A formalism for the calculus of variations with spinors

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Abstract

We develop a frame and dyad gauge-independent formalism for the calculus of variations of functionals involving spinorial objects. As part of this formalism we define a modified variation operator which absorbs frame and spin dyad gauge terms. This formalism is applicable to both the standard spacetime (i.e. $SL(2,\mathbb{C})$) 2-spinors as well as to space (i.e. $SU(2,\mathbb{C})$) 2-spinors. We compute expressions for the variations of the connection and the curvature spinors.

1 Introduction

Variational ideas play an important role in various areas of mathematical General Relativity —e.g. in the ADM formalism [1], in the analysis of Penrose-like inequalities [8] or in the analysis of area-angular momentum inequalities [6] to mention some. Similarly, spinorial methods constitute a powerful tool for the analysis and manipulation of the Einstein field equations and their solutions —most notably the proof of the positivity of the mass by Witten [12] and the analysis of linearised gravity, see e.g. [10].

To the best of our knowledge, all available treatments of calculus of variations and linerisations in spinorial settings make use of computations in terms of components with respect to a dyad. It is therefore of interest to have a setup for performing a dyad-independent calculus of variations and computation of linearisations with spinors. The purpose of the present article is to develop such a setup. We expect this formalism to be of great value in both the analysis of the notion of non-Kerrness introduced in [3, 4] and positivity of the mass in [5], as well as in a covariant analysis of linearised gravity.

The transformation properties of tensors and spinors pose some conceptual subtleties which have to be taken into account when computing variations of the basic tensorial and spinorial structures. It is possible to have variations of of these structures which are *pure gauge*. This difficulty is usually dealt with by a careful fixing of the gauge in some geometrically convenient manner. One thus makes calculus of variations in a specific gauge and has to be careful in distinguishing between properties which are specific to the particular gauge and those which are generic. This situation becomes even more complicated as, in principle, both the tensorial and spinorial structures are allowed to vary simultaneously.

In this article it is shown that it is possible to define a *modified variation operator* which absorbs gauge terms in the variation of spinorial fields and thus, allows to perform *covariant variations*. The idea behind this modified variation operator is similar to that behind the derivative operators in the GHP formalism which absorb terms associated to the freedom in a NP

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tetrad —see [7]. As a result of our analysis we are able to obtain expressions involving abstract tensors and spinors —thus, they are valid in any system of coordinates, and therefore invariant under diffeomorphisms which are constant with respect to variations. However, linearisations of diffeomorphisms do affect our variational quantities. This is discussed in Section 4.3, where we also find that the diffeomorphism freedom can be controled by a gauge source function.

Finally, we point out that although our primary concern in this article is the construction of a formalism for the calculus of variations of expressions involving spinors in a 4-dimensional Lorentzian manifold, the methods can be adapted to a space-spinor formalism on 3-dimensional Riemannian manifolds. This is briefly discussed in Section 6.

The calculations in this article have been carried out in the Mathematica based symbolic differential geometry suite *xAct* [9], in particular *SymManipulator* [2] developed by TB.

Notation and conventions

All throughout, we use abstract index notation to denote tensors and spinors. In particular, the indices $a, b, c, \ldots$ and $i, j, k, \ldots$ are abstract spacetime and spatial tensor indices respectively, while $A, B, C, \ldots$ denote abstract spinorial indices. The boldface indices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ will be used as tensor frame indices and spinor frame indices, respectively. We follow the tensorial and spinorial conventions of Penrose & Rindler [10].

Our signature convention for 4-dimensional Lorentzian metrics is $(+, -, -, -)$, and 3-dimensional Riemannian metrics have signature $(-, -, -)$.

The standard positions for the basic variations are δg_{ab}, $\delta \sigma_a{}^{AA'}$, $\delta \sigma_k{}^{AB}$, $\delta \omega^{\mathbf{a}}{}_b$, $\delta \epsilon^{\mathbf{A}}{}_B$, $\delta \epsilon_{AB}$, $\delta \gamma_a{}^b{}_c$. If any other index positions appear, this means that the indices are moved up or down with g_{ab} or ϵ_{AB} after the variation. The definitions of the above objects will be given in the main text.

2 Basic setup

In this section we discuss our basic geometric setup, which will be used in Section 3 to perform calculus of variations.

2.1 Families of metrics

In what follows, let $(\mathcal{M}, \mathring{g}_{ab})$ denote a 4-dimensional Lorentzian manifold (*spacetime*). The metric $\mathring{g}_{ab}$ will be known as the *background metric*. In what follows, in addition to $\mathring{g}_{ab}$, we consider *arbitrary* families of Lorentzian metrics $\{g_{ab}[\lambda]\}$ over $\mathcal{M}$ with $\lambda \in \mathbb{R}$ a parameter such that $g_{ab}[0] = \mathring{g}_{ab}$. Intuitively, a particular choice of family of metrics can be thought of as a curve in the moduli space of Lorentzian metrics over $\mathcal{M}$. The fact that we allow for arbitrary families of metrics enables us to probe all possible directions of this space in a neighbourhood of $\mathring{g}_{ab}$ and thus, we can compute *Fréchet* derivatives of functionals depending on the metric —see Section 3.1.

In order to make possible the discussion of spinors, it will be assumed that the spacetimes $(\mathcal{M}, g_{ab}[\lambda])$ for fixed λ are orientable and time orientable and admit a spinorial structure.

Notational warning. In what follows, for the ease of the presentation, we often suppress the dependence on λ from the various objects. Thus, unless otherwise stated, all objects not tagged with a *ring* ($\mathring{}$) are assumed to depend on a parameter λ.

2.2 Frames

In what follows, we assume that associated to each family of metrics $\{g_{ab}\}$ one has a family $\{e_{\mathbf{a}}{}^a\}$ of g_{ab}-orthonormal frames. Let $\{\omega^{\mathbf{a}}{}_a\}$ denote the family of associated cobases so that for fixed λ one has $e_{\mathbf{a}}{}^a \omega^{\mathbf{b}}{}_a = \delta_{\mathbf{a}}{}^{\mathbf{b}}$. Following the conventions of the previous section, we write $\mathring{e}_{\mathbf{a}}{}^a \equiv e_{\mathbf{a}}{}^a[0]$ and $\mathring{\omega}^{\mathbf{a}}{}_a \equiv \omega^{\mathbf{a}}{}_a[0]$. By assumption, one has that

$$g_{ab} e_{\mathbf{a}}{}^a e_{\mathbf{b}}{}^b = \eta_{\mathbf{ab}}, \qquad g_{ab} = \eta_{\mathbf{ab}} \omega^{\mathbf{a}}{}_a \omega^{\mathbf{b}}{}_b. \tag{1}$$

where, as usual, $\eta_{\mathbf{ab}} = \mathrm{diag}(1, -1, -1, -1)$.

Remark 1. *Observe that in view of the relations* (1) *any family of frames and coframes* $\{e'_{\mathbf{a}}{}^{a}\}$ *and* $\{\omega'^{\mathbf{a}}{}_{a}\}$ *related to* $\{e_{\mathbf{a}}{}^{a}\}$ *and* $\{\omega^{\mathbf{a}}{}_{a}\}$ *through a family of Lorentz transformations* $\{\Lambda^{\mathbf{a}}{}_{\mathbf{b}}\}$ *give rise to the the same family of metrics* $\{g_{ab}\}$ *—see Appendix A.1.*

2.3 Spinors

By assumption, the spacetimes $(\mathcal{M}, g_{ab})$ are endowed with a spinorial structure. Accordingly, we consider families of antisymmetric spinors $\{\epsilon_{AB}\}$ such that for fixed λ the spinor ϵ_{AB} gives rise to the spinor structure of $(\mathcal{M}, g_{ab})$. Moreover, we set $\mathring{\epsilon}_{AB} \equiv \epsilon_{AB}[0]$.

Associated to the family $\{\epsilon_{AB}\}$ one considers a family $\{\epsilon_{\mathbf{A}}{}^{A}\}$ of normalised spin dyads —that is, one has that

$$\epsilon_{AB}\epsilon_{\mathbf{A}}{}^{A}\epsilon_{\mathbf{B}}{}^{B} = \epsilon_{\mathbf{AB}}, \qquad \epsilon_{\mathbf{AB}} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

Let $\{\epsilon^{\mathbf{A}}{}_{A}\}$ denote the family of dual *covariant bases* for which the relation $\epsilon^{AB}\epsilon^{\mathbf{A}}{}_{A}\epsilon^{\mathbf{B}}{}_{B} = \epsilon^{\mathbf{AB}}$ with $(\epsilon^{\mathbf{AB}}) \equiv -(\epsilon_{\mathbf{AB}})^{-1}$ holds. It follows that one has

$$\delta_{A}{}^{B} = \epsilon_{A}{}^{\mathbf{A}}\epsilon_{\mathbf{A}}{}^{B}, \qquad \epsilon_{AB} = \epsilon_{\mathbf{AB}}\epsilon_{A}{}^{\mathbf{A}}\epsilon_{B}{}^{\mathbf{B}}, \qquad \epsilon^{AB} = \epsilon^{\mathbf{AB}}\epsilon_{\mathbf{A}}{}^{A}\epsilon_{\mathbf{B}}{}^{B}.$$

Remark 2. *As in the case of tensor frames, any family of dyads* $\{\epsilon'_{\mathbf{A}}{}^{A}\}$ *related to* $\{\epsilon_{\mathbf{A}}{}^{A}\}$ *through a family of Lorentz transformations* $\{\Lambda^{\mathbf{A}}{}_{\mathbf{B}}\}$ *gives rise to the same spinorial structures associated to the family of antisymmetric spinors* $\{\epsilon_{AB}\}$ *—see Appendix A.1.*

2.4 Infeld-van der Waerden and soldering forms

The well-known correspondence between tensors and spinors is realised by the *Infeld-van der Waerden symbols* $\sigma_{\mathbf{a}}{}^{\mathbf{AA}'}$ and $\sigma^{\mathbf{a}}{}_{\mathbf{AA}'}$. Given an arbitrary $v^{a} \in T\mathcal{M}$ and $\beta_{a} \in T^{*}\mathcal{M}$ one has that

$$v^{\mathbf{a}} \mapsto v^{\mathbf{AA}'} = v^{\mathbf{a}}\sigma_{\mathbf{a}}{}^{\mathbf{AA}'}, \qquad \beta_{\mathbf{a}} \mapsto \beta_{\mathbf{AA}'} = \beta_{\mathbf{a}}\sigma^{\mathbf{a}}{}_{\mathbf{AA}'}$$

where for *fixed* λ

$$v^{\mathbf{a}} \equiv v^{a}\omega^{\mathbf{a}}{}_{a}, \qquad \beta_{\mathbf{a}} \equiv \beta_{a}e_{\mathbf{a}}{}^{a},$$

denote the components of v^{a} and β_{a} with respect to the orthonormal basis $e_{\mathbf{a}}{}^{a}[\lambda]$ of $(\mathcal{M}, g_{ab}[\lambda])$. In more explicit terms, the correspondence can be written as

$$(v^{\mathbf{0}}, v^{\mathbf{1}}, v^{\mathbf{2}}, v^{\mathbf{3}}) \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} v^{\mathbf{0}} + v^{\mathbf{3}} & v^{\mathbf{1}} + \mathrm{i}v^{\mathbf{2}} \\ v^{\mathbf{1}} - \mathrm{i}v^{\mathbf{2}} & v^{\mathbf{0}} - v^{\mathbf{3}} \end{pmatrix}, \; (\beta_{\mathbf{0}}, \beta_{\mathbf{1}}, \beta_{\mathbf{2}}, \beta_{\mathbf{3}}) \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} \beta_{\mathbf{0}} + \beta_{\mathbf{3}} & \beta_{\mathbf{1}} - \mathrm{i}\beta_{\mathbf{2}} \\ \beta_{\mathbf{1}} + \mathrm{i}\beta_{\mathbf{2}} & \beta_{\mathbf{0}} - \beta_{\mathbf{3}} \end{pmatrix}.$$

From the *Infeld-van der Waerden symbols* we define the *soldering form* $\sigma_{a}{}^{AA'}$ and the dual of the soldering form $\sigma^{a}{}_{AA'}$ by

$$\sigma_{a}{}^{AA'} \equiv \epsilon_{\mathbf{A}}{}^{A}\bar{\epsilon}_{\mathbf{A}'}{}^{A'}\omega^{\mathbf{a}}{}_{a}\sigma_{\mathbf{a}}{}^{\mathbf{AA}'}, \tag{3a}$$

$$\sigma^{a}{}_{AA'} \equiv \epsilon^{\mathbf{A}}{}_{A}\bar{\epsilon}^{\mathbf{A}'}{}_{A'}e_{\mathbf{a}}{}^{a}\sigma^{\mathbf{a}}{}_{\mathbf{AA}'}. \tag{3b}$$

By direct calculation, we can then verify the relations

$$g_{ab} = \epsilon_{AB}\bar{\epsilon}_{A'B'}\sigma_{a}{}^{AA'}\sigma_{b}{}^{BB'}, \tag{4a}$$

$$\delta_{a}{}^{b} = \sigma_{a}{}^{BB'}\sigma^{b}{}_{BB'}. \tag{4b}$$

It is important to note that $\sigma_{a}{}^{AA'}$ and $\sigma^{a}{}_{AA'}$ are tensor frame and spin dyad dependent, while the relations (4a) and (4b) are universal.

Following our approach, in the sequel we consider families $\{\sigma_{a}{}^{AA'}\}$ and $\{\sigma^{a}{}_{AA'}\}$ of soldering forms such that $\mathring{\sigma}_{a}{}^{AA'} \equiv \sigma_{a}{}^{AA'}[0]$ and $\mathring{\sigma}^{a}{}_{AA'} \equiv \sigma^{a}{}_{AA'}[0]$ are the soldering forms associated to $(\mathring{\omega}^{\mathbf{b}}{}_{a}, \mathring{\epsilon}^{\mathbf{B}}{}_{A})$.

Remark 3. *In this article we adopt the point of view that the metric structure provided by* g_{ab} *and the spinorial structure given by* ϵ_{AB} *are independent from each other. After a choice of frame and spinor basis these structures are linked to each other —in an, admittedly, arbitrary manner— through the relations in* (3a) *and* (4a).

3 Calculus of variations

3.1 Basic formalism

The main objective of our calculus of variations is to describe how real valued functionals depend on their arguments —in particular, in the case the arguments are covariant spinors. To motivate our analysis, we first consider a real valued functional $\mathcal{F}[\omega^{\mathbf{a}}{}_a, \xi^{\mathbf{a}}]$, where ξ^a is a vector field and $\xi^{\mathbf{a}} = \omega^{\mathbf{a}}{}_a \xi^a$. Given a *particular* family of fields $\{\omega^{\mathbf{a}}{}_a[\lambda],\ \xi^{\mathbf{a}}[\lambda]\}$ depending on a parameter λ, we define the variations $\{\delta\omega^{\mathbf{a}}{}_a,\ \delta\xi^{\mathbf{a}}\}$ through the expressions

$$\delta\omega^{\mathbf{a}}{}_a \equiv \frac{\mathrm{d}\omega^{\mathbf{a}}{}_a}{\mathrm{d}\lambda}\bigg|_{\lambda=0}, \qquad \delta\xi^{\mathbf{a}} \equiv \frac{\mathrm{d}\xi^{\mathbf{a}}}{\mathrm{d}\lambda}\bigg|_{\lambda=0}.$$

In terms of the above fields over $\mathcal{M}$ we define *the Gâteux derivative of* $\mathcal{F}[\omega^{\mathbf{a}}{}_a, \xi^{\mathbf{a}}]$ *at* $\{\mathring{\omega}^{\mathbf{a}}{}_a,\ \mathring{\xi}^{\mathbf{a}}\}$ *in the direction of the family* $\{\omega^{\mathbf{a}}{}_a[\lambda],\ \xi^{\mathbf{a}}[\lambda]\}$ as

$$\begin{aligned}\delta_{\{\omega^{\mathbf{a}}{}_a,\, \xi^{\mathbf{a}}\}}\mathcal{F}[\mathring{\omega}^{\mathbf{a}}{}_a, \mathring{\xi}^{\mathbf{a}}] &\equiv \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}[\omega^{\mathbf{a}}{}_a[\lambda], \xi^{\mathbf{a}}[\lambda]]\bigg|_{\lambda=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}[\mathring{\omega}^{\mathbf{a}}{}_a + \lambda\delta\omega^{\mathbf{a}}{}_a, \mathring{\xi}^{\mathbf{a}} + \lambda\delta\xi^{\mathbf{a}}]\bigg|_{\lambda=0}.\end{aligned}$$

Now, if $\delta_{\{\omega^{\mathbf{a}}{}_a,\, \xi^{\mathbf{a}}\}}\mathcal{F}$ exists for *any* choice of family $\{\omega^{\mathbf{a}}{}_a,\ \xi^{\mathbf{a}}\}$ one then says that $\mathcal{F}[\omega^{\mathbf{a}}{}_a, \xi^{\mathbf{a}}]$ is *Fréchet differentiable* at $\{\mathring{\omega}^{\mathbf{a}}{}_a,\ \mathring{\xi}^{\mathbf{a}}\}$. If this is the case, there exists a functional $\delta\mathcal{F}$, the *Fréchet derivative*, from which $\delta_{\{\omega^{\mathbf{a}}{}_a,\, \xi^{\mathbf{a}}\}}\mathcal{F}$ can be computed if a particular choice of the family of the variations $\{\delta\omega^{\mathbf{a}}{}_a,\ \delta\xi^{\mathbf{a}}\}$ is considered. For more details concerning the notions of Gâteaux and Fréchet derivative and their relation see [11].

The functional $\mathcal{F}[\omega^{\mathbf{a}}{}_a, \xi^{\mathbf{a}}]$ considered in the previous paragraph depends on the coframe and components of a tensor fields in terms of this basis. As the particular choice of frame involves the specification of a gauge, instead of regarding the functional $\delta\mathcal{F}$ as depending on the fields $[\mathring{\omega}^{\mathbf{a}}{}_a, \mathring{\xi}^{\mathbf{a}}, \delta\omega^{\mathbf{a}}{}_a, \delta\xi^{\mathbf{a}}]$ it will be convenient to regard it as depending on $[\mathring{g}_{ab}, \mathring{\xi}^a, \delta g_{ab}, T_{ab}, \delta\xi^a]$, where the field T_{ab} describes the frame gauge choice and

$$\delta g_{ab} \equiv \frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda}\bigg|_{\lambda=0},$$

where $\{g_{ab}\}$ is a family of metrics over $\mathcal{M}$ such that for fixed λ the coframe $\mathring{\omega}^{\mathbf{a}}{}_a$ is g_{ab}-orthonormal.

Next, we consider real valued functionals depending on spinors. For concreteness consider the a functional of the form $\mathcal{F}[g_{ab}, \epsilon^{\mathbf{A}}{}_A, \kappa_{\mathbf{A}}]$. The Gâteaux and Fréchet derivatives of this functional are defined in the natural way by considering arbitrary families of fields $\{g_{ab},\ \epsilon^{\mathbf{A}}{}_A,\ \kappa_{\mathbf{A}}\}$ depending on a parameter λ. The variations implied by this family of fields is then defined by

$$\delta g_{ab} \equiv \frac{\mathrm{d}g_{ab}}{\mathrm{d}\lambda}\bigg|_{\lambda=0}, \qquad \delta\epsilon^{\mathbf{A}}{}_A \equiv \frac{\mathrm{d}\epsilon^{\mathbf{A}}{}_A}{\mathrm{d}\lambda}\bigg|_{\lambda=0}, \qquad \delta\kappa_{\mathbf{A}} \equiv \frac{\mathrm{d}\kappa_{\mathbf{A}}}{\mathrm{d}\lambda}\bigg|_{\lambda=0}.$$

In analogy to the example considered in the previous paragraphs, it will be convenient to regard the Fréchet derivative $\delta\mathcal{F}$, which in principle depends on $[\mathring{g}_{ab}, \mathring{\epsilon}^{\mathbf{A}}{}_A, \mathring{\kappa}_{\mathbf{A}}, \delta g_{ab}, \delta\epsilon^{\mathbf{A}}{}_A, \delta\kappa_{\mathbf{A}}]$, as a functional of the arguments $[\mathring{g}_{ab}, \mathring{\epsilon}_{AB}, \mathring{\kappa}_A, \delta g_{ab}, \delta\kappa_A, T_{ab}, S_{AB}]$ where the field S_{AB} describes the dyad gauge choice. In this way one obtains a formalism that separates the tensor frame and spin dyad gauge in the Fréchet derivatives. The main observation in the sequel is that is is possible to obtain a *modified variation operator* ϑ which absorbs the frame and dyad gauge terms so that the Fréchet derivative depends on the parameters $[\mathring{g}_{ab}, \mathring{\kappa}_A, \delta g_{ab}, \vartheta\kappa_A]$.

Notational warning. In what follows, for ease of presentation, we mostly suppress the ring $\mathring{}$ from the background quantities appearing in expressions involving variations. If an expression does not involves variations then it holds for both the background quantities and any other one in the family.

3.2 Basic formulae for frames

Consider first the expression for the metric g_{ab} in terms of the coframe $\{\omega^{\mathbf{a}}{}_a\}$ —namely

$$g_{ab} = \eta_{\mathbf{ab}}\omega^{\mathbf{a}}{}_a\omega^{\mathbf{b}}{}_b.$$

Applying the variational operator δ to the above expression, using the Leibnitz rule, and that $\eta_{\mathbf{ab}}$ are constants, yields

$$\delta g_{ab} = \eta_{\mathbf{ab}}\delta\omega^{\mathbf{a}}{}_a\omega^{\mathbf{b}}{}_b + \eta_{\mathbf{ab}}\omega^{\mathbf{a}}{}_a\delta\omega^{\mathbf{b}}{}_b. \tag{5}$$

In certain computations it is useful to be able to express $\delta\omega^{\mathbf{a}}{}_a$ in terms of δg_{ab}. In order to do this, it is noticed that from (5) it follows that

$$\delta g_{ab} = 2\eta_{\mathbf{ab}}\omega^{\mathbf{a}}{}_a\delta\omega^{\mathbf{b}}{}_b - 2T_{ab},$$

where

$$T_{ab} \equiv \eta_{\mathbf{cd}}\omega^{\mathbf{d}}{}_{[a}\delta\omega^{\mathbf{c}}{}_{b]}.$$

It then follows that

$$\delta(\omega^{\mathbf{a}}{}_a) = \tfrac{1}{2}e_{\mathbf{b}}{}^b\eta^{\mathbf{ba}}\delta g_{ab} - e_{\mathbf{b}}{}^b\eta^{\mathbf{ba}}T_{ab}. \tag{6}$$

A formula for the variation of the inverse metric can be computed by taking variations of the defining relation $\delta_a{}^b = g_{ac}g^{cb}$. One finds that

$$\delta(g^{dc}) = -\, g^{ad}g^{bc}\delta g_{ab}.$$

A formula for the variation of the frame vectors $\{e_{\mathbf{a}}{}^a\}$ in terms of the variation of $\delta\omega^{\mathbf{c}}{}_b$ is obtained by computing the variation of the expression $\delta_{\mathbf{a}}{}^{\mathbf{b}} = e_{\mathbf{a}}{}^a\omega^{\mathbf{b}}{}_a$. One finds that

$$\delta(e_{\mathbf{a}}{}^d) = -\, e_{\mathbf{a}}{}^b e_{\mathbf{c}}{}^d\delta\omega^{\mathbf{c}}{}_b.$$

The previous expressions can be used to compute a formula for the variation of a covector ξ_a. Writing $\xi_a = \xi_{\mathbf{a}}\omega^{\mathbf{a}}{}_a$, one obtains that

$$\delta\xi_a = \omega^{\mathbf{b}}{}_a\delta(\xi_{\mathbf{b}}) + \tfrac{1}{2}e_{\mathbf{c}}{}^b\eta^{\mathbf{cd}}\xi_{\mathbf{d}}\delta g_{ab} - e_{\mathbf{c}}{}^b\eta^{\mathbf{cd}}\xi_{\mathbf{d}}T_{ab}.$$

Remark 4. *An interpretation of the tensor T_{ab} appearing in equation (6) can be obtained by considering a situation where $\delta g_{ab} = 0$. In that case equation (6) reduces to*

$$\delta\omega^{\mathbf{a}}{}_a = -e_{\mathbf{b}}{}^b\eta^{\mathbf{ba}}T_{ab}.$$

Writing $T_{ab} = T_{\mathbf{ab}}\omega^{\mathbf{a}}{}_a\omega^{\mathbf{b}}{}_b$ where $T_{\mathbf{ab}}$ denote the components of T_{ab} with respect to the coframe $\{\omega^{\mathbf{a}}{}_a\}$ one has that

$$\delta\omega^{\mathbf{a}}{}_a = -e_{\mathbf{b}}{}^b\eta^{\mathbf{ba}}T_{\mathbf{cd}}\omega^{\mathbf{c}}{}_a\omega^{\mathbf{d}}{}_b = T^{\mathbf{a}}{}_{\mathbf{c}}\omega^{\mathbf{c}}{}_a,$$

where $T^{\mathbf{a}}{}_{\mathbf{c}} \equiv -\eta^{\mathbf{da}}T_{\mathbf{cd}}\omega^{\mathbf{c}}{}_a$. Comparing with the discussion in Section A.1 one sees that T_{ab} encodes a rotation of the basis. With this observation, in what follows we interpret the second term in equation (6) as a gauge term.

3.3 Basic formulae for spinors

The analysis in the previous section admits a straightforward spinorial analogue. Given a covariant spinorial dyad $\{\epsilon^{\mathbf{A}}{}_A\}$ one can write

$$\epsilon_{AB} = \epsilon_{\mathbf{AB}}\epsilon^{\mathbf{A}}{}_A\epsilon^{\mathbf{B}}{}_B.$$

Thus, one has that

$$\begin{aligned}\delta\epsilon_{AB} &= \epsilon_{\mathbf{AB}}\epsilon^{\mathbf{B}}{}_B\delta\epsilon^{\mathbf{A}}{}_A + \epsilon_{\mathbf{AB}}\epsilon^{\mathbf{A}}{}_A\delta\epsilon^{\mathbf{B}}{}_B \\ &= 2\epsilon_{\mathbf{AB}}\epsilon^{\mathbf{B}}{}_B\delta\epsilon^{\mathbf{A}}{}_A - 2S_{AB},\end{aligned}$$

where

$$S_{AB} \equiv \epsilon_{\mathbf{AB}}\epsilon^{\mathbf{B}}{}_{(B}\delta\epsilon^{\mathbf{A}}{}_{A)}.$$

The variation of the contravariant antisymmetric spinor ϵ^{AB} can be computed from the above formulae by first computing the variation of $\epsilon_{AB}\epsilon^{BC} = -\delta_A{}^C$ and then multiplying with ϵ^{AD}. We obtain that

$$\delta(\epsilon^{DC}) = -\epsilon^{AD}\epsilon^{BC}\delta\epsilon_{AB}.$$

As, $\delta\epsilon_{AB}$ is antisymmetric we can fully express it in terms of its trace as $\delta\epsilon_{AB} = -\frac{1}{2}\epsilon_{AB}\delta\epsilon^C{}_C$.

Now, if one wants to compute $\delta\epsilon^{\mathbf{A}}{}_A$ in terms of $\delta\epsilon_{AB}$ one has that

$$\delta\epsilon^{\mathbf{A}}{}_A = \tfrac{1}{2}\epsilon^{\mathbf{AB}}\epsilon_{\mathbf{B}}{}^B\delta\epsilon_{AB} + \epsilon^{\mathbf{AB}}\epsilon_{\mathbf{B}}{}^B S_{AB}. \tag{7}$$

If we compute the variation of $\delta_{\mathbf{A}}{}^{\mathbf{C}} = \epsilon^{\mathbf{C}}{}_B\epsilon_{\mathbf{A}}{}^B$ and multiply with $\epsilon_{\mathbf{C}}{}^D$ we get

$$\delta(\epsilon_{\mathbf{A}}{}^A) = -\epsilon_{\mathbf{A}}{}^B\epsilon_{\mathbf{C}}{}^A\delta\epsilon^{\mathbf{C}}{}_B.$$

Now consider a covariant spinor ϕ_A and expand it with respect to the spinor dyad $\{\epsilon^{\mathbf{A}}{}_A\}$ as

$$\phi_A = \phi_{\mathbf{A}}\epsilon^{\mathbf{A}}{}_A.$$

A calculation using equation (7) yields the expression

$$\begin{aligned}\delta\phi_A &= \delta\phi_{\mathbf{A}}\epsilon^{\mathbf{A}}{}_A + \phi_{\mathbf{A}}\delta\epsilon^{\mathbf{A}}{}_A \\ &= \delta\phi_{\mathbf{A}}\epsilon^{\mathbf{A}}{}_A + \tfrac{1}{2}\phi_{\mathbf{A}}\epsilon^{\mathbf{AP}}\epsilon_{\mathbf{P}}{}^B\delta\epsilon_{AB} + \phi_{\mathbf{A}}\epsilon^{\mathbf{AP}}\epsilon_{\mathbf{P}}{}^B S_{AB}.\end{aligned}$$

Using the identity $\epsilon^{\mathbf{AC}}\phi_{\mathbf{A}}\epsilon_{\mathbf{C}}{}^B = \epsilon^{CB}\phi_C$ the variation $\delta\phi_A$ can be reexpressed as

$$\delta\phi_A = (\delta\phi_{\mathbf{A}})\epsilon^{\mathbf{A}}{}_A + \tfrac{1}{4}(\delta\epsilon^Q{}_Q)\phi_A - S_A{}^B\phi_B.$$

Remark 5. *As in the case of equation* (6) *and the tensor* T_{ab}*, the spinor* S_{AB} *admits the interpretation of a rotation. Indeed, considering a situation where* $\delta\epsilon_{AB} = 0$*, writing* $S_{AB} = \epsilon^{\mathbf{A}}{}_A\epsilon^{\mathbf{B}}{}_B S_{\mathbf{AB}}$ *one finds that*

$$\begin{aligned}\delta\epsilon^{\mathbf{A}}{}_A &= \epsilon^{\mathbf{AB}}\epsilon_{\mathbf{B}}{}^B S_{AB} \\ &= \epsilon^{\mathbf{AB}}\epsilon_{\mathbf{B}}{}^B\epsilon^{\mathbf{P}}{}_A\epsilon^{\mathbf{Q}}{}_B S_{\mathbf{PQ}} \\ &= S^{\mathbf{A}}{}_{\mathbf{B}}\epsilon^{\mathbf{B}}{}_A.\end{aligned}$$

Comparing with Appendix A.1, we find that S_{AB} *encodes a rotation of the spin dyad.*

3.4 Variation of the soldering form

In the reminder of this article we will consider a more general setting in which both the metric g_{ab} and the antisymmetric spinor ϵ_{AB} can be varied simultaneously. To analyse the relation between the variations of these two structures it is convenient to consider the soldering form $\sigma_a{}^{AA'}$.

To compute the variation of the soldering form, one starts by computing the variation of the relation (3a). As we are treating the Infeld-van der Waerden symbols as constants, their variation vanishes — that is, although both the metric and spinor structure may vary, the formal relation between tetrads and spin dyads will be preserved. A direct combination of the methods of Sections 3.2 and 3.3 on formula (3a) lead, after a computation, to the expression

$$\begin{aligned}\delta\sigma_a{}^{AA'} = {}&\tfrac{1}{2}\delta\epsilon^A{}_B\sigma_a{}^{BA'} + \tfrac{1}{2}\delta\bar{\epsilon}^{A'}{}_{B'}\sigma_a{}^{AB'} + \tfrac{1}{2}g^{bc}\delta g_{ab}\sigma_c{}^{AA'} \\ &- \bar{S}^{A'}{}_{B'}\sigma_a{}^{AB'} - S^A{}_B\sigma_a{}^{BA'} - T_a{}^b\sigma_b{}^{AA'}.\end{aligned} \tag{8}$$

The terms in the second line of the previous expression are identified as *gauge terms*. Observe that in this case one has two types of gauge terms: one arising from the variation of the tensor frame and one coming from the variation of the spin frame.

If we compute the variation of equation (4b) and multiply with ${\sigma^a}_{AA'}$ we get

$$\delta({\sigma^b}_{AA'}) = -\delta({\sigma_a}^{BB'}){\sigma^a}_{AA'}{\sigma^b}_{BB'}.$$

Multiplying equation (8) with $g^{ac}{\sigma_c}^{BB'}$ and splitting into irreducible parts, we get the relations

$$\begin{aligned}
\delta{\sigma^a}_{(A}{}^{(A'}\sigma_{|a|B)}{}^{B')} &= \tfrac{1}{2}\delta g_{(AB)}{}^{(A'B')},\\
\delta\sigma^{a(A|B'|}{\sigma_a}^{B)}{}_{B'} &= T^{AB} - 2S^{AB},\\
\delta\sigma^{aB(A'}\sigma_{|a|B}{}^{B')} &= \bar{T}^{A'B'} - 2\bar{S}^{A'B'},\\
\delta\sigma^{aBB'}\sigma_{aBB'} &= \tfrac{1}{2}\delta {g^B}_B{}^{B'}{}_{B'} + \delta{\epsilon^B}_B + \delta{\bar{\epsilon}^{B'}}{}_{B'},
\end{aligned}$$

where we have defined

$$T_{AB} \equiv T_{ab}{\sigma^a}_A{}^{A'}{\sigma^b}_{BA'}, \qquad \delta g_{ABA'B'} \equiv \delta g_{ab}{\sigma^a}_{AA'}{\sigma^b}_{BB'}.$$

3.5 General variations of spinors

The formulae for the variations of the soldering form and its dual can now be used to compute the variation of arbitrary spinors under variations of the metric and spinor structures. To this end, consider spinors $\zeta^{AA'}$ and $\xi_{AA'}$. Making use of the Leibniz rule one obtains the expressions

$$\begin{aligned}
{\sigma_a}^{AA'}\delta\zeta^a = \delta(\zeta^{AA'}) &- \tfrac{1}{4}\delta{\epsilon^B}_B\zeta^{AA'} - \tfrac{1}{4}\delta{\bar{\epsilon}^{B'}}{}_{B'}\zeta^{AA'} - \tfrac{1}{2}\delta {g^A}_B{}^{A'}{}_{B'}\zeta^{BB'}\\
&- \tfrac{1}{2}{\bar{T}^{A'}}{}_{B'}\zeta^{AB'} + {\bar{S}^{A'}}{}_{B'}\zeta^{AB'} - \tfrac{1}{2}{T^A}_B\zeta^{BA'} + {S^A}_B\zeta^{BA'}, \qquad (9a)\\
{\sigma^a}_{AA'}\delta\xi_a = \delta(\xi_{AA'}) &+ \tfrac{1}{4}\delta{\epsilon^B}_B\xi_{AA'} + \tfrac{1}{4}\delta{\bar{\epsilon}^{B'}}{}_{B'}\xi_{AA'} + \tfrac{1}{2}\delta {g_A}^B{}_{A'}{}^{B'}\xi_{BB'}\\
&+ \tfrac{1}{2}{\bar{T}_{A'}}{}^{B'}\xi_{AB'} - {\bar{S}_{A'}}{}^{B'}\xi_{AB'} + \tfrac{1}{2}{T_A}^B\xi_{BA'} - {S_A}^B\xi_{BA'}, \qquad (9b)
\end{aligned}$$

where $\zeta^a \equiv {\sigma^a}_{BB'}\zeta^{BB'}$ and $\xi_a \equiv {\sigma_a}^{BB'}\xi_{BB'}$. We observe that both expressions contain a combination of *gauge terms* involving the spinors T_{AB} and S_{AB}.

In view of the discussion in the previous paragraph we introduce a general *modified variation operator*.

Definition 1. *The* modified variation operator ϑ *is for valence 1 spinors defined by*

$$\begin{aligned}
\vartheta\phi_A &\equiv \delta\phi_A + \tfrac{1}{4}\delta{\epsilon^B}_B\phi_A + \tfrac{1}{2}{T_A}^B\phi_B - {S_A}^B\phi_B,\\
\vartheta\phi^A &\equiv \delta\phi^A - \tfrac{1}{4}\delta{\epsilon^B}_B\phi^A - \tfrac{1}{2}{T^A}_B\phi^B + {S^A}_B\phi^B,\\
\vartheta\bar{\phi}_{A'} &\equiv \delta\bar{\phi}_{A'} + \tfrac{1}{4}\delta{\bar{\epsilon}^{B'}}{}_{B'}\bar{\phi}_{A'} + \tfrac{1}{2}{\bar{T}_{A'}}{}^{B'}\bar{\phi}_{B'} - {\bar{S}_{A'}}{}^{B'}\bar{\phi}_{B'},\\
\vartheta\bar{\phi}^{A'} &\equiv \delta\bar{\phi}^{A'} - \tfrac{1}{4}\delta{\bar{\epsilon}^{B'}}{}_{B'}\bar{\phi}^{A'} - \tfrac{1}{2}{\bar{T}^{A'}}{}_{B'}\bar{\phi}^{B'} + {\bar{S}^{A'}}{}_{B'}\bar{\phi}^{B'},
\end{aligned}$$

and extended to arbitrary valence spinors by the Leibniz rule.

In particular, using the above definitions in expressions (9a)-(9b) one finds that

$$\begin{aligned}
{\sigma_a}^{AA'}\delta\zeta^a &= \vartheta\zeta^{AA'} - \tfrac{1}{2}\delta {g^A}_B{}^{A'}{}_{B'}\zeta^{BB'},\\
{\sigma^a}_{AA'}\delta\xi_a &= \vartheta\xi_{AA'} + \tfrac{1}{2}\delta {g_A}^B{}_{A'}{}^{B'}\xi_{BB'},
\end{aligned}$$

showing that $\vartheta\zeta^{AA'}$ and $\vartheta\zeta_{AA'}$ are frame gauge independent. Moreover, a further calculation shows that

$$\vartheta\epsilon_{AB} = 0$$

so that the process of raising and lowering spinor indices commutes with the modified variation ϑ operator.

Remark 6. *Expanding the ϕ_A in terms of the spin dyad in the $\delta\phi_A$ term in Definition 1 gives*

$$\vartheta(\phi_A) = {\epsilon^{\mathbf{B}}}_A\delta(\phi_{\mathbf{B}}) + \tfrac{1}{2}{T_A}^B\phi_B. \qquad (10)$$

Observe that the S_{AB} and $\delta\epsilon_{AB}$ terms cancel out.

4 Variations and the covariant derivative

The purpose of this section is to to analyse te relation between the variation operators δ and ϑ and the Levi-Civita connection ∇_a of the metric g_{ab}.

4.1 Basic tensorial relations

Our analysis of the variations of expressions involving covariant derivatives is based on the following basic assumption:

Assumption. *For any scalar field f over $\mathcal{M}$ one has that*

$$\nabla_a \delta f = \delta(\nabla_a f) \tag{11}$$

In what follows, define the *frame dependent tensor*

$$\gamma_a{}^b{}_c \equiv -e_{\mathbf{c}}{}^b \nabla_a \omega^{\mathbf{c}}{}_c.$$

The tensor $\gamma_a{}^b{}_c$ can be regarded as a convenient way of grouping the connection coefficients $\gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}}$ of the connection ∇_a with respect to the frame $\{e_{\mathbf{a}}{}^a\}$. A calculation shows, indeed, that

$$\gamma_a{}^b{}_c = \gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}}\, \omega^{\mathbf{a}}{}_a \omega^{\mathbf{c}}{}_c e_{\mathbf{b}}{}^b.$$

We can express all covariant derivatives of the cobasis and the basis in terms of $\gamma_a{}^b{}_c$ via

$$\nabla_a \omega^{\mathbf{f}}{}_c = -\omega^{\mathbf{f}}{}_b \gamma_a{}^b{}_c, \qquad \nabla_d e_{\mathbf{f}}{}^b = e_{\mathbf{f}}{}^a \gamma_d{}^b{}_a.$$

Differentiating the orthonormality condition $\eta^{\mathbf{ab}} = \omega^{\mathbf{a}}{}_c \omega^{\mathbf{b}}{}_d g^{cd}$ and multiplying with $e_{\mathbf{a}}{}^h e_{\mathbf{b}}{}^l$ we get the relation

$$\gamma_f{}^{(a}{}_c g^{b)c} = 0 \tag{12}$$

encoding the metric compatibility of ∇_a. The variation of this gives

$$\delta\gamma_f{}^{(ab)} = \gamma_f{}^{(a|c|}\delta g^{b)}{}_c. \tag{13}$$

Now, for any covector ξ_a, its covariant derivative can be expanded in terms of the frame as

$$\nabla_a \xi_b = -\omega^{\mathbf{c}}{}_d \gamma_a{}^d{}_b \xi_{\mathbf{c}} + \omega^{\mathbf{c}}{}_b \nabla_a \xi_{\mathbf{c}}.$$

Computing the variation of this last expression, and using the relations above, gives after some straightforward calculations

$$\begin{aligned}\delta(\nabla_a\xi_b) = &-\delta\gamma_a{}^c{}_b\xi_c + T_c{}^d\gamma_{abd}\xi^c - T_b{}^d\gamma_{acd}\xi^c + \tfrac{1}{2}\gamma_{ac}{}^d\delta g_{bd}\xi^c + \tfrac{1}{2}\gamma_{ab}{}^d\delta g_{cd}\xi^c + \xi^c\nabla_a T_{bc}\\ &-\tfrac{1}{2}\xi^c\nabla_a\delta g_{bc} + \nabla_a\delta\xi_b.\end{aligned} \tag{14}$$

In the previous calculation Assumption (11) has been used. If we use relation (14) with $\xi_a = \nabla_a f$, antisymmetrize over a and b, and assume that the connection is torsion free, we get

$$0 = (T_c{}^d\gamma_{[ab]d} + \tfrac{1}{2}\delta g_c{}^d\gamma_{[ab]d} - \delta\gamma_{[a|c|b]} + \nabla_{[a}T_{b]c} - \tfrac{1}{2}\nabla_{[a}\delta g_{b]c} + T_{[a}{}^d\gamma_{b]cd} + \tfrac{1}{2}\gamma_{[a|c|}{}^d\delta g_{b]d})\nabla^c f.$$

Hence, the torsion free condition is encoded by

$$\delta\gamma_{[a|c|b]} = T_c{}^d\gamma_{[ab]d} + \tfrac{1}{2}\delta g_c{}^d\gamma_{[ab]d} + \nabla_{[a}T_{b]c} - \tfrac{1}{2}\nabla_{[a}\delta g_{b]c} + T_{[a}{}^d\gamma_{b]cd} + \tfrac{1}{2}\gamma_{[a|c|}{}^d\delta g_{b]d}. \tag{15}$$

Now, using the identity

$$\delta\gamma_{abc} = \delta\gamma_{[a|b|c]} - \delta\gamma_{[a|c|b]} + \delta\gamma_{[b|a|c]} + \delta\gamma_{a(bc)} - \delta\gamma_{b(ac)} + \delta\gamma_{c(ab)},$$

we can use equations (13) and (15) to compute

$$\delta\gamma_{abc} = -T_c{}^d\gamma_{abd} + T_b{}^d\gamma_{acd} + \tfrac{1}{2}\gamma_{ac}{}^d\delta g_{bd} + \tfrac{1}{2}\gamma_{ab}{}^d\delta g_{cd} - \nabla_a T_{bc} - \tfrac{1}{2}\nabla_b\delta g_{ac} + \tfrac{1}{2}\nabla_c\delta g_{ab}. \tag{16}$$

It follows then that equation (14) can therefore be simplified to

$$\delta(\nabla_a\xi_b) = \nabla_a(\delta\xi_b) - \tfrac{1}{2}g^{cd}(\nabla_a\delta g_{bc} + \nabla_b\delta g_{ac} - \nabla_c\delta g_{ab})\xi_d. \tag{17}$$

It is important to observe that this formula is a tensorial expression. Hence, it allows to define a *transition tensor*

$$Q_b{}^a{}_c \equiv \tfrac{1}{2}g^{ad}(\nabla_b\delta g_{dc} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}) \tag{18}$$

relating the connections ∇_a and $\delta\nabla_a$. This is not surprising as it is well known that the space of covariant derivatives on a manifold is an affine space. Making use the definition of $Q_b{}^a{}_c$, equation (17) takes the suggestive form

$$\delta(\nabla_a\xi_b) = \nabla_a(\delta\xi_b) - Q_b{}^d{}_a\xi_d. \tag{19}$$

Furthermore, making use of the Leibnitz rule one finds that for an arbitrary vector v^a one has

$$\delta(\nabla_a v^b) = \nabla_a(\delta v^b) + Q_c{}^b{}_a v^c.$$

The extension to higher valence tensors follows in a similar manner.

4.2 Spinorial expressions

In order to discuss the variations of the spinor covariant derivative $\nabla_{AA'}$ associated to the Levi-Civita connection ∇_a it is convenient to define a spinorial analogue of the tensor $\gamma_a{}^b{}_c$ —namely

$$\gamma_a{}^B{}_C \equiv -\epsilon_{\mathbf{C}}{}^B\nabla_a\epsilon^{\mathbf{C}}{}_C.$$

The hybrid $\gamma_a{}^B{}_C$ is related to $\gamma_a{}^{BB'}{}_{CC'} \equiv \gamma_a{}^b{}_c\sigma_b{}^{BB'}\sigma^c{}_{CC'}$ through the decomposition

$$\gamma_a{}^{BB'}{}_{CC'} = \gamma_a{}^B{}_C\delta_{C'}{}^{B'} + \bar{\gamma}_a{}^{B'}{}_{C'}\delta_C{}^B.$$

It follows then that

$$\gamma_a{}^B{}_C = \tfrac{1}{2}\gamma_a{}^c{}_b\sigma^b{}_{CB'}\sigma_c{}^{BB'}. \tag{20}$$

From this last expression can then be verified that

$$\gamma_{aBC} = \gamma_{aCB}.$$

The variational derivative of $\gamma_a{}^B{}_C$ can be computed using equation (20). One finds that

$$\begin{aligned}\delta(\gamma_a{}^B{}_C) = {}& \tfrac{1}{4}Q_{acd}\sigma^{dBB'}\sigma^c{}_{CB'} - \tfrac{1}{4}Q_{acd}\sigma^{dBB'}\sigma^c{}_{CB'} - \tfrac{1}{2}\gamma_{acd}S_{CD}\sigma^{cBB'}\sigma^{dD}{}_{B'} \\ & - \tfrac{1}{2}\gamma_{acd}S^B{}_D\sigma^c{}_C{}^{B'}\sigma^{dD}{}_{B'} - \tfrac{1}{2}\nabla_a T^B{}_C.\end{aligned} \tag{21}$$

In this last expression observe, in particular, the appearance of the gauge spinors S_{AB} and T_{AB}. In turn, equation (21) can be used to compute the variation of the covariant derivative of an arbitrary spinor κ_A. Expanding κ_A in terms of the spin dyad and differentiating we get

$$\nabla_a\kappa_B = \epsilon^{\mathbf{C}}{}_B\nabla_a\kappa_{\mathbf{C}} - \gamma_a{}^C{}_B\kappa_C.$$

It follows that the variation of this last expression is given by

$$\begin{aligned}\delta(\nabla_a\kappa_A) &= \nabla_a\delta\kappa_A - \tfrac{1}{2}\kappa^B\nabla_a T_{AB} + \kappa^B\nabla_a S_{AB} + \tfrac{1}{4}\kappa_A\nabla_a\delta\epsilon^B{}_B \\ &\quad + \tfrac{1}{4}Q_{abc}\kappa^B\sigma^b{}_A{}^{A'}\sigma^c{}_{BA'} - \tfrac{1}{4}Q_{acb}\kappa^B\sigma^b{}_A{}^{A'}\sigma^c{}_{BA'} \\ &= \nabla_a\vartheta\kappa_A - \tfrac{1}{4}\delta\epsilon^B{}_B\nabla_a\kappa_A + \tfrac{1}{2}T_{AB}\nabla_a\kappa^B - S_{AB}\nabla_a\kappa^B \\ &\quad + \tfrac{1}{4}Q_{abc}\kappa^B\sigma^b{}_A{}^{A'}\sigma^c{}_{BA'} - \tfrac{1}{4}Q_{acb}\kappa^B\sigma^b{}_A{}^{A'}\sigma^c{}_{BA'}.\end{aligned}$$

In order to write the spinorial derivative $\nabla_{AA'}\kappa_B$ (rather than $\nabla_a\kappa_B$) it is convenient to define the spinor

$$\Omega_{AA'BC} \equiv -\tfrac{1}{2}\sigma^a{}_{AA'}\sigma^b{}_B{}^{B'}\sigma^c{}_{CB'}Q_{[bc]a}. \tag{22}$$

Theorem 1. *The variation of the covariant derivative of a spinor is given by*

$$\vartheta(\nabla_{AA'}\kappa_B) = \nabla_{AA'}\vartheta\kappa_B + ♀_{AA'BC}\kappa^C - \tfrac{1}{2}\delta g_{ACA'B'}\nabla^{CB'}\kappa_B, \tag{23a}$$

$$\vartheta(\nabla_{AA'}\bar{\kappa}_{B'}) = \nabla_{AA'}\vartheta\bar{\kappa}_{B'} + \bar{♀}_{A'AB'C'}\bar{\kappa}^{C'} - \tfrac{1}{2}\delta g_{ABA'C'}\nabla^{BC'}\bar{\kappa}_{B'}. \tag{23b}$$

Proof. Using the expressions in the previous paragraphs one has that

$$\begin{aligned}
\delta(\nabla_{AA'}\kappa_B) &= \nabla_{AA'}\vartheta\kappa_B + ♀_{AA'BC}\kappa^C - \tfrac{1}{4}\delta\epsilon^C{}_C\nabla_{AA'}\kappa_B + \tfrac{1}{2}T_{BC}\nabla_{AA'}\kappa^C - S_{BC}\nabla_{AA'}\kappa^C \\
&\quad - \delta\sigma_a{}^{CB'}\sigma^a{}_{AA'}\nabla_{CB'}\kappa_B \\
&= \nabla_{AA'}\vartheta\kappa_B + ♀_{AA'BC}\kappa^C - \tfrac{1}{2}\delta\epsilon^C{}_C\nabla_{AA'}\kappa_B - \tfrac{1}{4}\delta\bar{\epsilon}^{B'}{}_{B'}\nabla_{AA'}\kappa_B + \tfrac{1}{2}T_{BC}\nabla_{AA'}\kappa^C \\
&\quad - S_{BC}\nabla_{AA'}\kappa^C + \tfrac{1}{2}\bar{T}_{A'B'}\nabla_A{}^{B'}\kappa_B - \bar{S}_{A'B'}\nabla_A{}^{B'}\kappa_B + \tfrac{1}{2}T_{AC}\nabla^C{}_{A'}\kappa_B \\
&\quad - S_{AC}\nabla^C{}_{A'}\kappa_B - \tfrac{1}{2}\delta g_{ACA'B'}\nabla^{CB'}\kappa_B.
\end{aligned}$$

Expressing the above formula in terms of the modified variation ϑ, we get (23a). The equation (23b) is given by complex conjugation. □

4.2.1 Decomposition of $♀_{AA'BC}$

Starting from the definition in equation (22), a calculation yields

$$\begin{aligned}
♀_{AA'BC} &= -\tfrac{1}{4}\sigma^a{}_{AA'}\sigma^b{}_B{}^{B'}\sigma^c{}_{CB'}\nabla_b\delta g_{ac} + \tfrac{1}{4}\sigma^a{}_{AA'}\sigma^b{}_B{}^{B'}\sigma^c{}_{CB'}\nabla_c\delta g_{ab} \\
&= -\tfrac{1}{2}\nabla_{(B}{}^{B'}\delta g_{C)AB'A'}.
\end{aligned}$$

The above expression can be conveniently decomposed in irreducible terms. To this end, one defines

$$G \equiv \delta g^C{}_C{}^{C'}{}_{C'}, \qquad G_{ABA'B'} \equiv \delta g_{(AB)(A'B')}.$$

If we also decompose $♀_{ABCA'}$ into irreducible parts, we get

$$♀_{AA'BC} = -\tfrac{1}{2}\nabla_{(A}{}^{B'}G_{BC)A'B'} + \tfrac{1}{8}\epsilon_{A(B}\nabla_{C)A'}G - \tfrac{1}{6}\epsilon_{A(B}\nabla^{DB'}G_{C)DA'B'}. \tag{24}$$

For future use we notice the following relations which follow from the decomposition in irreducible components of equation (24) and the reality of $\delta g_{ABA'B'}$:

$$\begin{aligned}
♀^B{}_{A'AB} &= -\tfrac{3}{16}\nabla_{AA'}G + \tfrac{1}{4}\nabla_{BB'}G_A{}^B{}_{A'}{}^{B'}, \\
\bar{♀}^{B'}{}_{AA'B'} &= ♀^B{}_{A'AB}, \\
\nabla_{BB'}G_{CDA'}{}^{B'} &= 2♀_{BA'CD} - 4♀^A{}_{A'(C|A|}\epsilon_{D)B} - \tfrac{1}{2}\epsilon_{(C|B|}\nabla_{D)A'}G, \\
\nabla_{BA'}G_A{}^B{}_{B'C'} &= 2\bar{♀}_{A'AB'C'} - 4♀^B{}_{(B'|AB|}\bar{\epsilon}_{C')A'} - \tfrac{1}{2}\bar{\epsilon}_{(B'|A'}\nabla_{A|C')}G.
\end{aligned}$$

We also define the field

$$\begin{aligned}
F^{AA'} &\equiv \nabla_{BB'}\delta g^{ABA'B'} - \tfrac{1}{2}\nabla^{AA'}\delta g^B{}_B{}^{B'}{}_{B'} \\
&= \nabla_{BB'}G^{ABA'B'} - \tfrac{1}{4}\nabla^{AA'}G.
\end{aligned} \tag{25}$$

In the next section we will see that this can be interpreted as a gauge source function for the linearised diffeomorphisms.

4.3 Diffeomorphism dependence

We will now briefly consider the dependence on diffeomorphisms. Let ϕ_λ be a one parameter group of diffeomorphisms generated by a vector field ξ^a and such that $g_{ab}[\lambda] = \phi^*_{-\lambda}\mathring{g}_{ab}$. The metrics in this family have the same geometric content and one readily finds that

$$\delta g_{ab} = \mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}. \tag{26}$$

Moreover, a further computation yields

$$\begin{aligned} \mho_{AA'BC} &= -\tfrac{1}{2}\nabla_{(C}{}^{B'}\nabla_{B)B'}\xi_{AA'} - \tfrac{1}{2}\nabla_{(C}{}^{B'}\nabla_{|AA'|}\xi_{B)B'}, \\ F^{AA'} &= \nabla_{BB'}\nabla^{BB'}\xi^{AA'} - 6\Lambda\xi^{AA'} + 2\Phi^{A}{}_{B}{}^{A'}{}_{B'}\xi^{BB'}. \end{aligned}$$

Given a general family of metrics $g_{ab}[\lambda]$, we can compute the field $F^{AA'}$ associated to the family. Given any $\widetilde{F}^{AA'}$, we can then solve the wave equation

$$\widetilde{F}^{AA'} - F^{AA'} = -6\Lambda\xi^{AA'} + 2\Phi^{A}{}_{B}{}^{A'}{}_{B'}\xi^{BB'} + \nabla_{BB'}\nabla^{BB'}\xi^{AA'}.$$

The solution $\xi^a = \sigma^a_{AA'}\xi^{AA'}$ to this equations will then give a one parameter group of diffeomorphisms ϕ_λ, such that $g_{ab}[\lambda] = \phi^*_{-\lambda}g_{ab}[\lambda]$ has the same geometric content, but with corresponding $\widetilde{F}^{AA'}$. With this observation, we can interpret (25) as a gauge source function for the linearised diffeomorphisms.

5 Variation of curvature

The purpose of this section is to compute the variation of the various spinorial components of the curvature tensor. As it will be seen below, the starting point of this computation is the commutator of covariant derivatives.

We start by computing the variation of

$$\square_{(AB}\kappa_{C)} = \nabla_{(A}{}^{A'}\nabla_{B|A'|}\kappa_{C)} = -\Psi_{ABCD}\kappa^D$$

for an arbitrary spinor κ_A. A direct calculation using the Leibnitz rule for the modified commutator ϑ gives

$$\begin{aligned} \Psi_{ABCD}\vartheta\kappa^D + \vartheta(\Psi_{ABCD})\kappa^D = &-\nabla_{(A}{}^{A'}\nabla_{B|A'|}\vartheta\kappa_{C)} + \tfrac{1}{4}G\nabla_{(A}{}^{A'}\nabla_{B|A'|}\kappa_{C)} \\ &- \tfrac{1}{2}G_{(A}{}^{DA'B'}\nabla_{B|A'}\nabla_{DB'|}\kappa_{C)} + \tfrac{1}{2}G_{(A}{}^{DA'B'}\nabla_{|DA'|}\nabla_{B|B'|}\kappa_{C)} \\ &+ \mho_{(A}{}^{A'}{}_{B}{}^{D}\nabla_{|DA'|}\kappa_{C)} + \bar{\mho}^{A'}{}_{(A|A'|}{}^{B'}\nabla_{B|B'|}\kappa_{C)} \\ &- \kappa^D\nabla_{(A}{}^{A'}\mho_{B|A'|C)D} + \tfrac{1}{8}\nabla_{(A}{}^{A'}G\nabla_{B|A'|}\kappa_{C)} \\ &+ \tfrac{1}{2}\nabla_{(A}{}^{A'}G_{B}{}^{D}{}_{|A'}{}^{B'}\nabla_{DB'|}\kappa_{C)} \\ = &\ \Psi_{ABCD}\vartheta\kappa^D - \tfrac{1}{4}G\Psi_{ABCD}\kappa^D - \kappa^D\nabla_{(A}{}^{A'}\mho_{B|A'|C)D} \\ &+ \tfrac{1}{2}\kappa^D G_{(AB}{}^{A'B'}\Phi_{C)DA'B'}. \end{aligned}$$

The above expression holds for all κ^A, and therefore we can conclude that

$$\vartheta(\Psi_{ABCD}) = -\tfrac{1}{4}G\Psi_{ABCD} - \nabla_{(A}{}^{A'}\mho_{B|A'|C)D} + \tfrac{1}{2}G_{(AB}{}^{A'B'}\Phi_{C)DA'B'}.$$

The symmetry of Ψ_{ABCD} can be used to simplify this last expression —the trace of the right hand side can be shown to vanish due to the commutators.

If we compute the variation of

$$\Phi_{BAA'B'}\kappa^A = -\nabla^A{}_{(A'}\nabla_{|A|B')}\kappa_B$$

we get

$$\begin{aligned} \Phi_{BAA'B'}\vartheta\kappa^A + \vartheta(\Phi_{BAA'B'})\kappa^A = &-\nabla^A{}_{(A'}\nabla_{|A|B')}\vartheta\kappa_B + \tfrac{1}{4}G\nabla^A{}_{(A'}\nabla_{|A|B')}\kappa_B \\ &- \tfrac{1}{2}G^{AC}{}_{(A'}{}^{C'}\nabla_{|A|B')}\nabla_{CC'}\kappa_B + \tfrac{1}{2}G^{AC}{}_{(A'}{}^{C'}\nabla_{|AC'}\nabla_{C|B')}\kappa_B \\ &+ \mho^A{}_{(A'|A}{}^{C}\nabla_{C|B')}\kappa_B + \bar{\mho}_{(A'}{}^{A}{}_{B')}{}^{C'}\nabla_{AC'}\kappa_B \\ &- \kappa^A\nabla^C{}_{(A'}\mho_{|C|B')BA} + \tfrac{1}{8}\nabla^A{}_{(A'}G\nabla_{|A|B')}\kappa_B \\ &+ \tfrac{1}{2}\nabla^A{}_{(A'}G_{|A|}{}^{C}{}_{B')}{}^{C'}\nabla_{CC'}\kappa_B \\ = &\ \Phi_{BAA'B'}\vartheta\kappa^A + G_{BAA'B'}\Lambda\kappa^A - \tfrac{1}{4}G\Phi_{BAA'B'}\kappa^A \\ &+ \tfrac{1}{2}G^{CD}{}_{A'B'}\Psi_{BACD}\kappa^A - \kappa^A\nabla^C{}_{(A'}\mho_{|C|B')BA}. \end{aligned}$$

The last relation holds for all κ^A, and therefore we can obtain an expression for $\vartheta\Phi_{ABA'B'}$.

Now, using the definition of $\Upsilon_{ABCA'}$, commuting derivatives and exploiting the irreducible decomposition of the various fields involved one gets

$$\begin{aligned}\nabla_{AA'}\Upsilon^{CA'}{}_{BC} = & -\tfrac{1}{2}G_B{}^{CA'B'}\Phi_{ACA'B'} - \tfrac{1}{2}G_A{}^{CA'B'}\Phi_{BCA'B'} \\ & + \nabla_{CA'}\Upsilon_A{}^{A'}{}_B{}^C + \epsilon_{AB}\nabla^{CA'}\Upsilon^D{}_{A'CD}.\end{aligned} \tag{27}$$

If we compute the variation of

$$\Lambda\kappa_A = \tfrac{1}{3}\nabla_{(A}{}^{A'}\nabla_{B)A'}\kappa^B$$

we get, after a lengthy computation, that

$$\begin{aligned}\Lambda\vartheta\kappa_A + \vartheta(\Lambda)\kappa_A = & \tfrac{1}{6}\kappa^B\nabla_{AA'}\Upsilon^{CA'}{}_{BC} - \tfrac{1}{6}\nabla_{AA'}\nabla_B{}^{A'}\vartheta\kappa^B + \tfrac{1}{24}G\nabla_{AA'}\nabla_B{}^{A'}\kappa^B \\ & + \tfrac{1}{6}\bar{\Upsilon}^{B'}{}_{BA'B'}\nabla_A{}^{A'}\kappa^B - \tfrac{1}{12}G_{BCA'B'}\nabla_A{}^{B'}\nabla^{CA'}\kappa^B - \tfrac{1}{48}\nabla_A{}^{A'}G\nabla_{BA'}\kappa^B \\ & - \tfrac{1}{6}\nabla_{BA'}\nabla_A{}^{A'}\vartheta\kappa^B + \tfrac{1}{24}G\nabla_{BA'}\nabla_A{}^{A'}\kappa^B + \tfrac{1}{6}\bar{\Upsilon}^{B'}{}_{AA'B'}\nabla_B{}^{A'}\kappa^B \\ & - \tfrac{1}{12}G_{ACA'B'}\nabla_B{}^{B'}\nabla^{CA'}\kappa^B + \tfrac{1}{48}\nabla_{AA'}\kappa_B\nabla^{BA'}G - \tfrac{1}{6}\kappa^B\nabla_{CA'}\Upsilon_A{}^{A'}{}_B{}^C \\ & - \tfrac{1}{6}\Upsilon_{AA'BC}\nabla^{CA'}\kappa^B - \tfrac{1}{6}\Upsilon_{BA'AC}\nabla^{CA'}\kappa^B + \tfrac{1}{12}\nabla_{AB'}G_{BCA'}{}^{B'}\nabla^{CA'}\kappa^B \\ & + \tfrac{1}{12}\nabla_{BB'}G_{ACA'}{}^{B'}\nabla^{CA'}\kappa^B + \tfrac{1}{12}G_{BCA'B'}\nabla^{CB'}\nabla_A{}^{A'}\kappa^B \\ & + \tfrac{1}{12}G_{ACA'B'}\nabla^{CB'}\nabla_B{}^{A'}\kappa^B \\ = & \Lambda\vartheta\kappa_A - \tfrac{1}{4}G\Lambda\kappa_A + \tfrac{1}{6}G_A{}^{CA'B'}\Phi_{BCA'B'}\kappa^B + \tfrac{1}{6}\kappa^B\nabla_{AA'}\Upsilon^{CA'}{}_{BC} \\ & - \tfrac{1}{6}\kappa^B\nabla_{CA'}\Upsilon_A{}^{A'}{}_B{}^C \\ = & \Lambda\vartheta\kappa_A - \tfrac{1}{4}G\Lambda\kappa_A + \tfrac{1}{12}G^{BCA'B'}\Phi_{BCA'B'}\kappa_A - \tfrac{1}{6}\kappa_A\nabla_{CA'}\Upsilon^{BA'}{}_B{}^C.\end{aligned}$$

In the last equality we have used the relation (27) and the irreducible decomposition of $G_A{}^{CA'B'}\Phi_{BCA'B'}$. From here we can deduce an expression for $\vartheta\Lambda$.

We summarise the discussion of this section in the following:

Theorem 2. *The modified variation of the curvature spinors is given by*

$$\begin{aligned}\vartheta\Psi_{ABCD} = & -\tfrac{1}{4}G\Psi_{ABCD} - \nabla_{(A}{}^{A'}\Upsilon_{B|A'|CD)} + \tfrac{1}{2}G_{(AB}{}^{A'B'}\Phi_{CD)A'B'}, \\ \vartheta\Phi_{ABA'B'} = & \, G_{ABA'B'}\Lambda - \tfrac{1}{4}G\Phi_{ABA'B'} + \tfrac{1}{2}G^{CD}{}_{A'B'}\Psi_{ABCD} - \nabla^C{}_{(A'}\Upsilon_{|C|B')AB}, \\ \vartheta\Lambda = & -\tfrac{1}{4}G\Lambda + \tfrac{1}{12}G^{BCA'B'}\Phi_{BCA'B'} - \tfrac{1}{6}\nabla_{CA'}\Upsilon^{BA'}{}_B{}^C.\end{aligned}$$

Remark 7. *For a pure gauge transformation* (26)*, we get after a lengthy but straightforward calculation using commutators, that*

$$\begin{aligned}\vartheta(\Lambda) = & (\mathcal{L}_\xi\Lambda), \\ \vartheta(\Phi_{AB}{}^{A'B'}) = & (\mathcal{L}_\xi\Phi)_{AB}{}^{A'B'} - \Phi^C{}_{(A}{}^{C'(A'}\nabla_{B)}{}^{B')}\xi_{CC'} - \Phi^C{}_{(A}{}^{C'(A'}\nabla_{|CC'|}\xi_{B)}{}^{B')}, \\ \vartheta(\Psi_{ABCD}) = & (\mathcal{L}_\xi\Psi)_{ABCD} - \Psi_{ABCD}\nabla^{FA'}\xi_{FA'},\end{aligned}$$

where [1]

$$\begin{aligned}(\mathcal{L}_\xi\Phi)_{AB}{}^{A'B'} \equiv & \, \xi^{CC'}\nabla_{CC'}\Phi_{AB}{}^{A'B'} + 2\Phi^C{}_{(A}{}^{C'(A'}\nabla_{B)}{}^{B')}\xi_{CC'}, \\ (\mathcal{L}_\xi\Psi)_{ABCD} \equiv & \, \xi^{FA'}\nabla_{FA'}\Psi_{ABCD} + 2\Psi_{(ABC}{}^F\nabla_{D)}{}^{A'}\xi_{FA'}.\end{aligned}$$

In this last calculation we have used the Bianchi identity in the form

$$\nabla^D{}_{A'}\Psi_{ABCD} = \nabla_{(A}{}^{B'}\Phi_{B)CA'B'} + \epsilon_{C(A}\nabla_{B)A'}\Lambda.$$

[1] The primed indices are moved up after the Lie derivative is taken to allow the symmetrizations to be written nicely.

6 Variations of space-spinor expressions

The analysis of Sections 3, 4 and 5 can be adapted to consider variations of spinorial fields in a space-spinor formalism. This formalism can be used to analyse variational problems in 3-dimensional Riemannian manifolds.

6.1 Basic formalism

In what follows, let $(\mathcal{S}, h_{ij})$ denote a 3-dimensional Riemannian manifold with negative-definite metric. On $(\mathcal{S}, h_{ij})$ we assume the existence of a spinor structure with an antisymmetric spinor ϵ_{AB}. In addition, we assume that the spinor structure is endowed with an Hermitian product. It follows from this assumption that there exists an Hermitian spinor $\varpi_{AA'}$ such given two spinors ξ_A and η_B the Hermitian inner product can be expressed as

$$\xi_A \hat{\eta}^A \equiv \varpi_{AA'} \bar{\eta}^{A'} \xi^A.$$

The spinor $\hat{\eta}^A$ defined by the above relation is called the Hermitian conjugate of η^A.

Let $e_{\mathbf{k}}{}^l$, $\omega^{\mathbf{k}}{}_l$ denote, respectively, an orthonormal frame and coframe of $(\mathcal{S}, h_{ij})$ and let $\epsilon^{\mathbf{A}}{}_B$ denote a normalised spin dyad such that the components of ϵ_{AB} and $\varpi_{AA'}$ are given, respectively, by

$$\epsilon_{\mathbf{AB}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varpi_{\mathbf{AA'}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The transformations of the spin dyad respecting the above expressions is given by $SU(2,\mathbb{C})$ matrices $O_{\mathbf{A}}{}^{\mathbf{B}}$.

The correspondence between spatial tensors and spinors is realised by the *spatial Infeld-van der Waerden symbols* $\sigma_{\mathbf{k}}{}^{\mathbf{AB}}$ and $\sigma^{\mathbf{k}}{}_{\mathbf{AB}}$. Given an arbitrary $v^k \in T\mathcal{S}$ and $\beta_k \in T^*\mathcal{S}$ one has that

$$v^{\mathbf{k}} \mapsto v^{\mathbf{AB}} = v^{\mathbf{k}} \sigma_{\mathbf{k}}{}^{\mathbf{AB}}, \qquad \beta_{\mathbf{k}} \mapsto \beta_{\mathbf{AB}} = \beta_{\mathbf{k}} \sigma^{\mathbf{k}}{}_{\mathbf{AB}},$$

where

$$v^{\mathbf{k}} \equiv v^k \omega^{\mathbf{k}}{}_k, \qquad \beta_{\mathbf{k}} \equiv \beta_k e_{\mathbf{k}}{}^k.$$

In more explicit terms, the correspondence is

$$(v^{\mathbf{1}}, v^{\mathbf{2}}, v^{\mathbf{3}}) \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} -v^{\mathbf{1}} - \mathrm{i}v^{\mathbf{2}} & v^{\mathbf{3}} \\ v^{\mathbf{3}} & v^{\mathbf{1}} - \mathrm{i}v^{\mathbf{2}} \end{pmatrix}, \qquad (\beta_{\mathbf{1}}, \beta_{\mathbf{2}}, \beta_{\mathbf{3}}) \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} -\beta_{\mathbf{1}} + \mathrm{i}\beta_{\mathbf{2}} & \beta_{\mathbf{3}} \\ \beta_{\mathbf{3}} & \beta_{\mathbf{1}} + \mathrm{i}\beta_{\mathbf{2}} \end{pmatrix}.$$

From these, we define the spatial soldering form to be

$$\sigma_k{}^{AB} \equiv \omega^{\mathbf{l}}{}_k \epsilon_{\mathbf{C}}{}^A \epsilon_{\mathbf{D}}{}^B \sigma_{\mathbf{l}}{}^{\mathbf{CD}}, \tag{29a}$$

$$\sigma^k{}_{AB} \equiv \epsilon^{\mathbf{C}}{}_A \epsilon^{\mathbf{D}}{}_B e_{\mathbf{l}}{}^k \sigma^{\mathbf{l}}{}_{\mathbf{CD}}. \tag{29b}$$

As we allow the spinor and tensor frames to be independent, the soldering form will therefore be frame dependent. However, we will always have the universal relations

$$\sigma_k{}^{CD} \sigma^l{}_{CD} = \delta_k{}^l, \tag{30a}$$

$$h_{kl} = \sigma_k{}^{AB} \sigma_l{}^{CD} \epsilon_{CA} \epsilon_{DB}. \tag{30b}$$

The Hermitian conjugate of

$$\phi_A = \phi_{\mathbf{0}} \epsilon^{\mathbf{0}}{}_A + \phi_{\mathbf{1}} \epsilon^{\mathbf{1}}{}_A$$

is given by

$$\hat{\phi}_A = -\bar{\phi}_{\mathbf{1}'} \epsilon^{\mathbf{0}}{}_A + \bar{\phi}_{\mathbf{0}'} \epsilon^{\mathbf{1}}{}_A.$$

It clearly follows that

$$\hat{\hat{\phi}}_A = -\phi_A.$$

The Hermitian conjugation can be extended to higher valence space spinors by requiring that the conjugate of a product equals the product of conjugates. We also get

$$\hat{\hat{\mu}}_{A_1 \cdots A_k} = (-1)^k \hat{\mu}_{A_1 \cdots A_k}.$$

Furthermore, it is important to note

$$\hat{\epsilon}_{AB} = \epsilon_{AB}, \qquad \hat{\sigma}^a{}_{AB} = -\sigma^a{}_{AB}. \tag{31}$$

6.2 Basic variational formulae

As in the case of standard spacetime spinors, we can compute the variations of the frames and the inverse metrics from the relations

$$\begin{aligned}\delta(e_{\mathbf{i}}{}^{l}) =& -e_{\mathbf{i}}{}^{j}e_{\mathbf{k}}{}^{l}\delta\omega^{\mathbf{k}}{}_{j},\\ \delta(\epsilon_{\mathbf{A}}{}^{D}) =& -\epsilon_{\mathbf{A}}{}^{B}\epsilon_{\mathbf{C}}{}^{D}\delta\epsilon^{\mathbf{C}}{}_{B},\\ \delta(h^{kl}) =& -(\delta h_{ij})h^{ik}h^{jl},\\ \delta(\epsilon^{CD}) =& -\delta\epsilon_{AB}\epsilon^{AC}\epsilon^{BD}.\end{aligned}$$

Likewise, from the relation (30a) we get

$$\delta(\sigma^{l}{}_{AB}) = -\sigma^{k}{}_{AB}\sigma^{l}{}_{CD}\delta\sigma_{k}{}^{CD}.$$

We can also split the variation of the coframes in terms of the variation of the metric and spin metric and gauge pieces

$$\begin{aligned}\delta\omega^{\mathbf{m}}{}_{a} =& -e_{\mathbf{h}}{}^{b}h^{\mathbf{hm}}T_{ab} + \tfrac{1}{2}e_{\mathbf{h}}{}^{b}h^{\mathbf{hm}}\delta h_{ab},\\ \delta\epsilon^{\mathbf{P}}{}_{A} =& -\epsilon_{\mathbf{H}}{}^{B}\epsilon^{\mathbf{HP}}S_{AB} - \tfrac{1}{2}\epsilon_{\mathbf{H}}{}^{B}\epsilon^{\mathbf{HP}}\delta\epsilon_{AB},\end{aligned}$$

where the tensor and spinor frame gauge fields are

$$T_{ab} \equiv h_{\mathbf{cd}}\omega^{\mathbf{d}}{}_{[a}\delta\omega^{\mathbf{c}}{}_{b]}, \qquad S_{AB} \equiv \omega^{\mathbf{D}}{}_{(A}\delta\omega^{\mathbf{C}}{}_{B)}\epsilon_{\mathbf{CD}}.$$

A calculation following the same principles as for the spacetime version starting from the relation (29a) gives the variation of the spatial soldering form:

$$\delta\sigma_{k}{}^{AB} = -T_{k}{}^{l}\sigma_{l}{}^{AB} + \tfrac{1}{2}\sigma^{lAB}\delta h_{kl} - 2\sigma_{k}{}^{(A|C|}S^{B)}{}_{C} + \sigma_{k}{}^{(A|C|}\delta\epsilon^{B)}{}_{C}.$$

The irreducible parts are given by

$$\begin{aligned}\sigma^{k(CD}\delta\sigma_{k}{}^{AB)} =& \tfrac{1}{2}\delta h^{(ABCD)},\\ \sigma^{k(C}{}_{B}\delta\sigma_{k}{}^{A)B} =& T^{AC} - 2S^{AC},\\ \sigma^{k}{}_{CD}\delta\sigma_{k}{}^{CD} =& \tfrac{1}{2}\delta h^{CD}{}_{CD} + \tfrac{3}{2}\delta\epsilon^{C}{}_{C},\end{aligned}$$

where

$$\begin{aligned}T_{AB} \equiv& T_{kl}\sigma^{k}{}_{A}{}^{C}\sigma^{l}{}_{BC},\\ \delta h_{ABCD} \equiv& \sigma^{k}{}_{AB}\sigma^{l}{}_{CD}\delta h_{kl}.\end{aligned}$$

We can now use this to see how the variation of vectors and covectors in space-spinor and tensor form differ:

$$\begin{aligned}\sigma_{k}{}^{AB}\delta\zeta^{k} =& \delta(\zeta^{AB}) - \tfrac{1}{2}\delta\epsilon^{C}{}_{C}\zeta^{AB} - \tfrac{1}{2}\delta h^{AB}{}_{CD}\zeta^{CD} + T^{(A|C|}\zeta^{B)}{}_{C} - 2S^{(A|C|}\zeta^{B)}{}_{C},\\ \sigma^{k}{}_{AB}\delta\xi_{k} =& \delta(\xi_{AB}) + \tfrac{1}{2}\delta\epsilon^{C}{}_{C}\xi_{AB} + \tfrac{1}{2}\delta h_{ABCD}\xi^{CD} + T_{(A}{}^{C}\xi_{B)C} - 2S_{(A}{}^{C}\xi_{B)C},\end{aligned}$$

where $\zeta^{k} = \sigma^{k}{}_{CD}\zeta^{CD}$ and $\xi_{k} = \sigma_{k}{}^{CD}\zeta_{CD}$. This leads us to define a modified variation that cancels the gauge terms and the variation of the spin metric.

Definition 2. *For valence 1 space spinors we define the modified variation operator ϑ via*

$$\begin{aligned}\vartheta(\phi_{A}) \equiv& \delta(\phi_{A}) + \tfrac{1}{4}\delta\epsilon^{B}{}_{B}\phi_{A} + \tfrac{1}{2}T_{A}{}^{B}\phi_{B} - S_{A}{}^{B}\phi_{B},\\ \vartheta(\phi^{A}) \equiv& \delta(\phi^{A}) - \tfrac{1}{4}\delta\epsilon^{B}{}_{B}\phi^{A} - \tfrac{1}{2}T^{A}{}_{B}\phi^{B} + S^{A}{}_{B}\phi^{B}.\end{aligned}$$

These relations extend to higher valence spinors via the Leibnitz rule.

In the same way as for the spacetime variations, we get a relation between ϑ and spin frame component variation:

$$\vartheta\phi_A = \epsilon^{\mathbf{B}}{}_A\delta(\phi_{\mathbf{B}}) + \tfrac{1}{2}T_A{}^B\phi_B. \tag{32}$$

The reality of T_{ab} and (31) gives

$$\widehat{T}_{AB} = T_{AB}.$$

Expanding the frame index in equation (32) and taking Hermitian conjugate yields

$$\begin{aligned}\widehat{\vartheta\phi}_A &= \epsilon^{\mathbf{1}}{}_A\delta(\bar{\phi}_{\mathbf{0}'}) - \epsilon^{\mathbf{0}}{}_A\delta(\bar{\phi}_{\mathbf{1}'}) + \tfrac{1}{2}T_A{}^B\hat{\phi}_B\\ &= \epsilon^{\mathbf{0}}{}_A\delta(\hat{\phi}_{\mathbf{0}}) + \epsilon^{\mathbf{1}}{}_A\delta(\hat{\phi}_{\mathbf{1}}) + \tfrac{1}{2}T_A{}^B\hat{\phi}_B\\ &= \vartheta(\hat{\phi})_A.\end{aligned}$$

Hence, the operation of Hermitian conjugation and the modified variation ϑ commute.

6.3 Variations of the spatial connection

Let $\mathcal{R}_{ABCD}$ denote the space spinor version of the trace free Ricci tensor, and let $\mathcal{R}$ be the Ricci scalar. Define

$$\begin{aligned}H^{ABCD} &\equiv \delta h^{(ABCD)},\\ H &\equiv \delta h_{AB}{}^{AB},\\ \Qoppa_{ABCD} &\equiv -\tfrac{1}{2}D_{(C}{}^F\delta h_{D)FAB},\\ F^{AB} &\equiv -\tfrac{1}{2}D^{AB}\delta h^{CD}{}_{CD} + D_{CD}\delta h^{ABCD}.\end{aligned}$$

Similarly to the case of spacetime spinors, we can compute the variation of a covariant derivative.

Theorem 3. *The variation of a covariant space-spinor derivative is given by*

$$\vartheta(D_{AB}\kappa_C) = D_{AB}\vartheta\kappa_C + \Qoppa_{ABCD}\kappa^D - \tfrac{1}{2}\delta h_{ABDF}D^{DF}\kappa_C.$$

We also get

$$\begin{aligned}\Qoppa_A{}^C{}_{BC} &= -\tfrac{1}{6}D_{AB}H + \tfrac{1}{4}D_{CD}H_{AB}{}^{CD},\\ F_{AB} &= -\tfrac{1}{6}D_{AB}H + D_{CD}H_{AB}{}^{CD},\\ D_{DF}H_{ABC}{}^F &= 2\Qoppa_{(ABC)D} + 2\epsilon_{D(A}\Qoppa_B{}^F{}_{C)F} + \tfrac{1}{2}\epsilon_{D(A}D_{BC)}H.\end{aligned}$$

6.4 Diffeomorphism dependence

To analyse the dependence of the formalism on diffeomorphisms, we proceed in the same way as in Section 4.3. Accordingly, let ϕ_λ be a one parameter group of diffeomorphisms generated by a vector field ξ^a. Now, let $h_{ab}[\lambda] = \phi^*_{-\lambda}\mathring{h}_{ab}$. All members of the family $h_{ab}[\lambda]$ will have the same geometric content and we get

$$\delta h_{ab} = \mathcal{L}_\xi h_{ab} = 2D_{(a}\xi_{b)}. \tag{33}$$

Moreover, one has that

$$\begin{aligned}\Qoppa_{ABCD} &= -\tfrac{1}{2}D_{(C}{}^FD_{D)F}\xi_{AB} - \tfrac{1}{2}D_{(C}{}^FD_{|AB|}\xi_{D)F},\\ F^{AB} &= D_{CD}D^{CD}\xi^{AB} - \tfrac{1}{3}\mathcal{R}\xi^{AB} - \mathcal{R}^{AB}{}_{CD}\xi^{CD}.\end{aligned}$$

Again, we see that F^{AB} can be interpreted as a gauge source function for the linearised diffeomorphisms, but this time one needs to solve an elliptic equation instead of a wave equation to obtain ξ^{AB} from F^{AB}.

6.5 Variations of the spatial curvature

By computing the variation of the commutator relations

$$\mathcal{R}\kappa_A = 8D_{(A}{}^{C}D_{B)C}\kappa^B, \tag{34a}$$
$$\mathcal{R}_{ABCD}\kappa^D = 2D_{(A}{}^{D}D_{B|D|}\kappa_{C)}, \tag{34b}$$

we get, after calculations similar to those carried out in the spacetime case, the variation of the curvature.

Theorem 4. *The variation of the spatial curvature spinors is given by*

$$\begin{aligned}\vartheta(\mathcal{R}) = {}& -\tfrac{1}{3}H\mathcal{R} - H^{BCDF}\mathcal{R}_{BCDF} - 4D_{CD}\Omega^{BC}{}_{B}{}^{D},\\ \vartheta(\mathcal{R}_{ABCD}) = {}& -\tfrac{1}{12}H_{ABCD}\mathcal{R} - \tfrac{1}{3}H\mathcal{R}_{ABCD} + 2D_{(A}{}^{F}\Omega_{B|F|CD)} + \tfrac{1}{2}H_{(AB}{}^{FH}\mathcal{R}_{CD)FH}.\end{aligned}$$

Proof. Computing the variation of relation (34a) gives

$$\begin{aligned}\mathcal{R}\vartheta\kappa_A + \vartheta(\mathcal{R})\kappa_A = {}& -4D_{AC}D_B{}^{C}\vartheta\kappa^B + \tfrac{4}{3}HD_{AC}D_B{}^{C}\kappa^B + 4\Omega_B{}^{D}{}_{CD}D_A{}^{C}\kappa^B + 4\kappa^B D_{AD}\Omega^{CD}{}_{BC}\\ & - 2H_{BCDF}D_A{}^{F}D^{CD}\kappa^B - \tfrac{2}{3}D_A{}^{B}HD_{BC}\kappa^C - 4D_{BC}D_A{}^{C}\vartheta\kappa^B\\ & + \tfrac{4}{3}HD_{BC}D_A{}^{C}\kappa^B + 4\Omega_A{}^{D}{}_{CD}D_B{}^{C}\kappa^B - 2H_{ACDF}D_B{}^{F}D^{CD}\kappa^B\\ & + \tfrac{2}{3}D_{AC}\kappa_B D^{BC}H - 4\kappa^B D_{CD}\Omega_A{}^{C}{}_{B}{}^{D} - 4\Omega_{ACBD}D^{CD}\kappa^B - 4\Omega_{BCAD}D^{CD}\kappa^B\\ & + 2D_{AF}H_{BCD}{}^{F}D^{CD}\kappa^B + 2D_{BF}H_{ACD}{}^{F}D^{CD}\kappa^B + 2H_{BCDF}D^{DF}D_A{}^{C}\kappa^B\\ & + 2H_{ACDF}D^{DF}D_B{}^{C}\kappa^B\\ = {}& \mathcal{R}\vartheta\kappa_A - \tfrac{1}{3}H\mathcal{R}\kappa_A - 2H_A{}^{CDF}\mathcal{R}_{BCDF}\kappa^B + 4\kappa^B D_{AD}\Omega^{CD}{}_{BC}\\ & - 4\kappa^B D_{CD}\Omega_A{}^{C}{}_{B}{}^{D}\\ = {}& \mathcal{R}\vartheta\kappa_A - \tfrac{1}{3}H\mathcal{R}\kappa_A - H^{BCDF}\mathcal{R}_{BCDF}\kappa_A - 4\kappa_A D_{CD}\Omega^{BC}{}_{B}{}^{D}.\end{aligned}$$

Computing the variation of relation (34b) gives

$$\begin{aligned}\mathcal{R}_{ABCD}\vartheta\kappa^D + \vartheta(\mathcal{R}_{ABCD})\kappa^D = {}& 2D_{(A}{}^{D}D_{B|D|}\vartheta\kappa_{C)} - \tfrac{2}{3}HD_{(A}{}^{D}D_{B|D|}\kappa_{C)}\\ & + H_{(A}{}^{DFH}D_{B|D}D_{FH|}\kappa_{C)} - H_{(A}{}^{DFH}D_{|DF|}D_{B|H|}\kappa_{C)}\\ & - 2\Omega_{(A}{}^{D}{}_{B}{}^{F}D_{|DF|}\kappa_{C)} - 2\Omega_{(A}{}^{D}{}_{|D|}{}^{F}D_{B|F|}\kappa_{C)}\\ & + 2\kappa^D D_{(A}{}^{F}\Omega_{B|F|C)D} - \tfrac{1}{3}D_{(A}{}^{D}HD_{B|D|}\kappa_{C)}\\ & - D_{(A}{}^{D}H_{B|D}{}^{FH}D_{FH|}\kappa_{C)}\\ = {}& \mathcal{R}_{ABCD}\vartheta\kappa^D - \tfrac{1}{12}H_{ABCD}\mathcal{R}\kappa^D - \tfrac{1}{3}H\mathcal{R}_{ABCD}\kappa^D\\ & + 2\kappa^D D_{(A}{}^{F}\Omega_{B|F|C)D} + \tfrac{1}{2}\kappa^D H_{(AB}{}^{FH}\mathcal{R}_{C)DFH}.\end{aligned}$$

□

Remark 8. *For a pure gauge transformation* (33)*, we get*

$$\begin{aligned}\vartheta(\mathcal{R}) &= \mathcal{L}_\xi\mathcal{R},\\ \vartheta(\mathcal{R}_{ABCD}) &= \mathcal{L}_\xi\mathcal{R}_{ABCD} - \mathcal{R}_{(AB}{}^{FH}D_{CD)}\xi_{FH} - \mathcal{R}_{(AB}{}^{FH}D_{|FH|}\xi_{CD)},\end{aligned}$$

where

$$\mathcal{L}_\xi\mathcal{R}_{ABCD} = \xi^{FH}D_{FH}\mathcal{R}_{ABCD} + 2\mathcal{R}_{(AB}{}^{FH}D_{CD)}\xi_{FH}.$$

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A Rotations

The purpose of this appendix is to discuss some issues related to the gauge in the frame and spin dyad formalisms.

A.1 Lorentz transformations

As it is well known, the metric g_{ab} is not determined in a unique way by the orthonormal coframe $\omega^{\mathbf{a}}{}_a$. Any other coframe related to $\omega^{\mathbf{a}}{}_a$ by means of a Lorentz transformation —i.e. a matrix $(\Lambda^{\mathbf{a}}{}_{\mathbf{b}})$ such that

$$\eta_{\mathbf{ab}}\Lambda^{\mathbf{a}}{}_{\mathbf{c}}\Lambda^{\mathbf{b}}{}_{\mathbf{d}} = \eta_{\mathbf{cd}}. \tag{35}$$

It follows that $\grave{\omega}^{\mathbf{a}}{}_a \equiv \Lambda^{\mathbf{a}}{}_{\mathbf{b}}\omega^{\mathbf{b}}{}_a$ is also orthonormal with respect to g_{ab} and one can write $g_{ab} = \eta_{\mathbf{ab}}\grave{\omega}^{\mathbf{a}}{}_a\grave{\omega}^{\mathbf{b}}{}_b$. The associated orthonormal frame is $\grave{e}_{\mathbf{a}}{}^a = \Lambda_{\mathbf{a}}{}^{\mathbf{b}}e_{\mathbf{b}}{}^a$ with $(\Lambda_{\mathbf{a}}{}^{\mathbf{b}}) \equiv (\Lambda^{\mathbf{a}}{}_{\mathbf{b}})^{-1}$ where the last expression is a relation between matrices.

The discussion in the previous paragraph can be extended to include spinors. Making use of the Infeld-van der Waerden symbols, equation (35) can be rewritten as

$$\epsilon_{\mathbf{AB}}\epsilon_{\mathbf{A'B'}}\Lambda^{\mathbf{AA'}}{}_{\mathbf{CC'}}\Lambda^{\mathbf{BB'}}{}_{\mathbf{DD'}} = \epsilon_{\mathbf{CD}}\epsilon_{\mathbf{C'D'}},$$

with $\Lambda^{\mathbf{AA'}}{}_{\mathbf{CC'}} \equiv \sigma_{\mathbf{a}}{}^{\mathbf{AA'}}\sigma^{\mathbf{c}}{}_{\mathbf{CC'}}\Lambda^{\mathbf{a}}{}_{\mathbf{c}}$. It can be shown that the spinorial components $\Lambda^{\mathbf{AA'}}{}_{\mathbf{CC'}}$ can be decomposed as

$$\Lambda^{\mathbf{AA'}}{}_{\mathbf{CC'}} = \Lambda^{\mathbf{A}}{}_{\mathbf{C}}\bar{\Lambda}^{\mathbf{A'}}{}_{\mathbf{C'}},$$

where $(\Lambda^{\mathbf{A}}{}_{\mathbf{C}})$ is a $SL(2,\mathbb{C})$ matrix. The latter naturally induces a change of spinorial basis via the relations

$$\grave{\epsilon}_{\mathbf{A}}{}^A = \Lambda_{\mathbf{A}}{}^{\mathbf{B}}\epsilon_{\mathbf{B}}{}^A, \qquad \grave{\epsilon}^{\mathbf{A}}{}_A = \Lambda^{\mathbf{A}}{}_{\mathbf{B}}\epsilon^{\mathbf{B}}{}_A,$$

with $(\Lambda_{\mathbf{A}}{}^{\mathbf{B}}) \equiv (\Lambda^{\mathbf{A}}{}_{\mathbf{B}})^{-1}$. Crucially, one has that

$$\epsilon_{\mathbf{AB}} = \Lambda^{\mathbf{C}}{}_{\mathbf{A}}\Lambda^{\mathbf{D}}{}_{\mathbf{B}}\epsilon_{\mathbf{CD}}, \qquad \epsilon^{\mathbf{AB}} = \Lambda_{\mathbf{C}}{}^{\mathbf{A}}\Lambda_{\mathbf{D}}{}^{\mathbf{B}}\epsilon^{\mathbf{CD}}.$$

A.2 $O(3)$-rotations

Given a 3-dimensional negative-definite Riemannian metric h_{ij} and an associated orthonormal coframe $\omega^{\mathbf{i}}{}_k$ one has that

$$h_{ij} = -\delta_{\mathbf{ij}}\omega^{\mathbf{i}}{}_i\omega^{\mathbf{j}}{}_j.$$

Any other coframe $\grave{\omega}^{\mathbf{i}}{}_k$ related to the coframe $\omega^{\mathbf{i}}{}_k$ through the relation $\grave{\omega}^{\mathbf{i}}{}_k = O_{\mathbf{j}}{}^{\mathbf{i}}\omega^{\mathbf{j}}{}_k$, where $(O_{\mathbf{j}}{}^{\mathbf{i}})$ is a $O(3)$-matrix, gives rise to the same metric. The defining condition for $(O^{\mathbf{i}}{}_{\mathbf{j}})$ can be expressed as

$$\delta_{\mathbf{ij}} = \delta_{\mathbf{kl}}O_{\mathbf{i}}{}^{\mathbf{k}}O_{\mathbf{j}}{}^{\mathbf{l}}.$$

A direct calculation using the definition of the Hermitian product shows that the changes of spin dyad preserving the Hermitian structure induced by the Hermitian spinor $\varpi_{AA'}$ are of the form $\grave{\epsilon}_{\mathbf{A}}{}^A = O_{\mathbf{A}}{}^{\mathbf{B}}\epsilon_{\mathbf{B}}{}^A$ where $(O_{\mathbf{A}}{}^{\mathbf{B}})$ are $SU(2,\mathbb{C})$ matrices. As $SU(2,\mathbb{C})$ is a subgroup of $SL(2,\mathbb{C})$, one has that $\epsilon_{\mathbf{AB}} = O_{\mathbf{A}}{}^{\mathbf{C}}O_{\mathbf{B}}{}^{\mathbf{D}}\epsilon_{\mathbf{CD}}$. The matrices $(O_{\mathbf{j}}{}^{\mathbf{i}})$ and $(O_{\mathbf{A}}{}^{\mathbf{B}})$ are related to each other via the spatial Infeld-van der Waerden symbols:

$$O_{\mathbf{i}}{}^{\mathbf{j}} = \sigma_{\mathbf{i}}{}^{\mathbf{AB}}\sigma^{\mathbf{j}}{}_{\mathbf{CD}}O_{\mathbf{A}}{}^{\mathbf{C}}O_{\mathbf{B}}{}^{\mathbf{D}}.$$

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