 - 3\gamma)\tau} \right\}.$$

We see that when  $\gamma \geq 7/3$  the Ricci scalar remains finite for all  $\tau$ , whilst for  $2 < \gamma < 7/3$  or  $\gamma < 5/3$  the Ricci scalar diverges to infinity every time  $\tau = (4 - 3\gamma)n\pi/6$ . The evolution of  $a(t)$  in the vicinity of these singularities goes like (8). Similarly, when  $5/3 < \gamma < 7/3$  there are curvature singularities at  $\tau = c_{29} + n\pi$ . In the vicinity of these singularities  $a(t) \sim t^{\frac{1}{2}}$ . These latter singularities correspond to  $a(t) \rightarrow 0$ , whereas the former correspond to the divergence  $a(t) \rightarrow \infty$  (except in the range  $4/3 < \gamma < 5/3$ , in which case  $a(t) \rightarrow 0$ ).

An instructive special case to consider is that of pressureless dust,  $\gamma = 0$ . In this case the coordinate transformation  $t = \cot(\tau - c_{29})$  allows the metric to be recast as

$$ds^2 = -dt^2 + \frac{\sqrt{t^2 + 1}}{\sin^{\frac{1}{6}}\{6(c_{29} + \cot^{-1}\{t\})\}} d\mathbf{x}^2.$$

It can now be seen directly that  $a(t) \rightarrow t^{\frac{1}{2}}$  as  $t \rightarrow \infty$  (or  $\tau \rightarrow c_{29} + n\pi$ ), and diverges to  $\infty$  as (8) when  $t \rightarrow -\cot\{c_{29} - n\pi/6\}$  (or  $\tau \rightarrow n\pi/6$ ).

We will now consider the solution  $\{a_2, b_2\}$ . In the limit  $\tau \rightarrow \pm\infty$  this solution approaches

$$\begin{aligned} ds^2 &\simeq -e^{-\frac{8\tau}{(5-3\gamma)}} d\tau^2 + e^{-\frac{4\tau}{(5-3\gamma)}} d\mathbf{x}^2 \\ &= -dt^2 + t d\mathbf{x}^2, \end{aligned}$$

which is the power-law solution (9). When  $\tau \rightarrow 0$  we see that we have the same limit as in the previous solution, where the power-law evolution (8) is approached. Similarly, as  $\tau \rightarrow c_{29}$  we approach the power-law evolution  $a \sim t^{\frac{1}{2}}$ , (9). The conditions for whether or not these limits correspond to singularities, and if these singularities correspond to  $a(t) \rightarrow 0$  or  $a(t) \rightarrow \infty$ , have exactly the same dependence on  $\gamma$  as with the previous solution.

The pressureless dust case is again instructive, and can be written in terms of proper time, by making the transformation  $t = \coth\{c_{29} - \tau\}$ , as

$$ds^2 = -dt^2 + \frac{\sqrt{t^2 - 1}}{\sinh^{\frac{1}{6}}\{6(c_{29} - \coth^{-1}\{t\})\}} d\mathbf{x}^2.$$

This solution displays  $t^{\frac{1}{2}}$  evolution as  $t \rightarrow \infty$ , and divergence to  $\infty$  as  $t \rightarrow \coth\{c_{29}\}$ .

The solution  $\{a_3, b_3\}$  will now be considered. The late-time evolution of this solution is the same as in the previous case. The evolution about  $\tau = c$  is also the same. Now, however, we no longer have the singular behaviour about  $\tau = 0$  that previously existed. Instead the scale-factor evolves as  $a \sim a_0 + t^{\frac{1}{2}}$  in the vicinity of this point, where  $a_0$  is some constant. The dust solution in this case is then

$$ds^2 = -dt^2 + \frac{\sqrt{t^2 - 1}}{\cosh^{\frac{1}{6}}\{6(c_{30} - \coth^{-1}\{t\})\}} d\mathbf{x}^2,$$

where  $t = \coth\{c_{30} - \tau\}$ . This metric has a scale factor that no longer diverges at any finite  $t$ .

### 6.2.2 $5 - 3\gamma < 0$

The late-time evolution of these two solutions is of the form  $a \sim t^{\frac{1}{2}}$ , as with the previous two solutions. Now, the form of solution  $\{a_4, b_4\}$  about  $\tau = 0$  displays the same divergences as  $\{a_2, b_2\}$ . Evolution towards the point  $\tau = c$  is non-singular, and goes as  $a \sim a_0 + t^{\frac{1}{2}}$ . The special case of pressureless dust is given in this case, in terms of proper time  $t$ , by

$$ds^2 = -dt^2 + \frac{\sqrt{1 - t^2}}{\sinh^{\frac{1}{6}}\{6(c_{31} + \tanh^{-1}\{t\})\}} d\mathbf{x}^2,$$

where  $t = \tanh\{\tau - c_{31}\}$ . The scale-factor here can be seen to vanish at  $t = \pm 1$  (corresponding to  $\tau \rightarrow \pm\infty$ ), and to diverge to  $\infty$  at  $t = \tanh\{-c_{31}\}$ . The evolution of the scale factor follows the late-time attractor towards the points at  $t = \pm 1$ .

We will now consider the remaining solution,  $\{a_5, b_5\}$ . This solution behaves as the solution  $\{a_3, b_3\}$  in the vicinity of  $\tau = 0$ , and as  $\{a_4, b_4\}$  in the vicinity of  $\tau = c$ . That is, about both of

these points the evolution of the scale factor goes as  $a \sim a_0 + t^{\frac{1}{2}}$  and is non-singular. In fact, at no finite  $\tau$  does this solution become singular. The pressureless dust solution is given here, in terms of proper time, as

$$ds^2 = -dt^2 + \frac{\sqrt{1-t^2}}{\cosh^{\frac{1}{6}}\{6(c_{32} + \tanh^{-1}\{t\})\}} d\mathbf{x}^2,$$

where  $t = \tanh\{\tau - c_{32}\}$ . This solution describes a space-time that evolves from  $a = 0$  at  $t = -1$  (corresponding to  $\tau = -\infty$ ) to  $a = 0$  at  $t = 1$  (corresponding to  $\tau = \infty$ ). The evolution between these two end points is at all times finite.

## 7 Discussion

We have investigated the homogeneous and isotropic cosmological solutions of  $R^n$  theories. It has been shown that in a number of cases the field equations can be integrated directly, allowing the general behaviour of these models to be found. For spatially flat vacuum universes the solutions for any  $n$  can be found. For spatially curved vacuum universes, and flat perfect fluid universes, the general solutions can be obtained for various particular values of  $n$ . These solutions were given explicitly in sections 3 and 4. It is interesting to note that unlike the general relativistic cosmologies, vacuum solutions can be either open, closed or flat. It was also found that there is generically not one unique solution for any given  $n$ , matter content and topology, as there usually is in GR. This was found previously in [17, 21], where a phase plane analysis showed an invariant sub-manifold in the phase space of solutions which could not be crossed by any trajectory. This result is reiterated here, where we find multiple ways to integrate the field equations, yielding multiple solutions. It was also shown in [17, 21] that the form of the general solution changes with  $n$ : Critical points in the phase space change their attractor nature as  $n$  is varied. Again, we confirm this result by finding explicit solutions for cosmologies with the same topology and matter content, but different  $n$ . These solutions have clearly different forms. In sections 5 and 6 we performed a brief analysis of the vacuum and perfect fluid solutions, respectively. The evolution of these universes, in terms of proper time, were found in different regimes. The nature of the initial singularity was investigated, or its lack thereof, as well as evolution at late-times. In summary, it was found that great variety is present in the cosmologies of these theories. Behaviour at both late and early times can vary wildly, depending on  $n$ , and even between different solutions with the same  $n$ , matter content and topology. Solutions can manifest initially singular or non-singular behaviour, and late-time evolution can either lead to a crunch or to monotonic expansion.

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## A Vacuum $\delta = -1/2$ cosmologies

We will now describe the evolution of universes described by the solution (16). Taking the positive branch of (16) we see that as  $\tau \rightarrow \infty$  we have  $d\tau \ll dt$ , so that in terms of proper time the universe evolves very slowly at late-times. However, for the negative branch of (16) we see that  $d\tau \gg dt$  in this limit. For the negative branch, therefore, we should expect increasingly rapid evolution at late times, and for the positive branch an evolution towards an asymptotic steady state. A power series expansion of (16) shows that the early-time behaviour of both branches goes like the power-law particular solution (9). Figure 5 shows graphically the evolution of the two branches of this solution.

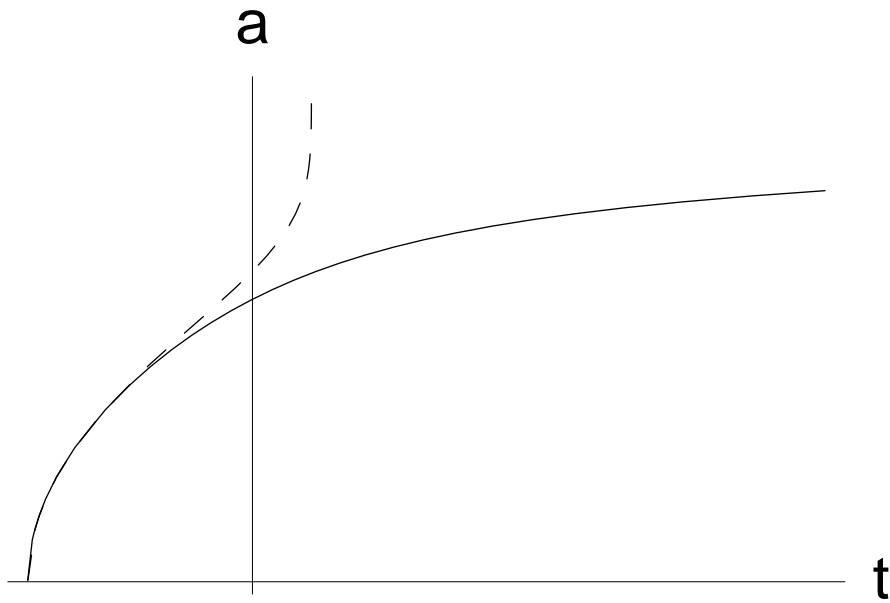


Figure 5: The evolution of the scale factor of solution (16) in terms of proper time,  $t$ . The solid lines correspond to the positive branch, and the dashed line to the negative branch.

## B Vacuum $\delta = 1/4$ cosmologies

We will now describe the evolution of universes described by the solution (17). A power series expansion of this solution shows that the scale factors of both branches approach a stationary state, in terms proper time  $t$ , as  $\tau \rightarrow 0$ . We can then see that as  $\tau$  increases the positive branch expands out of this stationary state, and the negative branch collapses out of it. The

positive branch therefore corresponds to an expanding universe evolving out of a bounce, and the negative universe to an initially expanding universe with a maximum of expansion and subsequent collapse. Figure 6 shows the evolution of the two branches of this solution.

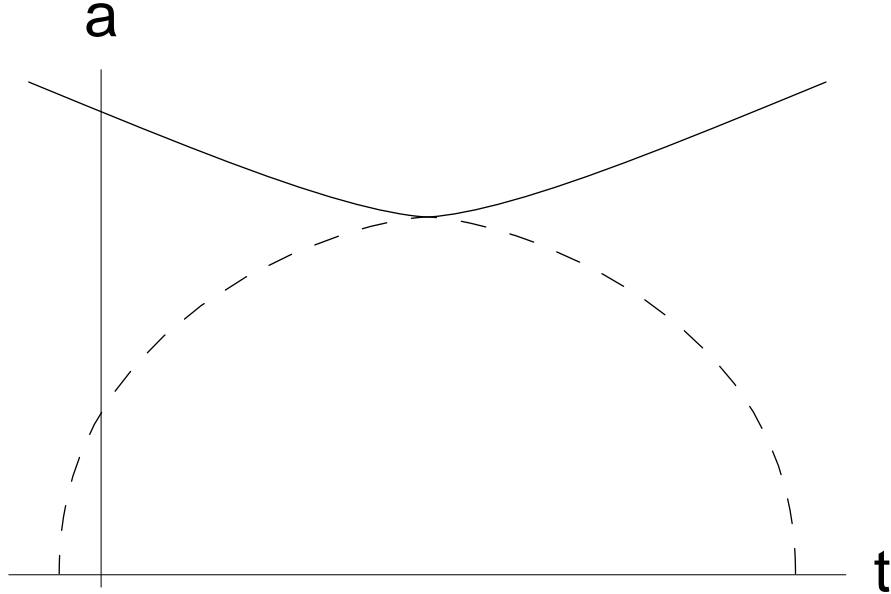


Figure 6: The evolution of the scale factor of solution (17) in terms of proper time,  $t$ . The solid lines correspond to the positive branch, and the dashed line to the negative branch.

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