Classification of subgroups isomorphic to $\mathrm{PSL}_2(27)$ in the Monster

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19th April 2013

Abstract

As part of the problem of the determination of the maximal subgroups of the Monster we classify subgroups isomorphic to $\mathrm{PSL}_2(27)$. Indeed, we prove that the Monster does not contain any subgroup isomorphic to $\mathrm{PSL}_2(27)$.

1 Introduction

The Monster is the largest of the 26 sporadic simple groups. The maximal subgroups of the other 25 are all known, so it would be satisfying to complete this project also for the Monster. The problem of determining the maximal subgroups of the Monster has a long history (see for example [9, 10, 12, 13, 5, 6, 7, 3, 11]). The cases left open by previous published work are normalizers of simple subgroups with trivial centralizer, and isomorphic to one of the groups

$$\mathrm{PSL}_2(8),\ \mathrm{PSL}_2(13),\ \mathrm{PSL}_2(16),\ \mathrm{PSL}_2(27),\ \mathrm{PSU}_3(4),\ \mathrm{PSU}_3(8),\ \mathrm{Sz}(8).$$

Of these, $\mathrm{PSL}_2(8)$ and $\mathrm{PSL}_2(16)$ have been classified in unpublished work of Holmes. The cases $\mathrm{PSL}_2(27)$ and $\mathrm{Sz}(8)$ are particularly interesting because no subgroup isomorphic to $\mathrm{PSL}_2(27)$ or $\mathrm{Sz}(8)$ is known. Here we consider the case $\mathrm{PSL}_2(27)$, and we show that in fact there is no subgroup isomorphic to $\mathrm{PSL}_2(27)$ in the Monster.

2 Theoretical results

The strategy we use here is the standard one for $\mathrm{PSL}_2(q)$, namely to classify the possibilities for the BN-pair, consisting in this case of $B \cong 3^3{:}13$ and $N \cong D_{26}$ intersecting in the 'torus' of order 13.

First we use the 3-local analysis to classify subgroups of the Monster isomorphic to $3^3{:}13$. Since neither $3A$-elements nor $3C$-elements can form a pure 3^3 (see [12]), the 3-elements in any $3^3{:}13$ must be in class $3B$, and we know from [12] that there are just three classes of pure $3B$-type 3^2. Clearly all the 3^2 subgroups of our 3^3 must be of the same type.

Consider first the case when they are of type $3B_4(i)$ in the notation of [12]. Such a 3^2 has centralizer $3^2.3^5.3^{10}.\mathrm{M}_{11}$. It is shown in Theorem 6.5 of [12] that there is a unique conjugacy class of such 3^3. The normalizer of this 3^3 has the shape

$3^3.3^2.3^6.3^6.(\mathrm{PSL}_3(3) \times SD_{16})$. Hence there is, up to conjugacy, a unique group $3^3{:}13$ of this type. It has centralizer $3^2{:}D_8$, and its normalizer is a group of shape

$$(3^3{:}(2 \times 13{:}3) \times 3^2{:}SD_{16}).\frac{1}{2} = (3^3{:}13{:}3 \times 3^2{:}D_8).2,$$

that is, a subgroup of index 2 in $3^3{:}(2 \times 13{:}3) \times 3^2{:}SD_{16}$.

Now this $3^3{:}13$ contains $13A$-elements, and the invertilizer of a $13A$-element has shape $13{:}2 \times \mathrm{PSL}_3(3)$. Hence there are just 118 copies of D_{26} containing a given element of class $13A$. Elementary calculations show that the subgroup $3^2{:}D_8$ of $\mathrm{PSL}_3(3)$ has orbits of sizes $9 + 6 + 6 + 12 + 12 + 36 + 36$ on the 117 involutions in $\mathrm{PSL}_3(3)$, and in particular has no regular orbit. It follows that any $\mathrm{PSL}_2(27)$ of this type would have non-trivial centralizer. This is a contradiction, as none of the element-centralizers in the Monster contains $\mathrm{PSL}_2(27)$.

Next consider the cases $3B_4(ii)$ and $3B_4(iii)$. By Propositions 6.1 and 6.2 of [12], the whole of the 3^3 lies inside a unique 3^{1+12}. Therefore $3^3{:}13$ lies inside $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$. Now in Suz:2 the Sylow 13-normalizer has the shape 13:12. Therefore the 13-element normalizes just four subgroups of shape 3^3 in the 3^{1+12}, and these are permuted by the 13-normalizer. It follows that there is a unique class of $3^3{:}13$ of this type in the Monster. Such a subgroup has centralizer of order 3, and normalizer of shape $3 \times 3^3{:}(2 \times 13{:}3)$.

To summarise the results of this section, we have proved the following.

Theorem 1 *There are exactly two conjugacy classes of subgroups isomorphic to* $3^3{:}13$ *in the Monster, one of which contains* $13A$*-elements, while the other contains* $13B$*-elements.*

Theorem 2 *There is no* $\mathrm{PSL}_2(27)$ *in the Monster containing* $13A$*-elements.*

3 Computational strategy

At this point we resorted to computer calculations to finish the job. The original calculations were done about ten years ago, using the mod 2 construction of the Monster [8], but subsequently lost. The calculations were therefore repeated, as described here, using the mod 3 construction [4]. The general methods of computation are described in [4, 5, 7], and summarized in [11], which also contains some improved methods. As in these references, we take a, b as generators for the subgroup $2^{1+24}{\cdot}\mathrm{Co}_1$, and T as a 'triality element', cycling the 3 central involutions in a subgroup $2^2.2^{11}.2^{22}.\mathrm{M}_{24}$ of $2^{1+24}{\cdot}\mathrm{Co}_1$.

By Theorems 1 and 2 we have reduced to the case in which $3^3{:}13$ contains $13B$-elements. The invertilizer of a $13B$-element is $13^{1+2}{:}4A_4$, and therefore there are exactly 78 copies of D_{26} containing a given $13B$-element.

We break the calculations down into a number of steps. First we make the part of the 13-normalizer that we can easily find inside $2^{1+24}{\cdot}\mathrm{Co}_1$. This is done in Section 4, where we obtain a group $13{:}(3 \times 4A_4) \cong (13{:}3 \times 2A_4){:}2$. Then in Section 5 we pick a non-central involution in this group, and find an element of the Monster conjugating it to the central involution. This allows us in Section 6 to find another element of order 13 commuting with the first one, thereby extending the subgroup to $13^{1+2}{:}(3 \times 4A_4)$. This group contains all the involutions which extend 13 to D_{26}.

Then in Section 7 we find an element of order 3 which, together with our original element of order 13, generates $3^3{:}13$. Finally, in Section 8 we show that under the action of the normalizer of $3^3{:}13$ the 78 ways of extending 13 to D_{26} fall into six orbits. Then we test the resulting 6 cases to see if the given element of order 3 in $3^3{:}13$, multiplied by one of the 13 involutions in D_{26}, gives an element of order 3. This is a criterion which distinguishes $\mathrm{PSL}_2(27)$ from all other groups.

4 Finding $(13{:}3 \times 2A_4){:}2$

The strategy here is to work first in Co_1, to find enough of the centralizer of a $2B$-element to obtain a group $2^2 \times G_2(4)$. Then we conjugate one of the central involutions to the other, in such a way that we obtain $A_4 \times G_2(4)$. Within this subgroup we find a copy of $\mathrm{PSL}_2(13)$ by random search, and then a copy of 13:6 inside $\mathrm{PSL}_2(13)$. Finally we apply the standard method known colloquially as 'applying the formula' in order to lift to elements of $2^{1+24}\cdot\mathrm{Co}_1$ which normalize a particular element of order 13 we choose.

4.1 Constructing $A_4 \times G_2(4)$ in Co_1

We take a, b to be the original pair of generators of $2^{1+24}\cdot\mathrm{Co}_1$, and first work in the quotient Co_1 to make the element

$$c_1 = (ab)^4(ab^2)^3$$

of order 26, so that

$$i_1 = (c_1)^{13}$$

is an element of class $2B$ in Co_1. We make

$$c_2 = abi_1[ab, i_1]^5$$

which centralizes i_1, and let

$$i_2 = (c_2)^{13}.$$

The elements c_1, c_2 then generate $2^2 \times G_2(4)$, in which the central 2^2 is generated by i_1, i_2. Then let

$$\begin{aligned} n_1 &= (ai_1)^5(ab)^{-2}i_2(ab)^2a(ab)^{-2} \\ n_2 &= (ai_1)^5(i_1i_2a)^5 \end{aligned}$$

to give elements which normalize the 2^2 and give us a group $A_4 \times G_2(4)$.

The normal subgroup A_4 is generated by

$$\begin{aligned} a_1 &= i_1 \\ a_2 &= (n_1n_2)^{13}, \end{aligned}$$

while the normal $G_2(4)$ is generated by

$$\begin{aligned} g_1 &= (c_1)^2 \\ g_2 &= (n_1n_2)^3. \end{aligned}$$

4.2 Constructing a subgroup $A_4 \times 13{:}6$

We then make standard generators of $G_2(4)$, as defined in [15], as

$$\begin{aligned} g_3 &= (g_1^4g_2)^4 \\ g_4 &= ((g_1g_2g_1g_2^2)^3)^{g_2^4} \end{aligned}$$

and generators for a subgroup $\mathrm{PSL}_2(13)$ can then be read off from [15] as

$$\begin{aligned} g_5 &= ((g_3g_4)^3g_4)^3((g_3g_4)^4g_4g_3g_4(g_3g_4^2)^2)^3((g_3g_4)^3g_4)^{-3} \\ g_6 &= (g_3g_4g_3g_4^2)^{-2}(g_3g_4(g_3g_4g_3g_4^2)^2)^5(g_3g_4g_3g_4^2)^2 \end{aligned}$$

Inside here we find that a subgroup 13:6 is generated by g_5 and

$$g_7 = (g_6)^{g_5g_6^2},$$

and we may take the element of order 13 to be

$$g_8 = [g_5, g_7].$$

4.3 Lifting to $2^{1+24}{}^{\cdot}\mathrm{Co}_1$

Now we 'apply the formula' to lift to $2^{1+24}{}^{\cdot}\mathrm{Co}_1$. That is, we replace the elements a_1, a_2, g_5, g_7 by new elements, in the same cosets of 2^{1+24}, which normalize the subgroup $\langle g_8 \rangle$ of order 13. Specifically, we make

$$\begin{array}{rcl} a_1' &=& g_8 a_1 (g_8 a_1^{-1} g_8 a_1)^6 \\ a_2' &=& (g_8 a_2 (g_8 a_2^{-1} g_8 a_2)^6)^2 \\ g_5' &=& g_8 g_5 (g_8^{-1} g_5^{-1} g_8 g_5)^6 \\ g_7' &=& (g_8 g_7 (g_8^9 g_7^{-1} g_8 g_7)^6)^2 \end{array}$$

so that a_1', a_2' generate $2A_4$ and g_5', g_7' generate $26{}^{\cdot}6 = (2 \times 13{:}3){}^{\cdot}2)$, commuting with each other. Thus they together generate

$$2{}^{\cdot}(A_4 \times 13{:}6).$$

An element of order 12 normalizing $\langle g_8 \rangle$ and commuting with $\langle a_1', a_2' \rangle \cong 2A_4$ and with g_5' may be obtained as

$$g_9 = g_5' g_8^3 g_7' g_8^{10}.$$

5 Changing post

The process of 'changing post' really consists of two parts. The first part consists of finding a word x in the generators of the Monster, which conjugates a given involution in $C(z)$, to z. This part is more or less algorithmic, especially as, in this case, we already in [11] found a word conjugating an involution in the desired $C(z)$-conjugacy class, to z. Here the involution which we want to map to z is $a_1' g_5'$.

The second part consists of 'shortening the word' for an element g^x, where both g and g^x lie in $C(z)$. This part is more *ad hoc*, and involves often quite laborious search for a word in a and b which is equal to the desired element. In this section, the element we want to write as a word in a and b is the appropriate conjugate of the element g_9 of order 12. Our strategy is to first find a word for its image in Co_1, and then to lift to $2^{1+24}{}^{\cdot}\mathrm{Co}_1$. Even within Co_1, the search is not easy, and we perform it in stages, first dealing with the involution which is its sixth power, and then its fourth power, before finally reaching the element itself. In the course of these calculations, we also identify two useful elements which centralize the given element of order 12.

5.1 Conjugating the involution $a_1' g_5'$ to z

The next step in extending to the full 13-normalizer $13^{1+2}{:}(3 \times 4S_4)$ is to conjugate the involution $i_3 = a_1' g_5'$ to z. First we make our 'standard' involution in this conjugacy class in $2^{1+24}{}^{\cdot}\mathrm{Co}_1$ as follows. As in [11] we make

$$\begin{array}{rcl} h &=& (ab)^{34}(abab^2)^3(ab)^6 \\ i &=& (ab^2)^{35}((ababab^2)^2 ab)^4 (ab^2)^5 \\ k_1 &=& hihi^2 \\ k_2 &=& hihihi^2 \\ k &=& (k_1 k_2)^3 k_2 k_1 k_2 \end{array}$$

Then we make

$$k' = ((a^2)^{(ab)^3} k^8)^{11} k^{11}$$

as our standard involution in this conjugacy class. This element is carefully chosen so that $T^{-1} k' T$ is an element of the normal 2^{1+24}.

We calculate once and for all how to conjugate this element to z. This calculation was already done in [11], and the result is that if

$$k_3 = (ab)^3(ab^2)^{20}(ababab^2abab^2)^8(ababab^2ab)^{12}(ababab^2)^5$$

then

$$(k')^{Tk_3T} = z.$$

It remains now to conjugate i_3 to k'. Now if two elements of Co_1-class $2C$ have product of order 13 or 35, then this product is fixed-point-free in its action on $2^{24} = 2^{1+24}/2$, and hence when we lift to $2^{24}{\cdot}\mathrm{Co}_1$ the product remains of odd order. Thus we can conjugate one to the other in $2^{24}{\cdot}\mathrm{Co}_1$ using the standard formula. Lifting to $2^{1+24}{\cdot}\mathrm{Co}_1$ is then easy. So we search for conjugates of i_3 and k' whose product has order 13 or 35, and thereby find that if

$$l_3 = (ab^2)^4(k'(i_3)^{(ab^2)^4})^6$$

then l_3 conjugates i_3 to k', and therefore l_3Tk_3T conjugates i_3 to z.

5.2 Finding the centralizer of $(g_9')^2$ in Co_1

This conjugation takes the element g_9 to an element

$$g_9' = g_9^{l_3Tk_3T}$$

which has order 12 in the quotient Co_1. We now want to find this element as a word in a, b, so as to eliminate the occurrences of T. This is by no means a simple process. In this subsection we obtain a word for an element which is congruent to $(g_9')^2$ modulo 2^{1+24}.

First note that $g_9^6 = z$, so that $(g_9')^6$ is (modulo 2^{1+24}) in the normal 2^{11} subgroup of the standard copy $\langle h, i\rangle$ of $2^{11}{:}\mathrm{M}_{24}$. By a random search we find a subgroup $2^{11}{:}\mathrm{M}_{12}$ centralizing $(g_9')^6$, generated by

$$\begin{array}{rcl} t_1 &=& (i^2)^{(hi^2)^6(hi)^{-6}} \\ t_2 &=& (i^2)^{(hi^2)^5(hi)^{-13}} \\ t_3 &=& (i^2)^{(hihihi^2hi)^3(hi)^{-2}} \end{array}$$

Moreover, the central involution of this group is

$$t_0 = (t_1t_3t_1t_3t_1t_3^2)^{11}.$$

Then we conduct another random search in this subgroup for elements commuting with the element $(g_9')^4$ of order 3. Writing

$$\begin{array}{rcl} u &=& t_1 \\ v &=& t_2t_3 \\ t_4 &=& ((uv)^4)^{(uvuvuv^2uvuv^2)^9(uvuvuv^2uv)^{10}} \\ t_5 &=& ((uv)^4)^{(uvuvuv^2)^{10}(uvuvuv^2uvuv^2)^2} \\ t_6 &=& (t_4)^{t_4t_5t_4t_5t_4t_5^2(t_4t_5)^7} \end{array}$$

we have that t_6 is in fact the inverse of $(g_9')^4$, modulo 2^{1+24}.

5.3 Finding the centralizer of g_9' in Co_1

Working first in the M_{12} quotient of $\langle t_1, t_2t_3\rangle$ we find that the following elements commute with t_6 modulo the 2-group:

$$\begin{aligned} t_8 &= (t_1)^{((t_1t_2t_3)^3t_2t_3)^3(t_2t_3)^5} \\ t_9 &= (t_1)^{((t_1t_2t_3)^3t_2t_3(t_1t_2t_3)^2t_2t_3)^7(t_2t_3)^6} \end{aligned}$$

Applying the formula we obtain

$$\begin{aligned} t_8' &= t_6^2t_8t_6^2t_8^2t_6^2t_8 \\ t_9' &= t_6^2t_9t_6^2t_9^2t_6^2t_9 \end{aligned}$$

and then

$$t_{10} = (t_9't_8't_9')^3$$

is congruent to $(g_9')^3$, modulo 2^{1+24}. We also make some elements commuting with g_9' modulo the 2-group, as follows:

$$\begin{aligned} t_{11} &= ((t_1t_3)^8t_6^2)^3t_8'(t_9')^2t_{10} \\ t_{12} &= (t_8')^2((t_1t_3)^8t_6^2)^3(t_8')^2(t_9')^2 \end{aligned}$$

5.4 Lifting to $2^{1+24}{\cdot}\mathrm{Co}_1$

Now we know that $t_{10}t_6$ is congruent modulo 2^{1+24} to the inverse of $(g_9')^{l_3Tk_3T}$. It remains to find the correct element of 2^{1+24} to multiply by. Using the method explained in [11], we obtain the element

$$w = t_{10}t_6p_1d_3d_4d_5d_7d_8d_9d_{10}d_{11}d_1p_3p_4p_6p_7p_8.$$

We lift the elements t_{11}, t_{12} to elements which commute with w, by the following method. First apply the formula, to get elements t_{11}', t_{12}' which commute with w^4:

$$\begin{aligned} t_{11}' &= w^4t_{11}w^4t_{11}^{-1}w^4t_{11} \\ t_{12}' &= w^4t_{12}w^4t_{12}^{-1}w^4t_{12} \end{aligned}$$

Then make the part of 2^{1+24} which commutes with w^4: by computing w^4 in the 24-dimensional $\mathbb{F}_2$-representation of Co_1, we find that this is generated by

$$\begin{aligned} q_1 &= d_9d_{12} \\ q_2 &= d_1d_4d_5d_6d_{10}d_{11} \\ q_3 &= d_3d_5d_7d_{10}d_{12} \\ q_4 &= d_2d_6d_7d_8d_{10}d_{12} \\ q_5 &= p_4p_6p_9p_{12}d_5d_8d_{10} \\ q_6 &= p_3p_6p_7p_9d_4d_7d_{12} \\ q_7 &= p_2p_5p_6p_7p_{10}p_{12}d_7d_{11} \\ q_8 &= p_1p_7p_8p_9p_{10}p_{12}d_4d_8 \end{aligned}$$

where $p_1, \ldots, p_{12}, d_1, \ldots, d_{12}$ are the generators for 2^{1+24} given in [11]. Finally test all multiples of t_{11}' and t_{12}' by products of the q_i. We find the following elements which commute with w:

$$\begin{aligned} t_{11}'' &= q_5q_6q_8t_{11}' \\ t_{12}'' &= q_4q_5q_6q_7t_{12}' \end{aligned}$$

Note also that w commutes with $q_2q_3q_4$, and modulo the central involution, also with q_4.

6 Finding the full $13B$-centralizer

In order to extend $(13{:}3 \times 2A_4){:}2$ to $13^{1+2}{:}(3 \times 2A_4){:}2$, we now seek an element of order 13 which is normalized by w. First we work in the quotient Co_1, and afterwards lift to $2^{1+24}{\cdot}\mathrm{Co}_1$.

6.1 Extending 12 to $13{:}12$ in Co_1

Now the element t_{11} maps to a $2B$-involution in the quotient Co_1, and the element of order 13 we are looking for centralizes *either* this involution, *or* t_0t_{11}. But conjugating by t_{12} interchanges these two cases, so we can assume the former. We therefore begin by making the centralizer of t_{11} in the quotient Co_1. Let

$$\begin{aligned} h_5 &= ((at_{11})^6 t_0 t_6)^4 \\ h_6 &= (t_0 t_6 (at_{11})^6)^4 \end{aligned}$$

which are elements of order 21 generating $G_2(4)$ in this centralizer. We search for a subgroup $\mathrm{PSL}_2(13)$ containing t_6, and find that $\langle h_7, t_6 \rangle$ is such a subgroup, where

$$h_7 = ((h_5h_6h_5h_6^2)^5)^{h_5h_6^7}.$$

Inside this copy of $\mathrm{PSL}_2(13)$, we find an element of order 13

$$h_8' = (h_7t_6^2)^4 t_6^2 h_7 t_6^2 (h_7t_6^4)^2$$

and the one normalized by t_6 is

$$h_8 = (h_8')^{h_7(h_8')^{10}}.$$

Then we work with the centralizer of t_6 in $G_2(4)$ to conjugate this 13-element to one which is normalized also by t_{10}. We first make this centralizer by a random search through $3A$-elements of $G_2(4)$ to find some which commute. We found

$$\begin{aligned} h_{10} &= ((h_5h_6h_5h_6h_5h_6^2h_5h_6)^7)^{h_5^{18}h_6^{10}} \\ h_{11} &= ((h_5h_6h_5h_6h_5h_6^2h_5h_6)^7)^{h_5^{18}h_6^{15}} \end{aligned}$$

which generate A_5. Conjugating by random elements of this centralizer we quickly find one of the 13-elements we are looking for, namely

$$h_{12} = (h_{10}h_{11}^2h_{10})^4 h_8 h_{10} h_{11}^2 h_{10}.$$

6.2 Lifting

The main lifting problem is to lift the element of order 13 to one which is normalized by w. Since there are 2^{24} elements of order 13 in the given coset of 2^{1+24}, only two of which are normalized by w, a brute force search is out of the question (or at least, unwieldy). We therefore do this in two stages, first finding an appropriate conjugating element to get a 13-element inverted by w^6. Since w^6 centralizes just 2^{12} out of the 2^{24} factor, this divides the problem into two searches, each in a population of size 2^{12}.

First we work in the 24-dimensional $\mathbb{F}_2$-representation of Co_1 and find that the fixed space of w^6 is spanned by vectors which lift to the following elements of 2^{1+24}: all even products of the d_i, together with

$$d_1p_1p_3p_6p_8p_{10}p_{12}.$$

Hence in the first search we may test conjugates just by the p_i. We find that the correct conjugating element is

$$p_1p_3p_5p_6p_7p_{12}.$$

In the second search we test conjugates by d_id_{i+1} and $d_1p_1p_3p_6p_8p_{10}p_{12}$. We find that exactly two conjugating elements work:

$$d_1d_2d_3d_5d_8d_9d_{11}p_1p_3p_6p_8p_{10}p_{12}$$
$$d_1d_3d_5d_6d_7d_9d_{10}d_{11}d_{12}p_1p_3p_6p_8p_{10}p_{12}.$$

Let h'_{12} and h''_{12} be the respective conjugates of h_{12}.

6.3 Finding the 12-normalizer

In order to obtain all possible 13-elements normalized by our element of order 12, we need to conjugate not just by elements of its centralizer, but by elements of its normalizer. Indeed it turns out that we need to replace w by its 7th power.

Such a normalizing element can be found inside the centralizer of w^3 as follows. First we conjugate a'_1 to t'_{11}, by conjugating by $(t'_{11}a'_1)^2$, so that we have the full $A_4 \times G_2(4)$ available. In particular, the element

$$(t'_{11}a'_1)^3(a'^2_2a'_1a'_2)(t'_{11}a'_1)^2$$

is an involution in the A_4, but not equal to t'_{11}. (In the end, we found we did not need to use this element.)

Within the A_5 generated by h_{10}, h_{11} we make the 15 involutions as conjugates, by powers of $h_{10}h_{11}$, of $(h_{10}h_{11})^2h_{11}$ and $h_{11}h_{10}h^2_{11}h_{10}$ and their product. We find that the first involution conjugated by $(h_{10}h_{11})^2$ commutes with w, and therefore the normalizing element we want is (modulo the 2-group)

$$t_{14} = (t'_{11}a'_1)^3(a'^2_2a'_1a'_2)(t'_{11}a'_1)^2(h_{10}h_{11})^3(h_{11}h_{10}h^2_{11}h_{10})(h_{10}h_{11})^2.$$

To lift to $2^{1+24}{\cdot}\mathrm{Co}_1$, we first apply the formula to get an element which commutes with w^4:

$$t'_{14} = w^4t_{14}w^4t^3_{14}w^4t_{14}.$$

Finally multiplying by combinations of the q_i we find that the element we want can be taken to be

$$t''_{14} = q_3q_7q_8t'_{14}.$$

We also made an element t'_{13} which conjugates w to its 5th power, but this turned out not to be necessary.

6.4 Testing commuting with the first 13-element

We are aiming to find the normalizer of g_8, so we have to test our candidate elements of order 13 to see which one(s) commute with $g_8^{l_3Tk_3T}$. The candidate elements are conjugates of h'_{12} and h''_{12} by combinations of $t''_{11}, t''_{12}, t'_{13}, t''_{14}$. Of these, we found that the one which works is

$$w_1 = (h''_{12})^{t''_{11}t''_{14}}.$$

We now have generators for $13^{1+2}{:}(3 \times 4A_4)$. These are best taken as the generators a'_1, a'_2, g'_5, g'_7 given above, together with the conjugate of w_1 by $(l_3Tk_3T)^{-1}$, that is

$$w'_1 = l_3Tk_3Tw_1T^{-1}k_3^{-1}T^{-1}l_3^{-1}.$$

7 Finding $3^3{:}13$

The element g_8 of order 13 lies inside a subgroup $6{}^{\cdot}\mathrm{Suz}$ of $2^{1+24}{}^{\cdot}\mathrm{Co}_1$. Now $6{}^{\cdot}\mathrm{Suz}$ also lies in a (unique) subgroup $3^{1+12}{}^{\cdot}3{}^{\cdot}\mathrm{Suz}$, of index 2 in a maximal subgroup $3^{1+12}{}^{\cdot}3{}^{\cdot}\mathrm{Suz}{:}2$ of the Monster. If we can find generators for this subgroup, then we can write down generators for a group $3^3{:}13$ containing g_8.

Our strategy is to find such a subgroup $6{}^{\cdot}\mathrm{Suz}$, which may be taken to be the centralizer in the Monster of the element za_2', and then move to the centralizer of a suitable non-central involution, where we can find an element of order 3 extending $3 \times 2^{1+6}{}^{\cdot}2{}^{\cdot}\mathrm{U}_4(2)$ to $(3^{1+4}{:}2 \times 2^{1+6}){}^{\cdot}\mathrm{U}_4(2)$, and thereby extending $6{}^{\cdot}\mathrm{Suz}$ to $3^{1+12}{}^{\cdot}2{}^{\cdot}\mathrm{Suz}$. It is then easy to write down a word for the element we want.

7.1 The subgroup $6{}^{\cdot}\mathrm{Suz}$

The element

$$s_1 = abababab^2abab^2ab^2$$

has order 66, so s_1^{22} is conjugate to a_2'. In the quotient Co_1, the elements s_1^{22} and a_2' generate a subgroup A_4, and $a_2's_1^{22}$ conjugates s_1^{22} to a_2' modulo the 2-group. Let s_1' be s_1 conjugated by $a_2's_1^{22}$. Then, modulo the 2-group, both s_1' and c_1^2 commute with a_2', and generate $3{}^{\cdot}\mathrm{Suz}$.

Hence, applying the formula, we get the following elements centralizing a_2', and generating $6{}^{\cdot}\mathrm{Suz}$:

$$\begin{aligned} s_1'' &= a_2's_1'a_2's_1'^{-1}a_2's_1' \\ s_2 &= a_2'c_1^2a_2'c_1^{-2}a_2'c_1^2 \end{aligned}$$

7.2 Changing post again

The element

$$j_2 = (s_2^2 s_1'')^6$$

turns out to be an involution mapping to Co_1-class $2A$, and forming a 2^2-group of Monster-type $2BAB$ with the central involution z of $2^{1+24}{}^{\cdot}\mathrm{Co}_1$. Our standard involution of this type is

$$j_0 = ((hi)^4 i)^{15},$$

and we can check that j_0^T lies in the 2^{1+24}. Moreover, j_0j_2 has order 5, so

$$j_3 = (j_0j_2)^2$$

conjugates j_2 to j_0.

The usual brute-force approach (used once and for all) then finds the element

$$j_4 = (ab)^{27}(ab^2)^4(abab^2)^4(ababab^2abab^2)^{13}(ababab^2ab)^9(ababab^2)^4$$

such that

$$(j_0)^{Tj_4T^{-1}} = z.$$

Hence

$$j_3Tj_4T^{-1}$$

conjugates j_2 to z.

7.3 Identifying the element of order 3

Writing

$$j_5 = ((s_1'')^{22})^{j_3 T j_4 T^{-1}}$$

we want to find words for the centralizer of the element zj_5 of order 6. This centralizer is a group of shape

$$(2^{1+6} \times 3^{1+4}{:}2){}^{\cdot}\mathrm{U}_4(2).$$

In particular, we want to find a non-central element of the normal 3^{1+4}.

We begin as usual in the Co_1 quotient, and look for $3A$-elements which commute with j_5. Writing

$$\begin{array}{rcl}
j_6 &=& (ab)(ab^2)^6(ababab^2ab)^9(ababab^2)^9 \\
j_7 &=& (ab)(ab^2)^8(ababab^2ab)^{11}(ababab^2)^4 \\
j_8 &=& (ab)^3(ab^2)^5(ababab^2ab)^{13}(ababab^2)^6 \\
j_9 &=& (ab)^3(ab^2)^{30}(ababab^2ab)^{10}(ababab^2)^4 \\
j_{10} &=& (ab)^6(ab^2)^{17}(ababab^2ab)^{12}(ababab^2)^9
\end{array}$$

we have that $j_n' = (a_2')^{j_n}$ is such a $3A$-element for $n \in \{6, 7, 8, 9, 10\}$. Moreover

$$j_5' = (j_6'j_7'j_8'j_9'j_{10}')^{12}$$

is congruent to j_5 modulo the 2-group.

To find out which of the p_n and d_n to multiply by, we apply the element $j_5(j_5')^{-1}$ to 13 carefully selected coordinate vectors, as described in [11], and read off the answer from the result. We find that the correct answer is

$$j_5'' = p_1p_6p_8p_9p_{10}p_{11}p_{12}d_1d_2d_5d_6d_{10}d_{11}d_{12}j_5'.$$

That is, j_5'' is actually equal to j_5 in the Monster.

7.4 Finding 3^{1+12}

Now apply the formula so that we get generators for the centralizer of j_5'' as follows. For $n \in \{6, 7, 8, 9, 10\}$, define

$$j_n'' = j_5''j_n'j_5''j_n'^{-1}j_5''j_n'$$

Then $j_6''j_7''(j_8''j_9''j_{10}'')^2$ has order 10 and we find that

$$j = (j_6''j_7''(j_8''j_9''j_{10}'')^2)^5(j_8''j_9''j_{10}''j_6''j_7''j_8''j_9''j_{10}'')^5$$

is an element in the normal 3^{1+4} as required.

We now know that, under the action of $6{}^{\cdot}\mathrm{Suz}$ generated by s_1'' and s_2, the element

$$j' = j^{Tj_4^{-1}T^{-1}j_3^{-1}}$$

and its conjugates generate a group 3^{1+12}.

7.5 Finding the right 3^3

Moreover, the element g_8 of order 13 acts fixed-point-freely on the natural 3^{12} quotient of 3^{1+12}. Elementary linear algebra then tells us that if we want a 3^3 on which the minimum polynomial of the action of g_8 is $x^3 - x - 1$, then we compute

$$\frac{x^{13} - 1}{(x-1)(x^3 - x - 1)} = x^9 + x^8 - x^7 + x^5 - x^3 - x^2 - 1,$$

and hence (modulo the central 3) the element

$$j'' = j'^{-1}g_8^4j'g_8j'g_8j'^{-1}g_8^2j'g_8^2j'^{-1}g_8j'^{-1}g_8^2$$

is the one we want. In fact, no correction for the centre is required.

8 Proof of the main theorem

8.1 Analysing the 78 cases

The 78 ways of extending 13 to D_{26} are obtained by taking the 6 non-central involutions in $4A_4$, and conjugating by suitable elements of 13^{1+2}. First take the involution $a_1'g_5'^3$ and check that $w_1a_1'g_5'^3$ has order 2, so that the 13 conjugates of $a_1'g_5'^3$ by powers of w_1' can be written as

$$(w_1')^n a_1' g_5'^3$$

for $0 \leq n \leq 12$. Then conjugate these 13 involutions by suitable elements of $\langle a_1', a_2' \rangle \cong 2A_4$ to get the full set of 78. For example, we may conjugate in turn by each of the six elements

$$1, a_2', a_2'^2, a_2'a_1', a_2'a_1'a_2', a_2'a_1'a_2'^2.$$

However, a subgroup $3\times2\times3$ of $13^{1+2}{:}(3\times4A_4)$ normalizes $3^3{:}13$, and it is easy to see that it fuses the 78 cases into 6 orbits, of lengths $3+3+18+18+18+18$. These six cases are represented by the cases $(w_1')^n a_1 g_5'^3$ for $n = 0, 1, 2$, and conjugates by $a_2'a_1'$. In each case we perform the following test. Given the fixed generators j'', g_8 for $3^3{:}13$, and the 6 involutions x, test each of the 13 words

$$j'' x g_8^m$$

(for $0 \leq m \leq 12$) on a random vector to see if it has order 3. Since j'' is a word involving exactly 28 occurrences of T or T^{-1}, and x averages just 4 such occurrences, each test involves on average 96 applications of T or T^{-1}, and a slightly larger number of applications of elements of $2^{1+24}{\cdot}\mathrm{Co}_1$. On my rather old laptop, such a test takes around 15 minutes, and therefore the total calculation takes around 1.5 hours.

8.2 Proofs

Most of the computer calculations were performed without proof, and therefore it is necessary to provide proofs for the few statements which we actually need in order to prove our main theorem.

Theorem 3 *There is no subgroup of the Monster isomorphic to* $\mathrm{PSL}_2(27)$.

By Theorems 1 and 2, there is a unique class of $3^3{:}13$ which we need to consider. We prove computationally that the elements j'' and g_8 generate a group $3^3{:}13$, by checking generators and relations on two vectors whose joint stabilizer in the Monster is known to be trivial. Moreover, g_8 lies in $2^{1+24}{\cdot}\mathrm{Co}_1$, so is in Monster-class $13B$.

Similarly, we check generators and relations for $(13{:}3 \times 2A_4){:}2$. Also, we check that w_1' is an element of order 13, and that it commutes with g_8 and with $a_1'g_5$. Since the latter element inverts g_8, we deduce that w_1' is not a power of g_8 (or this could be checked directly). Hence the given elements generate $13^{1+2}{:}(3 \times 4A_4)$, as required.

Therefore, the test runs through all the involutions inverting g_8, and since the test failed in every one of the six cases, the proof is complete.

Acknowledgements

Thanks are due to EPSRC, whose grant, number GR/S41319, paid for the laptop on which these calculations were performed.

References

[1] J. N. Bray, An improved method for generating the centralizer of an involution, *Arch. Math. (Basel)* **74** (2000), 241–245.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An Atlas of Finite Groups*, Oxford University Press, 1985.

[3] P. E. Holmes, A classification of subgroups of the Monster isomorphic to S_4 and an application, *J. Algebra* **319** (2008), 3089–3099.

[4] P. E. Holmes and R. A. Wilson, A new computer construction of the Monster using 2-local subgroups, *J. London Math. Soc.* **67** (2003), 349–364.

[5] P. E. Holmes and R. A. Wilson, A new maximal subgroup of the Monster, *J. Algebra* **251** (2002), 435–447.

[6] P. E. Holmes and R. A. Wilson, $\mathrm{PSL}_2(59)$ is a subgroup of the Monster, *J. London Math. Soc.* **69** (2004), 141–152.

[7] P. E. Holmes and R. A. Wilson, On subgroups of the Monster containing A_5's, *J. Algebra* **319** (2008), 2653–2667.

[8] S. A. Linton, R. A. Parker, P. G. Walsh and R. A. Wilson, Computer construction of the Monster, *J. Group Theory* **1** (1998), 307–337.

[9] S. Norton, Anatomy of the Monster: I, in *Proceedings of the* ATLAS *Ten Years On conference (Birmingham 1995)*, pp. 198–214, Cambridge Univ. Press, 1998.

[10] S. P. Norton and R. A. Wilson, Anatomy of the Monster: II, *Proc. London Math. Soc.* **84** (2002), 581–598.

[11] S. P. Norton and R. A. Wilson, A correction to the 41-structure of the Monster, a construction of a new maximal subgroup $\mathrm{L}_2(41)$, and a new Moonshine phenomenon, (19 pages), *J. London Math. Soc.*, published online Jan. 2013. doi: 10.1112/jlms/jds078

[12] R. A. Wilson, The odd-local subgroups of the Monster, *J. Austral. Math. Soc. (A)* **44** (1988), 1–16.

[13] R. A. Wilson, Is J_1 a subgroup of the Monster?, *Bull. London Math. Soc.* **18** (1986), 349–350.

[14] R. A. Wilson, *The finite simple groups*, Springer GTM 251, 2009.

[15] R. A. Wilson et al., *An Atlas of Group Representations*, http://brauer.maths.qmul.ac.uk/Atlas/.