THERE IS NO Sz(8) IN THE MONSTER

ROBERT A. WILSON

Abstract. As a contribution to an eventual solution of the problem of the determination of the maximal subgroups of the Monster we show that there is no subgroup isomorphic to Sz(8). This also completes the determination of exactly which simple groups are subgroups of which of the 26 sporadic simple groups. The proof is largely, though not entirely, computer-free.

1. Introduction

The Fischer–Griess Monster group $\mathbb{M}$ is the largest of the 26 sporadic simple groups, and was first constructed by Griess [6] in 1982. A simplified construction along the same general lines was given by Conway [2].

One of the major problems in group theory today is that of classifying the maximal subgroups of the finite simple groups and their automorphism groups (see, for example, [1]). Much work has been done over many years attempting to determine the maximal subgroups of $\mathbb{M}$, but it is still the only sporadic group whose maximal subgroups are not completely classified (see [23] and references therein).

The maximal p-local subgroups of the Monster were classified in [22, 15, 16], and much theoretical work on non-local subgroups was accomplished in [17, 18]. Following successful computer constructions of the Monster [14, 8] other techniques became available, and further progress was made [9, 11, 7, 19, 26, 27], including discovery of five previously unknown maximal subgroups, isomorphic to $\mathrm{PSL}_2(71)$, $\mathrm{PSL}_2(59)$, $\mathrm{PSL}_2(41)$, $\mathrm{PGL}_2(29)$, $\mathrm{PGL}_2(19)$.

The cases left open by this previous work are possible maximal subgroups with socle isomorphic to one of the following simple groups:

$$\mathrm{PSL}_2(8), \mathrm{PSL}_2(13), \mathrm{PSL}_2(16), \mathrm{PSU}_3(4), \mathrm{PSU}_3(8), \mathrm{Sz}(8).$$

Of these, $\mathrm{PSL}_2(8)$ and $\mathrm{PSL}_2(16)$ have been classified in unpublished work of P. E. Holmes. The case of Sz(8) is particularly interesting because it is not yet known whether Sz(8) is a subgroup of the Monster at all. Indeed, this is the last remaining case of the question of which particular simple groups are subgroups of one of the sporadic simple groups.

Throughout this paper, $\mathbb{M}$ denotes the Monster, and S denotes a subgroup of $\mathbb{M}$, isomorphic to Sz(8). The notation of the Atlas [3] is generally used for group names and structures, occasionally replaced by more traditional names as in [23]. In addition, $B \cong 2^{3+3}{:}7$ denotes the Borel subgroup of S.

The main result of this paper is the following.

Theorem 1. *There is no subgroup isomorphic to* Sz(8) *in the Monster sporadic simple group* $\mathbb{M}$.

Date: Version of 24 August 2015.

Table 1. The known maximal subgroups of the Monster

$2{\cdot}\mathbb{B}, 2^{2.2}E_6(2){:}S_3, 2^{10+16}{\cdot}\mathrm{P}\Omega_{10}^{+}(2),$
$2^{1+24}{\cdot}\mathrm{Co}_1, 2^{2+11+22}{\cdot}(S_3 \times \mathrm{M}_{24}), 2^{3+6+12+18}(\mathrm{L}_3(2) \times 3S_6), 2^{5+10+20}(\mathrm{L}_5(2) \times S_3),$
$3{\cdot}\mathrm{Fi}_{24}, (3^2{:}2 \times \mathrm{P}\Omega_8^{+}(3)){\cdot}S_4, 3^8{\cdot}\mathrm{P}\Omega_8^{-}(3){\cdot}2, S_3 \times \mathrm{Th},$
$3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2, 3^{2+5+10}{\cdot}(2S_4 \times \mathrm{M}_{11}), 3^{3+2+6+6}{:}(\mathrm{L}_3(3) \times SD_{16}),$
$(D_{10} \times \mathrm{HN}){\cdot}2, (5^2{:}4{\cdot}2^2 \times \mathrm{U}_3(5)){:}S_3, 5^4{:}(3 \times \mathrm{SL}_2(25)){:}2,$
$5^{1+6}{:}2\mathrm{J}_2.4, 5^{2+2+4}{:}(S_3 \times \mathrm{GL}_2(5)), 5^{3+3}{\cdot}(2 \times \mathrm{SL}_3(5)),$
$(7{:}3 \times \mathrm{He}){:}2, (7^2{:}(3 \times 2A_4) \times \mathrm{L}_2(7)).2, 7^{1+4}{:}(3 \times 2{\cdot}S_7), 7^{2+1+2}{:}\mathrm{GL}_2(7), 7^2{:}\mathrm{SL}_2(7),$
$11^2{:}(5 \times 2{\cdot}A_5), (13{:}6 \times \mathrm{L}_3(3)){\cdot}2, 13^{1+2}{:}(3 \times 4S_4), 13^2{:}2{\cdot}\mathrm{L}_2(13).4, 41{:}40,$
$(A_5 \times A_{12}){:}2, A_6^3{\cdot}(2 \times S_4), (A_7 \times A_5^2){:}D_8, (A_5 \times \mathrm{U}_3(8)){:}2, S_5^3{:}S_3,$
$(\mathrm{L}_2(11) \times \mathrm{M}_{12}){:}2, \mathrm{L}_2(11)^2{:}4, \mathrm{M}_{11} \times A_6{\cdot}2^2, (\mathrm{L}_3(2) \times \mathrm{S}_4(4)){:}4,$
$\mathrm{L}_2(19){:}2, \mathrm{L}_2(29){:}2, \mathrm{L}_2(41), \mathrm{L}_2(59), \mathrm{L}_2(71)$

The structure of the proof is as follows. First we prove the following.

Theorem 2. *If $B \cong 2^{3+3}{:}7$ is a subgroup of $\mathbb{M}$ isomorphic to the Borel subgroup of* Sz(8)*, then B lies in one of the maximal subgroups M_1 of shape $2^{1+24}{\cdot}\mathrm{Co}_1$ or M_2 of shape $2^3.2^6.2^{12}.2^{18}.(\mathrm{PSL}_3(2) \times 3S_6)$.*

Then we look more closely at the M_1 case and prove the following.

Theorem 3. *The Conway group* Co_1 *does not contain a subgroup isomorphic to the Borel subgroup of* Sz(8)*.*

This reduces the M_1 case to a classification of certain subgroups of 2^{1+24}, which yields exactly three classes of 2^3 which might lie in B. We then show that two of the three cases do not in fact extend to a copy of B. The other case may extend to B, but not to S. First we show that the M_2 case reduces to this last M_1 case. Then this possibility is eliminated using a small computation of orbits in the Held group to show that any group generated by the subgroup $2^3{:}7$ and an involution inverting the 7-element has non-trivial centralizer.

2. Locating the Borel subgroup

In this section, we consider all possibilities for known maximal subgroups of $\mathbb{M}$ which could contain a subgroup $B \cong 2^{3+3}{:}7$ extending to Sz(8) in $\mathbb{M}$. It is shown in [15, 16] that every 2-local subgroup of the Monster is contained in one of the known maximal subgroups. These papers do not however contain the stronger assertion that every 2-local subgroup of the Monster is contained in one of the known 2-local maximal subgroups. (That is, they classify 2-local maximal subgroups, not maximal 2-local subgroups.) We therefore need first of all to consider the other known maximal subgroups. A list of 43 of the currently known 44 classes of maximal subgroups can be found in Table 5.6 of [23]: the subgroup $\mathrm{PSL}_2(41)$ found in [19] was at that time thought not to exist. We reproduce the list here for convenience.

Lemma 1. *Every subgroup $B \cong 2^{3+3}{:}7$ of $\mathbb{M}$ lies in one of the known 2-local maximal subgroups.*

Proof. It is easy to see that B cannot lie in any of the known non-local subgroups. Most of the p-local subgroups for p odd are easy to eliminate, and we quickly reduce to those whose non-abelian composition factors are HN, Fi_{24}', Th, $\mathrm{P}\Omega_8^+(3)$ or $\mathrm{P}\Omega_8^-(3)$. In the case of HN, the subgroup lies in $2^3.2^2.2^6.\mathrm{PSL}_3(2)$, which embeds into $2^3.2^6.2^{12}.2^{18}.\mathrm{PSL}_3(2)$ in the Monster. In the case of Th, it either centralizes an involution, or lies in $2^5{}^{\cdot}\mathrm{PSL}_5(2)$, and in either case lies in a 2-local maximal subgroup of $\mathbb{M}$.

The group $\mathrm{P}\Omega_8^-(3)$ reduces to $\mathrm{P}\Omega_7(3)$, which does not contain a subgroup isomorphic to B. Similarly $\mathrm{P}\Omega_8^+(3)$ reduces to either $\mathrm{P}\Omega_7(3)$ or $\mathrm{P}\Omega_8^+(2)$, and the latter reduces to 2^6A_8. Moreover, since a triality automorphism of $\mathrm{P}\Omega_8^+(2)$ is realised in the Monster, this determines the group 2^6A_8 up to conjugacy, and it is easily seen to lie in $2^{10+16}{}^{\cdot}\mathrm{P}\Omega_{10}^+(2)$.

We are left with $3{}^{\cdot}\mathrm{Fi}_{24}'$. By inspection, all 2-local maximal subgroups thereof lie inside 2-local maximal subgroups of $\mathbb{M}$, which leaves Fi_{23} and $\mathrm{P}\Omega_{10}^-(2)$ to consider. Similar arguments in these groups rapidly conclude the proof. □

The next lemma is a restatement of Theorem 2.

Lemma 2. *Every subgroup* $B \cong 2^{3+3}{:}7$ *of* $\mathbb{M}$ *lies in one of the two maximal subgroups* $M_1 = 2^{1+24}\mathrm{Co}_1$ *or* $M_2 = 2^3.2^6.2^{12}.2^{18}(\mathrm{PSL}_3(2) \times 3S_6)$.

Proof. Since B is generated by elements of order 7, we reduce in each case to the normal subgroup of the relevant maximal subgroup, generated by the elements of order 7. This allows us to eliminate the case $2^2.2^{11}.2^{22}.\mathrm{M}_{24}$, which is contained in M_1; and the case $2^2.{}^2E_6(2)$, which is contained in $2.\mathbb{B}$; and the case $2^5.2^{10}.2^{20}.\mathrm{PSL}_5(2)$, which is contained in $2^{10}.2^{16}.\mathrm{P}\Omega_{10}^+(2)$.

Now inside $2{}^{\cdot}\mathbb{B}$ we easily reduce to 2-local maximal subgroups, most of which are contained in one of the 2-constrained maximal subgroups of the Monster considered above. Everything else reduces to $2^2.{}^2E_6(2)$. But modulo the centre this is a group of Lie type in characteristic 2, whose maximal 2-local subgroups are given by the Borel–Tits theorem, and again lie in the 2-constrained subgroups above.

Finally, we eliminate $2^{10}.2^{16}.\mathrm{P}\Omega_{10}^+(2)$. The centralizer of the action of 2^{3+3} on the 2^{10} orthogonal space must be a singular subspace, whose radical is acted on by the element of order 7. Now singular vectors are in class $2B$, while non-singular vectors are in class $2A$. As a module for 7, therefore, this radical is either irreducible 3-dimensional, in which case B is contained in M_2, or contains fixed points, in which case B is contained in M_1. □

3. Subgroups of Co_1 isomorphic to $2^{3+3}{:}7$

We begin with the M_1 case. As a first step, in this section we prove Theorem 3, that Co_1 does not contain a subgroup isomorphic to B. We use the list of maximal subgroups given in [23], and more particularly the 2-local maximal subgroups classified by Curtis [4]. Further information about maximal subgroups is taken from the Atlas [3].

Lemma 3. *Any subgroup of* Co_1 *isomorphic to the Borel subgroup of* $\mathrm{Sz}(8)$ *lies in a conjugate of* $2^{1+8}{}^{\cdot}\mathrm{P}\Omega_8^+(2)$. *Moreover, the subgroup* $2^3{:}7$ *is determined up to conjugacy.*

Proof. Inspection of the list of maximal subgroups of Co_1, as well as maximal subgroups of maximal subgroups, and so on as far as necessary, shows that any $2^{3+3}{:}7$ in Co_1 lies in one of the maximal 2-local subgroups $2^{1+8}{}^{\cdot}\mathrm{P}\Omega_8^+(2)$ or $2^{2+12}(A_8 \times S_3)$

or $2^{11}{:}\mathrm{M}_{24}$. But it is easy to see that in the latter two cases any such subgroup centralizes an involution of Co_1-class $2A$, so reduces to the first case.

Now $\mathrm{P}\Omega_8^+(2)$ does not contain $2^{3+3}{:}7$, so we must have a 2^3 subgroup of 2^{1+8}. This corresponds to a totally isotropic 3-space in the orthogonal 8-space. All such 3-spaces are equivalent. Each such 3-space has stabilizer $2^{3+6}{:}\mathrm{PSL}_3(2)$ in $\mathrm{P}\Omega_8^+(2)$, so up to conjugacy there is a unique 7 normalizing it. $\square$

Indeed, the full pre-image of this 3-space stabilizer in $2^{1+8}{\cdot}\mathrm{P}\Omega_8^+(2)$ lies inside the octad stabilizer in $2^{11}{:}\mathrm{M}_{24}$. Since the latter is a split extension, it is much easier to calculate in than the involution centralizer itself. The 2^3 itself consists of octads which are disjoint from the fixed octad.

Lemma 4. *The subgroup* $2^{11}{:}\mathrm{M}_{24}$ *of* Co_1 *does not contain a group isomorphic to* B.

Proof. The relevant subgroup of $2^{11}{:}\mathrm{M}_{24}$ is $2^{11}{:}2^{1+6}\mathrm{PSL}_3(2)$, that is the preimage of the involution centralizer in M_{24}. This is contained in $2^{11}{:}2^4A_8$, in which the 2^{11} is a uniserial modiule for 2^4A_8, with factors $1+4+6$. As a module for the cyclic group of order 7, therefore, the 2^{11} has structure $1a+1a+3a+3a+3b$, and the 2^3 which corresponds to the isotropic 3-space is one of the copies of $3a$. It follows that in a putative $2^{3+3}{:}7$, the top 2^3 is also of type $3a$. This identifies the $2^3{:}7$ up to conjugacy in $2^{1+6}{:}\mathrm{PSL}_3(2)$.

The module structure of the 2^{11} for this group $2^3{:}7$ can now be calculated. There can be gluing of a $3b$ under a $3a$, or of a $1a$ under a $3b$, but gluing a $3a$ under anything else is impossible. It follows that we can quotient by $1a+3b$, to get a group $2^6{:}2^3{:}7$ in which all three 2^3 chief factors are of type $3a$. A straightforward calculation now reveals that this group does not contain a copy of the group $2^{3+3}{:}7$ we are seeking. $\square$

This concludes the proof of Theorem 3.

4. Pure 2^3 subgroups of 2^{1+24}

By this stage we know that any embedding of B in M_1 involves a 2^3 in 2^{1+24}, and a quotient $2^3{:}7$ in Co_1. We next show that this forces the 7-elements to be in $\mathbb{M}$-class $7A$ (corresponding to Co_1-class $7B$).

Lemma 5. *There is no* $2^3{:}7$ *in* Co_1 *containing elements of* Co_1*-class* $7A$.

Proof. The $7A$-elements in Co_1 are fixed-point-free in the action of $2{\cdot}\mathrm{Co}_1$ on the Leech lattice. If they lie in $2^3{:}7$ in Co_1, then this lifts to $2^3{:}7$ in $2{\cdot}\mathrm{Co}_1$, acting faithfully on the Leech lattice. But then the element of order 7 would have a fixed point, which is a contradiction. This concludes the proof. $\square$

Lemma 6. *There are exactly three conjugacy classes of* $2^3{:}7$ *in* $2^{1+24}\mathrm{Co}_1$ *that have the properties that the* 2^3 *lies in* 2^{1+24} *and the* 7*-element lies in* Co_1*-class* $7B$*. Their centralizers in* $2^{1+24}\mathrm{Co}_1$ *are respectively*

(1) $2^{1+6}S_4$,
(2) $2^{1+6}.7$, *and*
(3) $2^{1+6}.2^2$.

Proof. First, the $7B$-normalizer in Co_1 is $(7{:}3\times\mathrm{PSL}_3(2)){:}2$, in which the two factors $7{:}3$ and $\mathrm{PSL}_3(2)$ both have two 3-dimensional representations, which we will denote $3a$ and $3b$. Then the representation of $7{:}3\times\mathrm{PSL}_3(2)$ on the 2^{24} is

$$1\otimes 3a+1\otimes 3b+3a\otimes 3a+3b\otimes 3b.$$

Since the outer half of the 7-normalizer swaps $3a$ with $3b$, we may assume that our 2^3 lies in the $3a\otimes 3a$ part of the representation.

Now we may interpret our 7-element as a scalar in the field $\mathbb{F}_8$ of order 8, so that $3a\otimes 3a$ becomes a 3-space over $\mathbb{F}_8$. Then we classify the orbits of $\mathrm{PSL}_3(2)$ on the $(8^3-1)/(8-1)=73$ one-dimensional subspaces of this 3-space. This is a straightforward calculation, and we find that the orbit lengths are 7, 24, and 42. Thus there are exactly three conjugacy classes of $2^3{:}7$ of this kind in $2^{1+24}\mathrm{Co}_1$, with centralizers respectively $2^{1+6}S_4$, $2^{1+6}.7$, and $2^{1+6}.2^2$. $\square$

5. Examples

The $2B$-elements in 2^{1+24}, modulo the central involution, correspond to crosses in the Leech lattice, that is congruence classes modulo 2 of lattice vectors of type 4. The 2^3 subgroups described in Lemma 6 can therefore be described by representative vectors of three such classes. We use the octonionic notation of [24] for the Leech lattice, and explicit generators for the Conway group given in [25]. In particular, we take the 7-element to rotate the imaginary units as $i_t\mapsto i_{t+1}$, with subscripts read modulo 7, and the $\mathrm{PSL}_3(2)$ to be generated modulo the central involution of $2^{\cdot}\mathrm{Co}_1$ by the sign-changes and permutations on the three octonionic coordinates, together with the matrix

$$g_1=\begin{pmatrix}0&\overline{s}&\overline{s}\\ s&-1&1\\ s&1&-1\end{pmatrix}$$

acting by right-multiplication on row vectors.

Now if $\mathrm{PSL}_3(2)$ acts in the usual way on $\mathbb{F}_2^2$, and η is a root of x^3+x+1 modulo 2, then the three orbits on 1-spaces have representatives respectively $(1,0,0)$, $(1,\eta,0)$ and $(1,\eta,\eta^2)$, giving orbit lengths 7, 42 and 24 respectively. This can be translated directly into the above situation, and enables us to write down representatives for the three orbits of $2^3{:}7$ described in Lemma 6.

Example 1. *In the first case, the* 2^3 *is centralized by an* S_4 *in the* $\mathrm{PSL}_3(2)$*, and this* S_4 *belongs to the so-called Suzuki chain of subgroups, and centralizes* A_8*. The resulting subgroup* $S_4\times A_8$ *lies in the stabilizer of a trio of three disjoint octads. We may take the* $7B$*-element to cycle the imaginary units* $i_0,i_1,\ldots,i_6$ *in the obvious way, and the* 2^3 *to consist of the crosses defined by the vector* $2(-1+i_0+i_1+i_3,0,0)$ *and its images under the* 7*-cycle.*

Adjoining the central involution of 2^{1+24} *and the cross defined by* $(4,0,0)$ *gives a copy of the* 2^5 *with normalizer* $2^5.2^{10}.2^{20}.(\mathrm{PSL}_5(2)\times S_3)$*. In particular, any copy of* B *containing this* $2^3{:}7$ *also lies in* M_2*.*

By applying the matrix g_1 *we obtain a spanning set for the* 3*-space over* $\mathbb{F}_8$*. A second basis vector may be taken modulo* 2 *to be* $(-2-i_0+i_3+i_5+i_6)(1,1,0)$*.*

Example 2. *In the second case, the* 2^3 *is centralized by an element of order* 7*. This is necessarily of* Co_1 *class* $7B$*, so can be conjugated to the element of class* $7B$ *described in the previous example. This element centralizes a* 2^{1+6} *in* 2^{1+24}*, which* *is acted on by a group* $\mathrm{PSL}_3(2)$ *which identifies the two invariant* 2^3 *subgroups. We can take either of them, since they are interchanged by an automorphism which inverts the* $7B$*-element.*

With the same notation as above, we find that an example is generated by the congruence classes of $(4,0,0)$ *and* $2(\overline{s},1,\pm 1)$*, and images under permutations of the three octads.*

Example 3. *We make the third example directly by translating* $(1,\eta,0)$ *into octonionic language, so that it is again normalized by the canonical element of order* 7*. It can be generated by the images of the congruence class of the vector* $(-2-i_0+i_3+i_5+i_6, 2i_4+i_0+i_3-i_5+i_6, 0)$*.*

6. IDENTIFYING THE 2^2 SUBGROUPS

It is well-known [15] that there are three classes of 2^2 of pure $2B$-type in the Monster, with the following properties with respect to the centralizer $2^{1+24}\mathrm{Co}_1$ of any one of its involutions.

(a) Contained in the normal subgroup 2^{1+24}, so having centralizer of the shape $(2\times 2^{1+22}).2^{11}\mathrm{M}_{24}$.
(b) Mapping onto an element of Co_1-class $2A$, whose centralizer is Co_1 has shape $2^{1+8}{\cdot}\mathrm{P}\Omega_8^+(2)$. The centralizer of this 2^2-group in the Monster is however only $(2^9\times 2^{1+6}).2^{1+8}.2^6A_8$.
(c) Mapping onto an element of Co_1-class $2C$, which has centralizer $2^{11}\mathrm{M}_{12}.2$ in Co_1. The centralizer of this 2^2 in the Monster is $2^{12}.2^{11}.\mathrm{M}_{12}.2$.

It is also proved in [15], and is in any case a straightforward calculation, that all three of these $2B^2$ subgroups are represented in 2^{1+24} modulo its centre, and that there is a unique conjugacy class in each case. In standard notation, if one of the involutions is taken to be the congruence class of $(8,0^{23})$, then the other is the congruence class of either $(4^4,0^{20})$ or $(2^8,4^2,0^{14})$ or $(2^{12},4,0^{11})$. These are of type (a), (b), (c) respectively. Examples in octonionic notation are $(4,0,0)$ with respectively $2(1+i_0+i_1+i_3,0,0)$ or $2(\overline{s},1,1)$ or $(1+i_0)(s-2,s,s)$. From this it is immediate that in the first two cases in Lemma 6 the 2^2-subgroups are respectively of type (a) and (b). A small calculation establishes that in the third case they are of type (c). As this calculation is somewhat tricky to carry out accurately, we give a sketch here.

Lemma 7. *The* 2^3 *of type (3) in Lemma 6 contains* 2^2 *subgroups of type (c).*

Proof. Let us take the example given in Lemma 3 above, spanned by the congruence classes of the vectors

$$\begin{array}{l}(-2-i_0+i_3+i_5+i_6, 2i_4+i_0+i_3-i_5+i_6, 0)\\(-2-i_1+i_4+i_6+i_0, 2i_5+i_1+i_4-i_6+i_0, 0).\end{array}$$

We aim to apply elements of the Conway group which map the first vector to a vector in the congruence class of $(4,0,0)$. First multiply the second and third coordinates by i_4, then i_6, then i_5, then i_1 to get

$$\begin{array}{l}(-2-i_0+i_3+i_5+i_6, -2-i_0+i_3+i_5+i_6, 0)\\(-2-i_1+i_4+i_6+i_0, -2i_0-1+i_2-i_3+i_5, 0).\end{array}$$

Now we can apply the matrix

$$\frac{1}{2}\begin{pmatrix} -1 & 1 & s \\ 1 & -1 & s \\ \overline{s} & \overline{s} & 0 \end{pmatrix}$$

to obtain

$$\begin{aligned}
&-2(0,0,i_0+i_3+i_5+i_6)\\
&\frac{1}{2}(1-3i_0+i_1+i_2-i_3-i_4+i_5-i_6,\\
&\qquad -1+3i_0-i_1-i_2+i_3+i_4-i_5+i_6,\\
&\qquad 1-i_0-i_1-i_2-i_3-i_4-i_5-5i_6)
\end{aligned}$$

We may, although this is not strictly necessary, tidy this up a little by multiplying the second and third coordinates by i_0 and then i_1, to obtain

$$\begin{aligned}
&2(0,0,1+i_1-i_2+i_4)\\
&\frac{1}{2}(1-3i_0+i_1+i_2-i_3-i_4+i_5-i_6,\\
&\qquad -1-i_0-3i_1-i_2-i_3-i_4+i_5+i_6,\\
&\qquad 1-i_0+i_1-i_2+i_3+5i_4+i_5-i_6)
\end{aligned}$$

and finally multiply by $(1-i_1)$ and then $(1+i_2)/2$ to obtain

$$\begin{aligned}
&(0,0,4)\\
&(-i_0+i_2+i_3-i_6,-1-i_0-i_2-i_4,2-i_1-i_2+i_3+i_4)
\end{aligned}$$

It is readily checked that this last vector lies in the Leech lattice, and that these two congruence classes determine a $2B^2$ subgroup of type (c) in the Monster. □

7. Eliminating the second and third cases

In these two cases we show that there is no embedding of B in M_1.

Lemma 8. *The group $2^3{:}7$ of type (2) considered in Lemma 6 cannot occur in a copy of B in the Monster.*

Proof. The second type of 2^3 has normaliser with order divisible by 7^2, and lying in $2^{1+24}{\cdot}\mathrm{Co}_1$. Now the only maximal subgroups of Co_1 whose order is divisible by 7^2 are $7^2{:}(3\times 2A_4)$ and $(A_7\times \mathrm{PSL}_3(2)){:}2$. Since neither of these groups contains $2^3{:}7$, the group in question cannot extend to $2^{3+3}{:}7$. □

Lemma 9. *The $2^3{:}7$ subgroup of type (3) in Lemma 6 cannot occur in a copy of B in the Monster.*

Proof. The third type of $2B$-pure 2^2 has centralizer $(2^2\times 2^{1+20}){\cdot}\mathrm{M}_{12}{\cdot}2$, which lies entirely within $2^{1+24}{\cdot}\mathrm{Co}_1$. Again, the subgroup $2^{3+3}{:}7$ of our putative Sz(8) projects onto a subgroup $2^3{:}7$ of Co_1. Moreover, the normal 2^{3+3} is in the centralizer of our 2^2 and projects to a pure 2^3 subgroup of Co_1. Since this 2^3 lies in $\mathrm{M}_{12}.2$, we need to look at the embedding of $\mathrm{M}_{12}{:}2$ in Co_1. We have that the classes $2A$ and $2C$ in $\mathrm{M}_{12}{:}2$ fuse to Co_1-class $2B$, while M_{12}-class $2B$ fuses to Co_1-class $2A$. But there is no pure 2^3 of Co_1-class $2B$, and no pure 2^3 of M_{12}-class $2B$. Therefore this case cannot arise. □

8. Eliminating the first case

In this case we adopt a different strategy, and show that any subgroup of $\mathbb{M}$ which is generated by a $2^3{:}7$ of this type and an involution which inverts an element of order 7 therein has non-trivial centralizer. Since $\mathrm{Sz}(8)$ can be generated in this way, and it is already known that every $\mathrm{Sz}(8)$ in $\mathbb{M}$ has trivial centralizer, this proves that this $2^3{:}7$ cannot lie in $\mathrm{Sz}(8)$.

Before we prove this, we show that the M_2 case also reduces to this case.

Lemma 10. *Every copy of the group* $B \cong 2^{3+3}{:}7$ *in* M_2 *contains the socle of* M_2*.*

Proof. If we label the two 3-dimensional representations of $\mathrm{PSL}_3(2)$ as $3a$ and $3b$, and label other representations by their degrees, then the representations of $\mathrm{PSL}_3(2) \times 3S_6$ on the chief factors of $N(2^3)$ are respectively $3a \otimes 1$, $1 \otimes 6$, $3b \otimes 4$, and $3a \otimes 6$. Now $3a$ and $3b$ remain distinct on restriction to the subgroup of order 7. But in $B \cong 2^{3+3}{:}7$, the 3-dimensional representations of the group of order 7 on B'' and B'/B'' are the same, and this can only occur in M_2 in the case when B contains the socle. $\square$

Lemma 11. *The* $2^3{:}7$ *subgroup of type (1) in Lemma 6 cannot occur in a copy of* $\mathrm{Sz}(8)$ *in the Monster.*

Proof. In this case the $2^3{:}7$ has centralizer $2^6{:}3S_6$, visible in M_2. The 7-element extends to exactly 266560 groups D_{14} inside the invertilizer $(7 \times \mathrm{He}){:}2$. It is easy to calculate (using a suitable computer algebra package such as GAP [5]) the orbits of $2^6{:}3S_6$ on these 266560 points, and to observe that there is no regular orbit. (This permutation representation was taken from [28].) Now $\mathrm{Sz}(8)$ can be generated by subgroups $2^3{:}7$ and D_{14} intersecting in 7. It follows that if $\mathrm{Sz}(8)$ is generated by one of these amalgams, with this particular $2^3{:}7$, then it is centralized by a non-trivial element. This is a contradiction. $\square$

This concludes the proof of Theorem 1.

Appendix: containments of simple groups in sporadic groups

The question of which simple groups are subgroups of the Monster was first considered in [18], which left open just the cases

$$\mathrm{L}_2(27), \mathrm{L}_2(29), \mathrm{L}_2(59), \mathrm{L}_2(71), \mathrm{L}_3(4), \mathrm{Sz}(8).$$

Subsequently, subgroups isomorphic to $\mathrm{L}_2(29)$(see [9]), $\mathrm{L}_2(41)$ (see [19]), $\mathrm{L}_2(59)$ (see [10]) and $\mathrm{L}_2(71)$ (see [11]) were explicitly found, and $\mathrm{L}_2(27)$ (see [26]) and $\mathrm{L}_3(4)$ (see [11]) eliminated, computationally. Combined with the main theorem of the present paper, we obtain the exact list as displayed in the Table.

The corresponding question for the other sporadic simple groups has already been answered, although it is not necessarily easy to extract the answer from the literature. Lists of maximal subgroups in [3] and [23] are sufficient to reconstruct easily the answers for most of the smaller groups, but for the Fischer groups and the Baby Monster it is necessary to go to the published papers [12, 13, 20, 21] which classify the maximal subgroups, and extract a number of other results from them.

Table 2. Containments of simple groups in sporadic groups

M_{11}	$A_5, A_6, \mathrm{L}_2(11)$
M_{12}	$A_5, A_6, \mathrm{L}_2(11), \mathrm{M}_{11}$
M_{22}	$A_5, A_6, A_7, \mathrm{L}_2(7), \mathrm{L}_2(11), \mathrm{L}_3(4)$
M_{23}	$A_5, A_6, A_7, A_8, \mathrm{L}_2(7), \mathrm{L}_2(11), \mathrm{L}_3(4), \mathrm{M}_{11}, \mathrm{M}_{22}$
M_{24}	$A_5, A_6, A_7, A_8, \mathrm{L}_2(7), \mathrm{L}_2(11), \mathrm{L}_2(23), \mathrm{L}_3(4), \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}$
J_2	$A_5, \mathrm{L}_2(7), \mathrm{U}_3(3)$
Suz	$A_5, A_6, A_7, \mathrm{L}_2(7), \mathrm{L}_2(11), \mathrm{L}_2(13), \mathrm{L}_2(25), \mathrm{L}_3(3), \mathrm{L}_3(4), \mathrm{U}_3(3), \mathrm{U}_3(4), \mathrm{U}_4(2), \mathrm{U}_5(2), G_2(4), \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{J}_2$
HS	$A_5, A_6, A_7, A_8, \mathrm{L}_2(7), \mathrm{L}_2(11), \mathrm{L}_3(4), \mathrm{U}_3(5), \mathrm{M}_{11}, \mathrm{M}_{22}$
McL	$A_5, A_6, A_7, \mathrm{L}_2(7), \mathrm{L}_2(11), \mathrm{L}_3(4), \mathrm{U}_3(3), \mathrm{U}_3(5), \mathrm{U}_4(2), \mathrm{U}_4(3), \mathrm{M}_{11}, \mathrm{M}_{22}$
Co_3	$A_5, A_6, A_7, A_8, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_2(23), \mathrm{L}_3(4), \mathrm{U}_3(3), \mathrm{U}_3(5), \mathrm{U}_4(2), \mathrm{U}_4(3), \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{HS}, \mathrm{McL}$
Co_2	$A_5, A_6, A_7, A_8, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_3(4), \mathrm{U}_3(3), \mathrm{U}_3(5), \mathrm{U}_4(2), \mathrm{U}_4(3), \mathrm{U}_5(2), \mathrm{U}_6(2), \mathrm{S}_6(2), \mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{HS}, \mathrm{McL}$
Co_1	$A_5, A_6, A_7, A_8, A_9, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_2(13), \mathrm{L}_2(23), \mathrm{L}_2(25), \mathrm{L}_3(3), \mathrm{L}_3(4), \mathrm{U}_3(3), \mathrm{U}_3(4), \mathrm{U}_3(5), \mathrm{U}_4(2), \mathrm{U}_4(3), \mathrm{U}_5(2), \mathrm{U}_6(2), \mathrm{S}_6(2), G_2(4), \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{J}_2, \mathrm{HS}, \mathrm{McL}, \mathrm{Co}_3, \mathrm{Co}_2$
Fi_{22}	$A_5, A_6, A_7, A_8, A_9, A_{10}, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_2(13), \mathrm{L}_2(25), \mathrm{L}_3(3), \mathrm{L}_3(4), \mathrm{L}_4(3), \mathrm{U}_3(3), \mathrm{U}_4(2), \mathrm{U}_4(3), \mathrm{U}_5(2), \mathrm{S}_6(2), \mathrm{P}\Omega_7(3), \mathrm{P}\Omega_8^+(2), G_2(3), {}^2F_4(2)', \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}$
Fi_{23}	$A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_2(13), \mathrm{L}_2(16), \mathrm{L}_2(17), \mathrm{L}_2(23), \mathrm{L}_2(25), \mathrm{L}_3(3), \mathrm{L}_4(3), \mathrm{U}_3(3), \mathrm{U}_4(2), \mathrm{U}_5(2), \mathrm{S}_4(4), \mathrm{S}_6(2), \mathrm{S}_8(2), \mathrm{P}\Omega_7(3), \mathrm{P}\Omega_8^+(2), \mathrm{P}\Omega_8^-(2), \mathrm{P}\Omega_8^+(3), G_2(3), {}^2F_4(2)', \mathrm{M}_{11}, \mathrm{M}_{12}$
Fi_{24}'	$A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_2(13), \mathrm{L}_2(16), \mathrm{L}_2(17), \mathrm{L}_2(23), \mathrm{L}_2(25), \mathrm{L}_3(3), \mathrm{L}_4(3), \mathrm{U}_3(3), \mathrm{U}_4(2), \mathrm{U}_5(2), \mathrm{S}_4(4), \mathrm{S}_6(2), \mathrm{S}_8(2), \mathrm{P}\Omega_7(3), \mathrm{P}\Omega_8^+(2), \mathrm{P}\Omega_8^-(2), \mathrm{P}\Omega_8^+(3), \mathrm{P}\Omega_{10}^-(2), G_2(3), {}^2F_4(2)', \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{Fi}_{23}, \mathrm{He}$
He	$A_5, A_6, \mathrm{L}_2(7), \mathrm{L}_2(16), \mathrm{S}_4(4)$
HN	$A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{U}_3(5), \mathrm{U}_3(8), \mathrm{M}_{11}, \mathrm{M}_{12}$
Th	$A_5, A_6, \mathrm{L}_2(8), \mathrm{L}_2(13), \mathrm{L}_2(19), \mathrm{L}_3(3), \mathrm{U}_3(3), \mathrm{U}_3(8), G_2(3), {}^3D_4(2)$
$\mathbb{B}$	$A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_2(13), \mathrm{L}_2(16), \mathrm{L}_2(17), \mathrm{L}_2(19), \mathrm{L}_2(23), \mathrm{L}_2(25), \mathrm{L}_2(31), \mathrm{L}_2(49), \mathrm{L}_3(3), \mathrm{L}_3(4), \mathrm{L}_4(3), \mathrm{U}_3(3), \mathrm{U}_3(5), \mathrm{U}_3(8), \mathrm{U}_4(2), \mathrm{U}_4(3), \mathrm{U}_5(2)\mathrm{S}_4(4), \mathrm{S}_6(2), \mathrm{S}_8(2), G_2(3), {}^3D_4(2), {}^2F_4(2)', F_4(2), \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{HS}, \mathrm{Fi}_{22}, \mathrm{HN}, \mathrm{Th}, \mathrm{Fi}_{23}$
$\mathbb{M}$	$A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_2(13), \mathrm{L}_2(16), \mathrm{L}_2(17), \mathrm{L}_2(19), \mathrm{L}_2(23), \mathrm{L}_2(25), \mathrm{L}_2(29), \mathrm{L}_2(31), \mathrm{L}_2(41), \mathrm{L}_2(49), \mathrm{L}_2(59), \mathrm{L}_2(71), \mathrm{L}_3(3), \mathrm{L}_4(3), \mathrm{U}_3(3), \mathrm{U}_3(4), \mathrm{U}_3(5), \mathrm{U}_3(8), \mathrm{U}_4(2), \mathrm{U}_5(2), \mathrm{P}\Omega_7(3), \mathrm{P}\Omega_8^+(2), \mathrm{P}\Omega_8^-(2), \mathrm{P}\Omega_8^+(3), \mathrm{P}\Omega_{10}^-(2), \mathrm{S}_4(4), \mathrm{S}_6(2), \mathrm{S}_8(2), G_2(3), {}^3D_4(2), {}^2F_4(2)', \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{Fi}_{23}, \mathrm{He}, \mathrm{HN}, \mathrm{Th}$
J_1	$A_5, \mathrm{L}_2(11)$
J_3	$A_5, A_6, \mathrm{L}_2(16), \mathrm{L}_2(17), \mathrm{L}_2(19)$
Ru	$A_5, A_6, A_7, A_8, \mathrm{L}_2(7), \mathrm{L}_2(13), \mathrm{L}_2(25), \mathrm{L}_2(29), \mathrm{L}_3(3), \mathrm{U}_3(3), \mathrm{U}_3(5), \mathrm{Sz}(8), {}^2F_4(2)'$
O′N	$A_5, A_6, A_7, \mathrm{L}_2(7), \mathrm{L}_2(11), \mathrm{L}_2(31), \mathrm{L}_3(7), \mathrm{M}_{11}$
Ly	$A_5, A_6, \mathrm{L}_2(7), \mathrm{L}_3(5), \mathrm{U}_3(3), G_2(5), \mathrm{M}_{11}$
J_4	$A_5, A_6, A_7, A_8, \mathrm{L}_2(7), \mathrm{L}_2(11), \mathrm{L}_2(23), \mathrm{L}_2(32), \mathrm{L}_3(4), \mathrm{U}_3(3), \mathrm{U}_3(11), \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}$

References

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, *Invent. Math.* **76** (1984),469–514.

[2] J. H. Conway, A simple construction for the Fischer–Griess monster group, *Invent. Math.* **79** (1985), 513–540.

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An Atlas of Finite Groups*, Oxford University Press, 1985.

[4] R. T. Curtis, On subgroups of ·0. II. Local structure, *J. Algebra* **63** (1980), 413–434.

[5] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.4, 2004.

[6] R. L. Griess, The Friendly Giant, *Invent. Math.* **69** (1982), 1–102.

[7] P. E. Holmes, A classification of subgroups of the Monster isomorphic to S_4 and an application, *J. Algebra* **319** (2008), 3089–3099.

[8] P. E. Holmes and R. A. Wilson, A new computer construction of the Monster using 2-local subgroups, *J. London Math. Soc.* **67** (2003), 349–364.

[9] P. E. Holmes and R. A. Wilson, A new maximal subgroup of the Monster, *J. Algebra* **251** (2002), 435–447.

[10] P. E. Holmes and R. A. Wilson, $\mathrm{PSL}_2(59)$ is a subgroup of the Monster, *J. London Math. Soc.* **69** (2004), 141–152.

[11] P. E. Holmes and R. A. Wilson, On subgroups of the Monster containing A_5's, *J. Algebra* **319** (2008), 2653–2667.

[12] P. B. Kleidman, R. A. Parker and R. A. Wilson, The maximal subgroups of the Fischer group Fi_{23}, *J. London Math. Soc.* **39** (1989), 89–101.

[13] S. A. Linton and R. A. Wilson, The maximal subgroups of the Fischer groups Fi_{24} and Fi_{24}', *Proc. London Math. Soc.* **63** (1991), 113–164.

[14] S. A. Linton, R. A. Parker, P. G. Walsh and R. A. Wilson, A computer construction of the Monster, *J. Group Theory* **1** (1998), 307–337.

[15] U. Meierfrankenfeld and S. V. Shpectorov, Maximal 2-local subgroups of the Monster and Baby Monster, Preprint, Michigan State University, 2002. http://www.mth.msu.edu/~meier/Preprints/2monster/maxmon.pdf

[16] U. Meierfrankenfeld, The maximal 2-local subgroups of the Monster and Baby Monster, II, Preprint, Michigan State University, 2003. http://www.mth.msu.edu/~meier/Preprints/2monster/2MNC.pdf

[17] S. Norton, Anatomy of the Monster: I, in *Proceedings of the* ATLAS *Ten Years On conference (Birmingham 1995)*, pp. 198-214, Cambridge Univ. Press, 1998.

[18] S. P. Norton and R. A. Wilson, Anatomy of the Monster: II, *Proc. London Math. Soc.* **84** (2002), 581–598.

[19] S. P. Norton and R. A. Wilson, A correction to the 41-structure of the Monster, a construction of a new maximal subgroup $L_2(41)$, and a new Moonshine phenomenon, *J. London Math. Soc.* **87** (2013), 943–962.

[20] R. A. Wilson, Some subgroups of the Baby Monster, *Invent. Math.* **89** (1987), 197–218.

[21] R. A. Wilson, The maximal subgroups of the Baby Monster. I, *J. Algebra* **211** (1999), 1–14.

[22] R. A. Wilson, The odd-local subgroups of the Monster, *J. Austral. Math. Soc. (A)* **44** (1988), 1–16.

[23] R. A. Wilson, *The finite simple groups*, Springer GTM 251, 2009.

[24] R. A. Wilson, Octonions and the Leech lattice, *J. Algebra* **322** (2009), 2186–2190.

[25] R. A. Wilson, Conway's group and octonions, *J. Group Theory* **14** (2011), 1–8.

[26] R. A. Wilson, Classification of subgroups isomorphic to $\mathrm{PSL}_2(27)$ in the Monster, *LMS J. Comput. Math.* **17** (2014), 33–46.

[27] R. A. Wilson, Every $\mathrm{PSL}_2(13)$ in the Monster contains $13A$-elements, *LMS J. Comput. Math.*, to appear.

[28] R. A. Wilson et al., *An Atlas of Group Representations*, http://brauer.maths.qmul.ac.uk/Atlas/.

School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, U.K.

E-mail address: `R.A.Wilson@qmul.ac.uk`