

Poisson approximation of counts of subgraphs in random intersection graphs.

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Abstract

Random intersection graphs are characterized by three parameters: n , m and p , where n is the number of vertices, m is the number of objects, and p is the probability that a given object is associated with a given vertex. Two vertices in a random intersection graph are adjacent if and only if they have an associated object in common. When $m = \lfloor n^\alpha \rfloor$ for constant α , we provide a condition, called *strictly α -balanced*, for the Poisson convergence of the number of induced copies of a fixed subgraph.

1. Introduction

The random intersection graph $\mathcal{G}(n, m, p)$ is a probability distribution on labelled graphs. The set of vertices of the random intersection graph \mathcal{V} is of size $|\mathcal{V}| = n$ and a second set \mathcal{W} of size $|\mathcal{W}| = m$, called the set of *objects*, is used to determine the adjacencies in the graph. Each vertex $v \in \mathcal{V}$ is associated with a set of objects $\mathcal{W}_v \subseteq \mathcal{W}$ and two vertices $v_1, v_2 \in \mathcal{V}$ are adjacent if and only if $\mathcal{W}_{v_1} \cap \mathcal{W}_{v_2} \neq \emptyset$. The randomness in the graph comes by setting $\mathbb{P}\{w \in \mathcal{W}_v\} = p$ independently for all $w \in \mathcal{W}$, $v \in \mathcal{V}$. The preceding description characterizes the random intersection graph denoted by $\mathcal{G}(n, m, p)$. The model $\mathcal{G}(n, m, p)$ was introduced in [6].

Let H_0 be a given graph on $h \geq 2$ vertices and with at least one edge and let $K_{\mathcal{V}}$ denote the complete graph on the vertex set \mathcal{V} . Let \mathcal{H}_0 denote the set of subgraphs of $K_{\mathcal{V}}$ isomorphic to H_0 . A copy $H \in \mathcal{H}_0$ is *induced* in $\mathcal{G}(n, m, p)$ if all of its edges are edges in $\mathcal{G}(n, m, p)$ and none of its non-edges are edges in $\mathcal{G}(n, m, p)$. In this paper we find conditions on H_0 , n , m and p which imply that the number of induced copies of H_0 in $\mathcal{G}(n, m, p)$ has an approximately Poisson distribution.

Poisson approximation for the number of induced copies of subgraphs has already been studied in detail for the Erdős-Rényi model of random graphs $\mathcal{G}(n, \hat{p})$, in which edges between n vertices appear independently and with identical probability \hat{p} ; see Chapter 6 of [4]. Let $e = |E(H_0)|$

We denote the number of induced copies of H_0 in $\mathcal{G}(n, m, p)$ by $X = X(H_0)$. In order to facilitate our Poisson approximation of the distribution of X , we will express X as a sum of indicator random variables. Given an integer N , define the set $[N]$ to be $[N] = \{1, \dots, N\}$ and define $\text{aut}(H_0)$ to be the set of

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automorphisms of H_0 . The number of subgraphs of K_V isomorphic to H_0 is

$$N_n := |\mathcal{H}_0| = \binom{n}{h} \frac{h!}{|\text{aut}(H_0)|}$$

and we may index the subgraphs in \mathcal{H}_0 by

$$\mathcal{H}_0 = \{H_i : i \in [N_n]\}.$$

We decompose X as

$$X = \sum_{i \in [N_n]} X_i, \tag{1}$$

where X_i is the indicator random variable of the event $\{H_i \text{ is induced in } \mathcal{G}(n, m, p)\}$. The intention is that the X_i 's should be approximately independent and therefore X should approach a Poisson distribution as $n \rightarrow \infty$ for appropriate choices of m and p .

The total variation distance between a random variable taking nonnegative integer values and a random variable P_λ with the Poisson distribution with parameter λ is defined to be

$$d_{TV}(X, P_\lambda) = \frac{1}{2} \sum_{k=0}^{\infty} |\mathbb{P}\{X = k\} - e^{-\lambda} \lambda^k / k!|.$$

As was done in [6, 8], we parametrise $m = m(n)$ by

$$m = \lfloor n^\alpha \rfloor \tag{2}$$

for some constant $\alpha > 0$. Our method of proof will be to apply Stein's method to show that $d_{TV}(X, P_\lambda) \rightarrow 0$ as $n \rightarrow \infty$ under suitable conditions.

Poisson approximation for the number of induced copies of subgraphs has already been studied in detail for the Erdős-Rényi model of random graphs, in which edges appear independently and with identical probability; \hat{p} ; see Chapter 6 of [4]. Let H_0 be a graph with e edges and h vertices. Given $S \subseteq V(H_0)$, we define $E(S)$ to be the set of edges of H_0 having both vertices in S . A graph H_0 is called *strictly balanced* if

$$\max_{\emptyset \subsetneq S \subsetneq V(H_0)} \frac{|E(S)|}{|S|} < \frac{e}{h}. \tag{3}$$

Let W denote the number of not necessarily induced copies of H_0 in $\mathcal{G}(n, \hat{p})$ and let

$$\lambda = \mathbb{E}(W) = \binom{n}{h} \frac{h!}{|\text{aut}(H_0)|} \hat{p}^e.$$

Define $\kappa = \kappa(H_0)$ by

$$\kappa = \min_{\emptyset \subsetneq S \subsetneq V(H_0)} |E(S)| \left(\frac{|S|}{|E(S)|} - \frac{h}{e} \right).$$

Bollobás [2] shows Poisson convergence of W through the method of moments. Theorem 5.B of [1] gives the bound

$$d_{TV}(W, P_\lambda) = \begin{cases} O(1) \lambda^{1-1/e} n^{-\kappa} & \text{if } \lambda \geq 1; \\ O(1) \lambda n^{-\kappa} & \text{if } \lambda < 1. \end{cases} \tag{4}$$

When \hat{p} is such that $\lambda \rightarrow \lambda_0$ for a constant λ_0 , then (4) implies that the distribution of W converges in total variation distance to a $\text{Poisson}(\lambda_0)$ distribution. That is not the case for subgraphs which are not strictly balanced.

The only subgraphs for which the asymptotic distribution of $X(H_0)$ has been determined for $\mathcal{G}(n, m, p)$ are $H_0 = K_h$, the complete graphs on h vertices, in [8], in which $X(K_h)$ was shown to have a limiting Poisson distribution at the threshold for the appearance of K_h . Theorem 1 from [8] for complete graphs is the kind of result we have in mind to extend to general H_0 . For a constant $c > 0$, we parametrise $p = p(n)$ by

$$p(n) \sim \begin{cases} c n^{-1} m^{-\frac{1}{h}} & \text{for } 0 < \alpha < \frac{2h}{h-1}; \\ c n^{-\frac{h+1}{h-1}} & \text{for } \alpha = \frac{2h}{h-1}; \\ c n^{-\frac{1}{h-1}} m^{-\frac{1}{2}} & \text{for } \alpha > \frac{2h}{h-1}. \end{cases} \quad (5)$$

We focus on asymptotic values thus we will use standard Landau notation $O(\cdot)$, $o(\cdot)$, $\Omega(\cdot)$, \sim , and \asymp as in [4]. The following theorem was proved in [8].

Theorem 1. *Let $\mathcal{G}(n, m, p)$ be a random intersection graph defined with m and p given in terms of n by (2) and (5) and let $h \geq 3$ be a fixed integer. Let X_n be the random variable counting the number of instances of K_h in $\mathcal{G}(n, m, p)$.*

(i) *If $\alpha < \frac{2h}{h-1}$, then $\lambda_n = \mathbb{E}X_n \sim c^h/h!$ and*

$$d_{TV}(X_n, P_{\lambda_n}) = O(n^{-\frac{\alpha}{h}});$$

(ii) *If $\alpha = \frac{2h}{h-1}$, then $\lambda_n = \mathbb{E}X_n \sim (c^h + c^{h(h-1)})/h!$ and*

$$d_{TV}(X_n, P_{\lambda_n}) = O(n^{-\frac{2}{h-1}});$$

(iii) *If $\alpha > \frac{2h}{h-1}$, then for $\lambda_n = \mathbb{E}X_n \sim c^{h(h-1)}/h!$ and*

$$d_{TV}(X_n, P_{\lambda_n}) = O\left(n^{\left(h - \frac{\alpha(h-1)}{2} - \frac{2}{h-1}\right)} + n^{-1}\right).$$

The different cases in Theorem 1 arise from the ways copies of K_h can appear in $\mathcal{G}(n, m, p)$. The main ways are either that a single object is responsible for the existence of every edge in the clique or that each edge appears because of a different object associated with it. There are other ways in which copies of K_h can appear, but asymptotically they are unimportant. It may happen that one of the two main ways is dominant or that both ways contribute. In case (i), the first way dominates; in case (iii), the second way dominates; and in case (ii), both cases contribute.

The ways copies can appear in K_h are described in [6] by using the notion of clique covers, to be defined in Section 2. Using clique covers, [6] proves a theorem for the model $\mathcal{G}(n, m, p)$ showing how to compute the threshold for the appearance of subgraphs of $\mathcal{G}(n, m, p)$. We will extend the idea behind the definition of strictly balanced graphs to clique covers and thereby derive a Poisson approximation result for subgraph counts in $\mathcal{G}(n, m, p)$. In Section 2 we define strictly balanced clique covers and state Theorem 2, our main result.

2. Strictly balanced clique covers

The idea of categorising the various ways a subgraph can appear was formalised in [6] through the notion of clique covers. The following definitions are taken from [6]. Given a fixed subgraph H_0 of $K_{\mathcal{V}}$, define $V(H_0)$ and $E(H_0)$ to be the vertex and edge sets of H_0 , respectively.

Definition 1. A *clique cover* $\mathbf{C} = \{C_1, \dots, C_t\}$ of H_0 is a set of non-empty subsets of $V(H_0)$ such that

- (i) each $C_i \in \mathbf{C}$ induces a clique in H_0 ;
- (ii) for any $\{v_1, v_2\} \in E(H_0)$ there exists $C_i \in \mathbf{C}$ such that $v_1, v_2 \in C_i$.

If in addition

- (iii) $|C_i| \geq 2$ for all $C_i \in \mathbf{C}$

we call a clique cover *proper*.

We call t the *size* of the clique cover. By the definition of clique cover, the cliques induced in $K_{\mathcal{V}}$ by the sets in \mathbf{C} cover all the edges of H_0 and no other edges. There are clearly only a finite number of clique covers of H_0 . We denote the finite set of *proper* clique covers of H_0 by $\mathcal{C}(H_0)$.

If $w \in \mathcal{W}_v$, then we say that w has been *chosen* by v . In $\mathcal{G}(n, m, p)$, the set of vertices which have chosen a particular object $w \in \mathcal{W}$ always form a clique in $\mathcal{G}(n, m, p)$ and, therefore, the set of edges in $\mathcal{G}(n, m, p)$ is the union of the m edge-sets of the cliques generated by the elements of \mathcal{W} .

Definition 2. We say that $H_0 \subseteq \mathcal{G}(n, m, p)$ is *induced* by a clique cover $\mathbf{C} = \{C_1, \dots, C_t\}$ of H_0 if there is a family of disjoint non-empty subsets $\{W_1, \dots, W_t\}$ of \mathcal{W} , such that,

- (i) for all $i \in [t]$, each element of W_i is an object chosen by all the vertices of C_i and no other vertices from $V(H_0)$;
- (ii) each $w \in \mathcal{W} \setminus \bigcup_{i=1}^t W_i$ is chosen by at most one vertex from $V(H_0)$.

Clearly, if H_0 is an induced subgraph of $\mathcal{G}(n, m, p)$, then it is induced by exactly one clique cover from $\mathcal{C}(H_0)$ in $\mathcal{G}(n, m, p)$.

Letting $\mathbf{C} = \{C_1, \dots, C_t\}$ be any clique cover of H_0 (not necessarily in $\mathcal{C}(H_0)$), we denote: $|\mathbf{C}| = t$ and $\sum \mathbf{C} = \sum_{i=1}^t |C_i|$. We define the clique cover derived from \mathbf{C} containing only cliques of size at least two by $\mathbf{C}' = \{C_i : |C_i| \geq 2, i \in [t]\}$. Given $\emptyset \subsetneq S \subsetneq V(H)$, we define two different types of *restricted clique covers*, which are multisets defined by

$$\mathbf{C}[S] := \{C_i \cap S : |C_i \cap S| \geq 1, i \in [t]\}$$

and

$$\mathbf{C}'[S] := \{C_i \cap S : |C_i \cap S| \geq 2, i \in [t]\}.$$

Let $H_0[S]$ be the subgraph of H_0 induced by S . We say that restricted clique cover $\mathbf{C}[S]$ (respectively $\mathbf{C}'[S]$) induces $H_0[S]$ if Definition 2 is satisfied with H_0 replaced by $H_0[S]$ and the set \mathbf{C} replaced by the multiset $\mathbf{C}[S]$ (respectively $\mathbf{C}'[S]$). If \mathbf{C} induces H_0 , then $\mathbf{C}[S]$ and $\mathbf{C}'[S]$ induce $H_0[S]$. The subset S of vertices plays a similar role here as it does in definition (3) of strictly balanced subgraphs. The restricted clique covers are defined to be multisets because, if $C_i \cap S = C_j \cap S$ for $i \neq j$, and if H_0 is induced by \mathbf{C} , then vertices from $C_i \cap S = C_j \cap S$ must still choose objects from disjoint non-empty subsets W_i, W_j . Moreover, if $|C_i \cap S| = 1$, then there must still be an object chosen by the single vertex in $C_i \cap S$ in order for \mathbf{C} to induce H_0 . This explains why restricted cliques of size 1 are included in the definition of $\mathbf{C}[S]$. It is shown

in [6] that the order of the expected number of copies of $H_0[S]$ induced by $\mathbf{C}[S]$ does not depend on the number of restricted cliques of size 1 when mp is bounded below, because in that case the probability that at least one object will choose any given vertex is bounded below. Thus, \mathbf{C} is important when $mp = o(1)$ and \mathbf{C}' is important when mp is bounded below.

Define the sizes of the multisets $\mathbf{C}[S]$, $\mathbf{C}'[S]$ by

$$\sum \mathbf{C}[S] = \sum_{\substack{i \in [t] \\ |C_i \cap S| \geq 1}} |C_i \cap S|$$

and

$$\sum \mathbf{C}'[S] = \sum_{\substack{i \in [t] \\ |C_i \cap S| \geq 2}} |C_i \cap S|.$$

Let $X(H_0, \mathbf{C}, S)$ denote the number of copies of $H_0[S]$ induced by \mathbf{C} and \mathbf{C}' . It is shown in [6] that, assuming $mp^2 = o(1)$,

$$\mathbb{E}(X(H_0, \mathbf{C}, S)) \asymp \psi(H_0, \mathbf{C}, S) := \min \left\{ n^{|\mathbf{C}[S]| + \alpha |\mathbf{C}'[S]|} p^{\sum \mathbf{C}[S]}, n^{|\mathbf{C}'[S]| + \alpha |\mathbf{C}[S]|} p^{\sum \mathbf{C}'[S]} \right\}.$$

We are interested in $p = p(n)$ such that $\mathbb{E}(X(H_0, \mathbf{C}, S)) \asymp 1$. For this purpose define

$$\eta_2(H_0, \mathbf{C}, S) := \begin{cases} \frac{|\mathbf{C}[S]| + \alpha |\mathbf{C}'[S]|}{\sum \mathbf{C}[S]} & \text{if either } \alpha < \frac{|S|}{\sum \mathbf{C}[S] - |\mathbf{C}[S]|} \text{ or } \sum \mathbf{C}[S] = |\mathbf{C}[S]|; \\ \frac{|\mathbf{C}'[S]| + \alpha |\mathbf{C}[S]|}{\sum \mathbf{C}'[S]} & \text{otherwise,} \end{cases} \quad (6)$$

so that $\psi(H_0, \mathbf{C}, S) = 1$ when $p = n^{-\eta_2(H_0, \mathbf{C}, S)}$, and define

$$\eta_1(H_0, \mathbf{C}) := \min_{\emptyset \subsetneq S \subseteq V(H_0)} \eta_2(H_0, \mathbf{C}, S). \quad (7)$$

The previous comments indicate that $p = n^{-\eta_1(H_0, \mathbf{C})}$ should be a threshold for \mathbf{C} inducing copies of H_0 . In other words, if $p \ll n^{-\eta_1(H_0, \mathbf{C})}$, then \mathbf{C} will induce 0 copies of H_0 almost surely (as $n \rightarrow \infty$), but, if $p \asymp n^{-\eta_1(H_0, \mathbf{C})}$, then \mathbf{C} will induce some copies of H_0 with positive probability. Thus, if we define

$$\eta_0 = \eta_0(H_0) := \max_{\mathbf{C} \in \mathcal{C}(H_0)} \eta_1(H_0, \mathbf{C}), \quad (8)$$

then it is natural to expect that under suitable conditions $p = n^{-\eta_0(H_0)}$ should be the threshold for the appearance of H_0 in $\mathcal{G}(n, m, p)$. The main result of [6], which applies for general m , not just m of the form (2), shows that $n^{-\eta_0(H_0)}$ actually is the threshold for the appearance of H_0 in $\mathcal{G}(n, m, p)$.

We now proceed with our results for Poisson approximation. We call a clique cover $\mathbf{C} \in \mathcal{C}(H_0)$ *strictly α -balanced* if $\eta_2(H_0, \mathbf{C}, S) > \eta_2(H_0, \mathbf{C}, V(H_0))$ for all $\emptyset \subsetneq S \subsetneq V(H_0)$. The clique covers which will induce copies of H_0 at threshold $p = n^{-\eta_0(H_0)}$ are those in the set

$$\mathcal{C}_0 = \mathcal{C}_0(H_0) := \{\mathbf{C} \in \mathcal{C}(H_0) : \eta_1(H_0, \mathbf{C}) = \eta_0\}. \quad (9)$$

We call H_0 strictly α -balanced if all $\mathbf{C} \in \mathcal{C}_0$ are strictly α -balanced.

Our main result, Theorem 2, gives new conditions for the Poisson convergence of $X(H_0)$ at the threshold of appearance of H_0 . Theorem 2 can be applied, for example, when H_0 is a K_h , a C_h (cycle), or triangle-free.

Theorem 2. *Let H_0 be a given graph, $m = \lfloor n^\alpha \rfloor$, for $\alpha > 0$, η_0 be given by (6), (7) and (8), and \mathcal{C}_0 be defined by (9). Suppose that $p = cn^{-\eta_0}$ for a constant $c > 0$ and that $mp^2 = o(1)$. If H_0 is strictly α -balanced, then*

$$d_{TV}(X, P_{\lambda_0}) = o(1)$$

for

$$\lambda_0 = \frac{1}{|\text{aut}(H_0)|} \sum_{\mathbf{C} \in \mathcal{C}_0(H_0)} c^{\sum \mathbf{C}},$$

where X is the number of induced copies of H_0 in $\mathcal{G}(n, m, p)$.

The meaning of λ_0 is that it is the limit of the number of copies of H_0 induced by clique covers in \mathcal{C}_0 . We do not know if it is possible that $\mathbb{E}(X) \not\rightarrow \lambda_0$.

Some of our results might be obtained using reasoning from [5] to the random bipartite graph with bipartition $(\mathcal{V}, \mathcal{W})$, however, due to double counting of ways subgraphs can be generated, it would seem to require great effort. Moreover, the treatment of the problem in this article might be useful in the developing arguments for other problems concerning subgraph counts in $\mathcal{G}(n, m, p)$.

To illustrate Theorem 2, we apply it to a triangle-free H_0 and compare the bounds obtained to related results for $\mathcal{G}(n, \hat{p})$ with $\hat{p} = mp^2$. We obtain the following bound:

Corollary 1. *Let H_0 be strictly balanced triangle-free graph on h vertices and with e edges. If $\alpha > h/e$ then for $p = cn^{\frac{h+\alpha e}{2e}}$ we have*

$$d_{TV}(X, P_{\lambda_0}) = o(1), \tag{10}$$

for $\lambda_0 = \frac{c^{2e}}{|\text{aut}(H_0)|}$, where X is the number of induced copies of H_0 .

For the proof of Corollary 1 see the Appendix.

The bound (10) gives Poisson convergence analagous to (4) for $\mathcal{G}(n, \hat{p})$ with $\hat{p} = mp^2$, but only applies for $\alpha > h/e$. It is typical for results in $\mathcal{G}(n, m, p)$ to show behaviour similar to that in $\mathcal{G}(n, \hat{p})$ for α large enough or small p . A result of [3] and [7] show equivalence in total variation distance between $\mathcal{G}(n, m, p)$ and $G(n, \hat{p})$ for \hat{p} chosen appropriately when $\alpha > 6$ or $p = o(n^{-1}m^{-1/3})$. Note that the results of [7] concerning $\alpha > 3$ do not apply because the event of having an induced subgraph is not an increasing event.

For the case $\alpha \leq h/e$, recall that for a triangle-free graph H_0 , the set $\mathcal{C}(H_0)$ contains only one clique cover \mathbf{C} consisting of 2-element sets. Moreover, for a strictly balanced H_0 , if $\alpha \leq h/e$, then for every $S \subseteq V(H_0)$ such that $H_0[S]$ has at least one edge

$$\frac{|S|}{\sum \mathbf{C}[S] - |\mathbf{C}|} = \frac{|S|}{E(S)} \geq \frac{h}{e} \geq \alpha.$$

Thus $\mathbf{C}[S]$ always contributes to η_2 and, in general, we should not expect similarity with $\mathcal{G}(n, \hat{p})$. However the following special cases show that, depending on the graph the value h/e is not always critical. Theorem 2, Corollary 1, and simple calculations lead to the following.

Example 1. *Let C_t be a cycle on $t \geq 4$ vertices and $p = cn^{-\frac{1}{2} - \frac{\alpha}{2}}$. Then for any $\alpha > 0$ and $\lambda_0 = \frac{c^{2t}}{2t}$ we have $d_{TV}(X(C_k), Po(\lambda_0)) = o(1)$*

Example 2. *Let $K_{k,t}$ be a bipartite graph with $t > k$. Then for $p = cn^{-\frac{k+t}{2kt} - \frac{\alpha}{2}}$ and $\lambda_0 = \frac{c^{2kt}}{k!t!}$ we have $d_{TV}(X(K_{k,t}), Po(\lambda_0)) = o(1)$ for $\alpha > \frac{t-k}{tk}$.*

Results on the asymptotic probabilities of clique covers which give the asymptotics of $\mathbb{E}(X)$ are derived in Section 3. Section 4 contains results regarding the second moment of X . In Section 5, we use Stein's method to prove Theorem 2.

3. Asymptotic probabilities

Lemma 1, which shows that the numbers of objects inducing cliques are asymptotically independent, is similar to Lemma 1 of [6]. One difference between Lemma 1 and Lemma 1 of [6] is that our conditions on A_0 are stated explicitly and another is that we present the proof in full detail. In Lemma 1 and Lemma 2 we allow the possibility that $|C_i| = 1$. Thus $\mathbf{C}[S]$ and $\mathbf{C}'[S]$ are written as \mathbf{C} in those lemmas to simplify notation.

Lemma 1. *Given a clique cover $\mathbf{C} = \{C_1, C_2, \dots, C_r\}$ of H_0 , let $c_{\min} = \min_{1 \leq i \leq r} |C_i|$. Moreover, let N_i be the number of objects, which have been chosen by all vertices from C_i and no vertex from $V(H_0) \setminus C_i$ in $\mathcal{G}(n, m, p)$ and let \tilde{N}_i have Poisson distribution $Po(mp^{|C_i|})$. If $mp^{c_{\min}+1} = o(1)$ then*

$$\mathbb{P} \left\{ \bigcap_{i=1}^r \{N_i = a_i\} \right\} \sim \prod_{i=1}^r \mathbb{P} \left\{ \tilde{N}_i = a_i \right\}$$

uniformly over all $a_i \leq A_0$ for any $A_0 = A_0(n)$ satisfying $A_0 = o(\sqrt{m})$ and $A_0 = o(p^{-c_{\min}})$.

Proof. Define

$$p_i = p^{|C_i|}(1-p)^{h-|C_i|}, \quad 1 \leq i \leq r, \quad (11)$$

which is the probability that a given object is chosen by all vertices in C_i and no other vertices in $V(H_0)$. Note that $p_i \leq p^{c_{\min}}$. Let $p_0 = 1 - \sum_{i=1}^r p_i$. Define $a_0 = m - \sum_{i=1}^r a_i$. If $1 \leq a_i \leq A_0$ for all $1 \leq i \leq r$, then

$$\begin{aligned} \mathbb{P} \left\{ \bigcap_{i=1}^r \{N_i = a_i\} \right\} &= \binom{m}{a_0, \dots, a_r} p_0^{a_0} \prod_{i=1}^r p_i^{a_i} \\ &= \frac{m!}{(m - \sum_{i=1}^r a_i)!} \left(1 - \sum_{i=1}^r p_i\right)^{m - \sum_{i=1}^r a_i} \prod_{i=1}^r \frac{p_i^{a_i}}{a_i!} \\ &= m^{\sum_{i=1}^r a_i} \exp\left(O\left(\frac{(rA_0)^2}{m}\right)\right) \exp\left(-m \sum_{i=1}^r p_i + O(r^2 A_0 p^{c_{\min}} + mr^2 p^{2c_{\min}})\right) \prod_{i=1}^r \frac{p_i^{a_i}}{a_i!} \\ &\sim \exp\left(-m \sum_{i=1}^r p_i\right) \prod_{i=1}^r \frac{(mp_i)^{a_i}}{a_i!} \\ &= \exp\left(-\sum_{i=1}^r mp^{|C_i|} + O(mp^{c_{\min}+1})\right) \prod_{i=1}^r \frac{(mp^{|C_i|})^{a_i}}{a_i!} (1-p_i)^{a_i(h-|C_i|)} \\ &\sim \prod_{i=1}^r e^{-mp^{|C_i|}} \frac{(mp^{|C_i|})^{a_i}}{a_i!} \exp(O(hA_0 p^{c_{\min}})) \\ &\sim \prod_{i=1}^r e^{-mp^{|C_i|}} \frac{(mp^{|C_i|})^{a_i}}{a_i!} = \prod_{i=1}^r \mathbb{P} \left\{ \tilde{N}_i = a_i \right\}. \end{aligned}$$

□

We let $\pi(H_0, \mathbf{C})$ denote the probability that H_0 is induced by clique cover \mathbf{C} . Using this definition, the fact that $H_i \in \mathcal{H}_0$ can be induced by at most one clique cover from $\mathcal{C}(H_0)$, and symmetry, we calculate that,

for each $i \in [N_n]$, the expectation of the random variable X_i appearing in (1) equals

$$\mathbb{E}(X_i) = \sum_{\mathbf{C} \in \mathcal{C}(H_i)} \pi(H_i, \mathbf{C}) = \sum_{\mathbf{C} \in \mathcal{C}(H_0)} \pi(H_0, \mathbf{C}).$$

In (12) below, we give asymptotics for $\pi(H_0, \mathbf{C})$. The similar results on pages 138–139 of [6] only provide asymptotics when $mp = o(1)$ and \mathbf{C} has no cliques of size 1. It is indeed the case that $mp \rightarrow \infty$ in the examples after Corollary 1 when $\alpha > 1$. The order estimate (13) was obtained in [6]. Recall that if $mp^2 = o(n^{-2})$ then with probability tending to one as $n \rightarrow \infty$ $\mathcal{G}(n, m, p)$ is edgeless and if $\ln n = o(mp^2)$, then with probability tending to one $\mathcal{G}(n, m, p)$ is the complete graph (see [3]).

Lemma 2. *Given a clique cover $\mathbf{C} = \{C_1, C_2, \dots, C_r\}$ of H_0 , let $\mathcal{I}_1 = \{1 \leq i \leq r : |C_i| = 1\}$ and $\mathcal{I}_2 = \{1 \leq i \leq r : |C_i| \geq 2\}$. Then,*

$$\pi(H_0, \mathbf{C}) \sim (1 - e^{-mp})^{|\mathcal{I}_1|} \prod_{i \in \mathcal{I}_2} mp^{|C_i|}, \quad \text{for } \Omega(n^{-1}) = mp^2 = o(1). \quad (12)$$

It follows that

$$\pi(H_0, \mathbf{C}) \asymp \min \left\{ m^{|\mathcal{C}|} p^{\Sigma \mathbf{C}}, m^{|\mathcal{C}'|} p^{\Sigma \mathbf{C}'} \right\}. \quad (13)$$

Proof. Let $(C_{r+1}, C_{r+2}, \dots, C_t)$ list the subsets of $V(H_0)$ not in \mathbf{C} of cardinality of at least 2. For each $1 \leq i \leq t$, let N_i be the number of objects which have been chosen by every vertex in C_i , and by no vertex in $V(H) \setminus C_i$. Let \tilde{N}_i be a random variable with the Poisson distribution $\text{Po}(mp^{|C_i|})$. The distribution of N_i is Binomial(m, p_i), where p_i is defined by (11). Chernoff's bound (see for example Theorem 2.1 [4]) implies that for A large enough $\mathbb{P}\{N_i \geq A_0\} = o(\pi(H_0, \mathbf{C}))$ and $\mathbb{P}\{\tilde{N}_i \geq A_0\} = o(\pi(H_0, \mathbf{C}))$, where $A_0(n) = A \max\{mp, \log n\}$. Therefore,

$$\begin{aligned} \pi(H_0, \mathbf{C}) &= \mathbb{P} \left\{ \bigcap_{i=1}^r \{N_i \geq 1\} \cap \bigcap_{j=r+1}^t \{N_j = 0\} \right\} \\ &= \sum_{a_i \geq 1 \text{ for } 1 \leq i \leq r} \mathbb{P} \left\{ \bigcap_{i=1}^r \{N_i = a_i\} \cap \bigcap_{j=r+1}^t \{N_j = 0\} \right\} \\ &= \sum_{1 \leq a_i \leq A_0 \text{ for } 1 \leq i \leq r} \mathbb{P} \left\{ \bigcap_{i=1}^r \{N_i = a_i\} \cap \bigcap_{j=r+1}^t \{N_j = 0\} \right\} + o(\pi(H_0, \mathbf{C})), \end{aligned}$$

We know that $c_{\min} \geq 1$, where c_{\min} is defined as in Lemma 1, and $mp^{c_{\min}+1} = O(mp^2) = o(1)$. Moreover,

$A_0 = o(\sqrt{m})$ and $A_0 p = o(1)$. By Lemma 1, we now have

$$\begin{aligned}
\pi(H_0, \mathbf{C}) &\sim \sum_{1 \leq a_i \leq A_0 \text{ for } 1 \leq i \leq r} \prod_{i=1}^r \mathbb{P}\{\tilde{N}_i = a_i\} \prod_{j=r+1}^t \mathbb{P}\{\tilde{N}_j = 0\} \\
&\sim \prod_{i=1}^r \mathbb{P}\{1 \leq \tilde{N}_i \leq A_0\} \prod_{j=r+1}^t \mathbb{P}\{\tilde{N}_j = 0\} \\
&\sim \prod_{i=1}^r \mathbb{P}\{\tilde{N}_i \geq 1\} \prod_{j=r+1}^t \mathbb{P}\{\tilde{N}_j = 0\} \\
&= \prod_{i=1}^r (1 - \exp(-mp^{|C_i|})) \prod_{j=r+1}^t \exp(-mp^{|C_j|}) \\
&\sim \prod_{i=1}^r (1 - \exp(-mp^{|C_i|})) \\
&\sim \prod_{i \in \mathcal{I}_1} (1 - e^{-mp}) \prod_{i \in \mathcal{I}_2} mp^{|C_i|},
\end{aligned}$$

proving (12).

To show (13), we observe that if $mp \geq 1$ then $1 - e^{-mp} \asymp 1$ and if $mp < 1$ then $1 - e^{-mp} \asymp mp$. Thus $1 - e^{-mp} \asymp \min\{mp, 1\}$. \square

Remark 1. Similar techniques lead to the following equation which might be useful in the study of subgraph counts above the threshold for subgraph appearance. For any clique cover $\mathbf{C} = \{C_1, C_2, \dots, C_r\}$ of H_0 , if $\Omega(1) = mp^2 = O(\log n)$ then

$$\pi(H_0, \mathbf{C}) \sim (1 - e^{-mp^2})^{|\mathcal{I}_2|} (e^{-mp^2})^{\binom{h}{2} - |\mathcal{I}_2|} \prod_{i \in \mathcal{I}_3} mp^{|C_i|},$$

where $\mathcal{I}_2 = \{1 \leq i \leq r : |C_i| = 2\}$ and $\mathcal{I}_3 = \{1 \leq i \leq r : |C_i| \geq 3\}$.

4. Asymptotic moments

We now define

$$\omega(H_0, \mathbf{C}) = \min_{\emptyset \subsetneq S \subsetneq V(H_0)} \psi(H_0, \mathbf{C}, S). \quad (14)$$

Lemma 3 is a second moment calculation for copies of H_0 induced by given clique covers $\mathbf{C}_1, \mathbf{C}_2$. A calculation resulting in a special case of Lemma 3 was made in [6].

Lemma 3. *Suppose that $mp^2 = o(1)$. Let G_1 and G_2 be two, not necessarily isomorphic, subgraphs of K_V on $g_1 := |V(G_1)|$ and $g_2 := |V(G_2)|$ labelled vertices, respectively, which intersect on $\ell := |V(G_1) \cap V(G_2)|$ vertices and such that $G_1 \cap G_2$ is an induced subgraph of both of G_1 and G_2 . Let \mathbf{C}_1 and let \mathbf{C}_2 be proper clique covers of G_1 and G_2 , respectively. Define $X(G_i, \mathbf{C}_i)$ to be the indicator random variable of the event that G_i is induced by a clique cover \mathbf{C}_i in $\mathcal{G}(n, m, p)$. Then, for $\mathcal{G}(n, m, p)$,*

$$\mathbb{E}(X(G_1, \mathbf{C}_1)X(G_2, \mathbf{C}_2)) = O(1)\mathbb{E}(X(G_1, \mathbf{C}_1))\mathbb{E}(X(G_2, \mathbf{C}_2)) \frac{n^\ell}{\omega(G_2, \mathbf{C}_2)}. \quad (15)$$

Proof. Suppose that $\mathbf{C}_1 = \{C_{1,1}, C_{1,2}, \dots, C_{1,r}\}$ and $\mathbf{C}_2 = \{C_{2,1}, C_{2,2}, \dots, C_{2,s}\}$. Let $\mathbf{C}_1 + \mathbf{C}_2$ denote the set of clique covers on $V(G_1 \cup G_2)$ such that $\mathbf{C} \in \mathbf{C}_1 + \mathbf{C}_2$ implies $\mathbf{C}[V(G_1)] = \mathbf{C}_1$ and $\mathbf{C}[V(G_2)] = \mathbf{C}_2$. If $\mathbf{C} \in \mathbf{C}_1 + \mathbf{C}_2$ and $C \in \mathbf{C}$, then C must be one of the following three forms:

- (i) $C = C_{1,i}$ for some $1 \leq i \leq r$ but $\forall_{1 \leq j \leq s} (C \neq C_{2,j})$;
- (ii) $C = C_{2,j}$ for some $1 \leq j \leq s$ but $\forall_{1 \leq i \leq r} (C \neq C_{1,i})$;
- (iii) $C = C_{1,i} \cup C_{2,j}$ for some $1 \leq i \leq r, 1 \leq j \leq s$ (including the case $C = C_{1,i} = C_{2,j}$).

Given $\mathbf{C} \in \mathbf{C}_1 + \mathbf{C}_2$, let $\mathcal{J}_1 = \mathcal{J}_1(\mathbf{C}) \subseteq [r]$, $\mathcal{J}_2 = \mathcal{J}_2(\mathbf{C}) \subseteq [s]$ and $\mathcal{J}_3 = \mathcal{J}_3(\mathbf{C}) \subseteq [r] \times [s]$ be the sets of indices for which (i), (ii) and (iii), respectively, are true for some $C \in \mathbf{C}$. Define

$$\mathcal{J}_4 = \{i \in [r] : \exists j \in [s] \text{ such that } (i, j) \in \mathcal{J}_3\}$$

and

$$\mathcal{J}_5 = \{j \in [s] : \exists i \in [r] \text{ such that } (i, j) \in \mathcal{J}_3\}$$

As $\mathbf{C}[V(G_1)] = \mathbf{C}_1$ and $\mathbf{C}[V(G_2)] = \mathbf{C}_2$, it must be true that

$$\mathcal{J}_1 \cup \mathcal{J}_4 = [r] \text{ and } \mathcal{J}_2 \cup \mathcal{J}_5 = [s]. \quad (16)$$

Note that the clique covers in $\mathbf{C}_1 + \mathbf{C}_2$ are proper because \mathbf{C}_1 and \mathbf{C}_2 are proper. Therefore, if G_1 is induced by \mathbf{C}_1 on $V(G_1)$ and G_2 is induced by \mathbf{C}_2 on $V(G_2)$, then $G_1 \cup G_2$ is induced by a unique element of $\mathbf{C}_1 + \mathbf{C}_2$ and

$$\mathbb{E}X(G_1, \mathbf{C}_1)X(G_2, \mathbf{C}_2) = \sum_{\mathbf{C} \in \mathbf{C}_1 + \mathbf{C}_2} \mathbb{E}X(G_{\mathbf{C}}, \mathbf{C}).$$

Moreover, for any $\mathbf{C} \in \mathbf{C}_1 + \mathbf{C}_2$, Lemma 2 implies

$$\begin{aligned} \mathbb{E}X(G_{\mathbf{C}}, \mathbf{C}) &\sim \prod_{\substack{i \in \mathcal{J}_1 \\ |C_{1,i}| > 1}} mp^{|C_{1,i}|} \prod_{\substack{i \in \mathcal{J}_1 \\ |C_{1,i}| = 1}} (1 - e^{-mp}) \prod_{\substack{i \in \mathcal{J}_1 \\ |C_{2,i}| > 1}} mp^{|C_{2,i}|} \prod_{\substack{i \in \mathcal{J}_1 \\ |C_{2,i}| = 1}} (1 - e^{-mp}) \\ &\times \prod_{\substack{(k,l) \in \mathcal{J}_3 \\ |C_{1,k} \cup C_{2,l}| > 1}} mp^{|C_{1,k} \cup C_{2,l}|} \prod_{\substack{(k,l) \in \mathcal{J}_3 \\ |C_{1,k} \cup C_{2,l}| = 1}} (1 - e^{-mp}). \end{aligned} \quad (17)$$

We will analyze (17) separately for the cases $mp \leq 1$ and $mp > 1$.

When $mp \leq 1$, we have $1 - e^{-mp} \leq mp$, and so

$$\mathbb{E}X(G_{\mathbf{C}}, \mathbf{C}) = O(1) \prod_{i \in \mathcal{J}_1} mp^{|C_{1,i}|} \prod_{j \in \mathcal{J}_2} mp^{|C_{2,j}|} \prod_{(k,l) \in \mathcal{J}_3} mp^{|C_{1,k} \cup C_{2,l}|}.$$

Now, (16) and $mp \leq e(1 - e^{-mp})$ give

$$\begin{aligned} \mathbb{E}X(G_{\mathbf{C}}, \mathbf{C}) &= O(1) \prod_{i \in \mathcal{J}_1} mp^{|C_{1,i}|} \prod_{j \in \mathcal{J}_2} mp^{|C_{2,j}|} \prod_{(k,l) \in \mathcal{J}_3} \frac{mp^{|C_{1,k}|} mp^{|C_{2,l}|}}{mp^{|C_{1,k} \cap C_{2,l}|}} \\ &= O(1) \prod_{i \in [r]} mp^{|C_{1,i}|} \prod_{j \in [s]} mp^{|C_{2,j}|} \prod_{(k,l) \in \mathcal{J}_3} \frac{1}{mp^{|C_{1,k} \cap C_{2,l}|}}. \end{aligned}$$

It must be the case that $(C_{1,k} \cup C_{2,l}) \cap V(G_1) = C_{1,k}$ and that $(C_{1,k} \cup C_{2,l}) \cap V(G_2) = C_{2,l}$, which implies that $C_{2,l} \cap (V(G_1) \cap V(G_2)) = C_{1,k} \cap C_{2,l}$. Therefore, by definition (14),

$$\prod_{(k,l) \in \mathcal{J}_3} mp^{|C_{1,k} \cap C_{2,l}|} \geq \prod_{C \in \mathcal{C}_2[V(G_1) \cap V(G_2)]} mp^{|C|} \geq \frac{\omega(G_2, \mathbf{C}_2)}{n^\ell},$$

proving (15).

In case $mp > 1$, by (17) we have

$$\mathbb{E}X(G_{\mathbf{C}}, \mathbf{C}) = O(1) \prod_{\substack{i \in \mathcal{J}_1 \\ |C_{1,i}| > 1}} mp^{|C_{1,i}|} \prod_{\substack{i \in \mathcal{J}_1 \\ |C_{2,i}| > 1}} mp^{|C_{2,i}|} \prod_{\substack{(k,l) \in \mathcal{J}_3 \\ |C_{1,k} \cup C_{2,l}| > 1}} mp^{|C_{1,k} \cup C_{2,l}|}.$$

We now bound terms of the form $mp^{|C_{1,k} \cup C_{2,l}|}$ in the expression above. If $|C_{1,k}| = |C_{2,l}| = 1$, then $mp^{|C_{1,k} \cup C_{2,l}|} = mp^2 = o(1)$. If $|C_{1,k}| = 1$ and $|C_{2,l}| > 1$, then $mp^{|C_{1,k} \cup C_{2,l}|} \leq mp^{|C_{2,l}|}$. If $|C_{1,k}| > 1$ and $|C_{2,l}| = 1$, then $mp^{|C_{1,k} \cup C_{2,l}|} \leq mp^{|C_{1,k}|}$. If $|C_{1,k}| > 1$ and $|C_{2,l}| > 1$, then $mp^{|C_{1,k} \cup C_{2,l}|} = mp^{|C_{1,k}|} mp^{|C_{2,l}|} / mp^{|C_{1,k} \cap C_{2,l}|}$. Using these bounds results in

$$\begin{aligned} \mathbb{E}X(G_{\mathbf{C}}, \mathbf{C}) &= O(1) \prod_{\substack{i \in [r] \\ |C_{1,i}| > 1}} mp^{|C_{1,i}|} \prod_{\substack{i \in [s] \\ |C_{2,i}| > 1}} mp^{|C_{2,i}|} \prod_{\substack{(k,l) \in \mathcal{J}_3 \\ |C_{1,k}| > 1, |C_{2,l}| > 1}} \frac{1}{mp^{|C_{1,k} \cap C_{2,l}|}} \\ &= O(1) \mathbb{E}(X(G_1, \mathbf{C}_1)) \mathbb{E}X(G_2, \mathbf{C}_2) \prod_{\substack{(k,l) \in \mathcal{J}_3 \\ |C_{1,k}| > 1, |C_{2,l}| > 1}} \frac{1}{mp^{|C_{1,k} \cap C_{2,l}|}}. \end{aligned}$$

Therefore, as in the argument for $mp \leq 1$, we have

$$\prod_{\substack{(k,l) \in \mathcal{J}_3 \\ |C_{1,k}| > 1, |C_{2,l}| > 1}} mp^{|C_{1,k} \cap C_{2,l}|} \geq \prod_{C \in \mathcal{C}'_2[V(G_1) \cap V(G_2)]} mp^{|C|} \geq \frac{\omega(G_2, \mathbf{C}_2)}{n^\ell},$$

resulting in (15) for this case. \square

5. Subgraph counts

In this section we prove Theorem 2. For copies of H_0 induced by clique covers in $\mathcal{C}_0(H_0)$, the proof is an application of Stein's method, using the estimates we have already obtained. We must also show that the number of copies of H_0 induced by clique covers in $\mathcal{C}(H_0) \setminus \mathcal{C}_0(H_0)$ converges to 0 in probability.

Proof. For each $i \in [N_n]$, each $\mathbf{C} \in \mathcal{C}(H_0)$ has a corresponding clique cover in $\mathcal{C}(H_i)$ induced by the isomorphism between H_0 and H_i . We write $X(H_i, \mathbf{C})$ for the indicator random variable that the clique cover in $\mathcal{C}(H_i)$ corresponding to $\mathbf{C} \in \mathcal{C}(H_0)$ induces H_i . Suppose that H_0 is strictly α -balanced and that $p = cn^{-\gamma_0}$ for $c > 0$. Write $X = \sum_{i \in [N_n]} X_i$ as $X = Y_0 + Y_1$, where

$$Y_0 = \sum_{i \in [N_n]} \sum_{\mathbf{C} \in \mathcal{C}_0(H_0)} X(H_i, \mathbf{C}),$$

and

$$Y_1 = \sum_{i \in [N_n]} \sum_{\mathbf{C} \in \mathcal{C}(H_0) \setminus \mathcal{C}_0(H_0)} X(H_i, \mathbf{C}).$$

For each $\mathbf{C} \in \mathcal{C}_0(H_0)$, $\eta_0(H_0) = \eta_1(H_0, \mathbf{C}) = \eta_2(H_0, \mathbf{C}, V(H_0))$ and by Lemma 2,

$$\begin{aligned} \mathbb{E}(Y_0) &\sim \frac{n^h}{|\text{aut}(H_0)|} \sum_{\mathbf{C} \in \mathcal{C}_0(H_0)} \pi(H_0, \mathbf{C}) \\ &\sim \lambda_0 := \frac{1}{|\text{aut}(H_0)|} \sum_{\mathbf{C} \in \mathcal{C}_0(H_0)} c^{\Sigma \mathbf{C}}. \end{aligned}$$

For any $\mathbf{C} \in \mathcal{C}_0$ and $\emptyset \subsetneq S \subsetneq V(H_0)$, $\psi(H_0, \mathbf{C}, S) \rightarrow \infty$ and so $\Phi \rightarrow \infty$, where Φ is defined by

$$\Phi := \Phi(H_0) = \min_{\mathbf{C} \in \mathcal{C}_0} (\omega(H_0, \mathbf{C})). \quad (18)$$

A *dependency graph* L is a graph with vertex set $[N_n] \times \mathcal{C}_0$ having the property that whenever $A \subseteq [N_n] \times \mathcal{C}_0$ and $B \subseteq [N_n] \times \mathcal{C}_0$ satisfy the property that there are no edges between A and B in L , it follows that $\{X(H_i, \mathbf{C}_1) : (i, \mathbf{C}_1) \in A\}$ and $\{X(H_j, \mathbf{C}_2) : (j, \mathbf{C}_2) \in B\}$ are mutually independent sets of random variables. We define a dependency graph L with a vertex set $[N_n] \times \mathcal{C}_0$ and such that for $(i, \mathbf{C}_1), (j, \mathbf{C}_2) \in [N_n] \times \mathcal{C}_0$, $\{(i, \mathbf{C}_1), (j, \mathbf{C}_2)\} \in E(L)$ if and only if $V(H_i) \cap V(H_j) \neq \emptyset$. Since subgraphs with disjoint vertex sets appear independently in $\mathcal{G}(n, m, p)$, this is well defined dependency graph. We have

$$\mathbb{E}(X(H_i, \mathbf{C})) = \pi(H_i, \mathbf{C}).$$

From Theorem 6.23 of [4], we have

$$\begin{aligned} d_{TV}(Y_0, P_{\mathbb{E}(Y_0)}) &\leq \min(\lambda_0^{-1}, 1) \left(\sum_{(i, \mathbf{C}) \in V(L)} \pi(H_i, \mathbf{C})^2 \right. \\ &\quad \left. + \sum_{(i, \mathbf{C}_1)(j, \mathbf{C}_2) \in E(L)} (\pi(H_i, \mathbf{C}_1)\pi(H_j, \mathbf{C}_2) + \mathbb{E}(X(H_i, \mathbf{C}_1)X(H_j, \mathbf{C}_2))) \right), \end{aligned}$$

where the sum $\sum_{(i, \mathbf{C}_1)(j, \mathbf{C}_2) \in E(L)}$ means summing over ordered pairs $((i, \mathbf{C}_1)(j, \mathbf{C}_2))$ such that $\{(i, \mathbf{C}_1)(j, \mathbf{C}_2)\} \in E(L)$. Observe that

$$\sum_{(i, \mathbf{C}) \in V(L)} \pi(H_i, \mathbf{C})^2 = O \left(\sum_{\mathbf{C} \in \mathcal{C}_0} n^h m^{2|\mathbf{C}|} p^{2\Sigma \mathbf{C}} \right) = O(n^{-h}).$$

and that

$$\sum_{(i, \mathbf{C}_1)(j, \mathbf{C}_2) \in E(L)} \pi(H_i, \mathbf{C}_1)\pi(H_j, \mathbf{C}_2) = O \left(n^{2h-1} m^{2|\mathbf{C}|} p^{2\Sigma \mathbf{C}} \right) = O(n^{-1})$$

Note that

$$\mathbb{E}(X(H_i, \mathbf{C}_1)X(H_i, \mathbf{C}_2)) = 0$$

when $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}_0(H_i)$ and $\mathbf{C}_1 \neq \mathbf{C}_2$. By Lemma 3 and (18)

$$\begin{aligned}
& \sum_{(i, \mathbf{C}_1)(j, \mathbf{C}_2) \in E(L)} \mathbb{E}(X(H_i, \mathbf{C}_1)X(H_j, \mathbf{C}_2)) \\
&= \sum_{\substack{i, j \in [N_n] \\ V(H_i) \cap V(H_j) \neq \emptyset}} \sum_{\mathbf{C}_1 \in \mathcal{C}_0(H_i)} \sum_{\mathbf{C}_2 \in \mathcal{C}_0(H_j)} \mathbb{E}(X(H_i, \mathbf{C}_1)X(H_j, \mathbf{C}_2)) \\
&= O(1) \sum_{\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}_0(H_0)} n^h \sum_{l=1}^{h-1} n^{h-l} \frac{\pi(H_0, \mathbf{C}_1)\pi(H_0, \mathbf{C}_2)n^l}{\Phi} \\
&= O(\Phi^{-1}) = o(1).
\end{aligned}$$

This proves that $d_{TV}(Y_0, P_{\mathbb{E}(Y_0)}) = o(1)$. Since, as is well known,

$$d_{TV}(P_{\mathbb{E}(Y_0)}, P_{\lambda_0}) = O(|\mathbb{E}(Y_0) - \lambda_0|) = o(1),$$

we have

$$d_{TV}(Y_0, P_{\lambda_0}) \leq d_{TV}(Y_0, P_{\mathbb{E}(Y_0)}) + d_{TV}(P_{\mathbb{E}(Y_0)}, P_{\lambda_0}) = o(1). \quad (19)$$

If $\mathbf{C} \in \mathcal{C}(H_0) \setminus \mathcal{C}_0(H_0)$, then $\eta_1(H_0, \mathbf{C}) < \eta_0(H_0)$ and it must be the case that there exists $\emptyset \subsetneq S \subseteq V(H_0)$ such that $\eta_2(H_0, \mathbf{C}, S) < \eta_0(H_0)$. For this $S \subset V(H_0)$, let $H_i[S]$ be the subgraph of H_i induced by those vertices of $V(H_i)$ which are the image of S under the isomorphism between H_0 and H_i . For this S and $p = cn^{-\eta_0}$, by definitions of η_0 , η_1 , and η_2 we have

$$\mathbb{P} \left\{ \sum_{i \in [N_n]} X(H_i, \mathbf{C}) > 0 \right\} \leq \mathbb{P} \{ \exists_{i \in [N_n]} X(H_i[S], \mathbf{C}[S]) > 0 \} \leq \psi(H_0, \mathbf{C}, S) = o(1).$$

Therefore, since there is a finite number of clique covers we have $\mathbb{P} \{Y_1 > 0\} = o(1)$. We conclude that

$$\begin{aligned}
d_{TV}(X, P_{\lambda_0}) &\leq d_{TV}(X, Y_0) + d_{TV}(Y_0, P_{\lambda_0}) \\
&\leq \mathbb{P} \{Y_1 > 0\} + d_{TV}(Y_0, P_{\lambda_0}) \\
&= o(1).
\end{aligned}$$

□

We make some further remarks pertaining to the possibility that $\mathbb{E}(X) \neq \lambda_0$. We call a clique cover $\mathbf{C} \in \mathcal{C}_0$ α -unbalanced if $\eta_2(H_0, \mathbf{C}, S) < \eta_1(H_0, \mathbf{C}, V(H_0))$ for some $\emptyset \subsetneq S \subsetneq V(H_0)$. If all $\mathbf{C} \in \mathcal{C}_0$ are strictly α -balanced, then Theorem 2 holds and $\lim_{n \rightarrow \infty} \mathbb{E}(X) = \lambda_0$. If all of the $\mathbf{C} \in \mathcal{C}_0$ are either strictly α -balanced or α -unbalanced, and at least one of them is strictly α -balanced, then Theorem 2 holds with $\lim_{n \rightarrow \infty} \mathbb{E}(X) > \lambda_0$. If all $\mathbf{C} \in \mathcal{C}_0$ are α -unbalanced, then $X \rightarrow 0$ in probability when $p = cn^{-\eta_0}$.

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Appendix

Proof of Corollary 1. Let \mathbf{C} be the sole clique cover of the strictly balanced subgraph H_0 consisting of cliques of size two. Then for any $S \subseteq V(H_0)$ we have $\sum \mathbf{C}'[S] = 2|\mathbf{C}'[S]| = 2|E(S)|$ and $\frac{|S|}{|E(S)|} > \frac{h}{e}$ if $E(S)$ is non-empty. Recall that $\alpha > \frac{h}{e}$.

Therefore, for any $\emptyset \subsetneq S \subsetneq V(H_0)$ such that $E(S)$ is non-empty

$$\frac{|S| + \alpha|\mathbf{C}'[S]|}{\sum \mathbf{C}'[S]} = \frac{|S|}{2|E(S)|} + \frac{\alpha}{2} > \frac{h}{2e} + \frac{\alpha}{2} = \frac{|V(H_0)| + \alpha|\mathbf{C}'[V(H_0)]|}{\sum \mathbf{C}'[V(H_0)]}.$$

Now, notice that $|E(S)| = \sum \mathbf{C}[S] - |\mathbf{C}[S]|$. Therefore, for any $\emptyset \subsetneq S \subsetneq V(H_0)$, such that $E(S)$ is non-empty and $\alpha \leq \frac{|S|}{\sum \mathbf{C}[S] - |\mathbf{C}[S]|}$, we have $|S| - \alpha|E(S)| \geq 0$. Thus

$$\begin{aligned} \frac{|S| + \alpha|\mathbf{C}[S]|}{\sum \mathbf{C}[S]} &= \frac{|S| + \alpha(\sum \mathbf{C}[S] - |E(S)|)}{\sum \mathbf{C}[S]} = \frac{|S| - \alpha|E(S)|}{\sum \mathbf{C}[S]} + \alpha \geq \alpha = \\ &= \frac{\alpha}{2} + \frac{\alpha}{2} > \frac{h}{2e} + \frac{\alpha}{2} = \frac{|V(H_0)| + \alpha|\mathbf{C}'[V(H_0)]|}{\sum \mathbf{C}'[V(H_0)]}. \end{aligned}$$

Finally, in the case $\sum \mathbf{C}[S] - |\mathbf{C}[S]| = 0$, i.e. $E(S) = \emptyset$, we have

$$\frac{|S| + \alpha|\mathbf{C}[S]|}{\sum \mathbf{C}[S]} > \alpha > \frac{h}{2e} + \frac{\alpha}{2} = \frac{|V(H_0)| + \alpha|\mathbf{C}'[V(H_0)]|}{\sum \mathbf{C}'[V(H_0)]}.$$

Therefore H_0 is strictly α -balanced and Theorem 2 applies. □